Quest Journals
Journal of Research in Applied Mathematics
Volume 7 ~ Issue 5 (2021) pp: 01-06
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

Core theorems of double sequences through the generalized de la Vallée-Poussin Mean

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ABSTRACT
The convergence of double sequences was a natural extension of the convergence of sequences. The transform of a sequence by a matrix A gives rise to the A-summability. It is natural that in order to find analogue of A-summability for double sequences, the matrix A is taken four-dimensional. It is pertinent to find the analogue of core of sequences for the double sequences. The aim of this paper is to use the generalized double de la Vallée-Pousin mean to find analogues of some results related to the Pringsheim P-core of double sequences.
KEY WORDS: *Double sequences, Bounded double sequences, Almost Convergence, Core theorems, Pringsheim core, Double De la Vallée-Pousin Mean.*

Received 14 May, 2021; Revised: 28 May, 2021; Accepted 30 May, 2021 © The author(s) 2021.
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I. INTRODUCTION

The concept of the core of a sequence was first introduced by Knopp [1], now known as the Knopp core. Let $x = \{x_k\}$ be a sequence in $\mathbb{C}$, the set of all complex numbers and C_k be the least convex closed region of complex plane containing $x_k, x_{k+1}, x_{k+2}, \dots$. The Knopp core of x (K-core of x or core of x) is defined by the intersection of all C_k $(k = 1,2,..)$. In the real case the K-core of x is reduced to the closed interval $[lim\ inf\ x, lim\ sup\ x]$. If A is a non-negative regular matrix, then the core of x is contained in the core of Ax, provided that Ax exists. Rhoades [2] gave a slight generalization of Knopp's core theorem in 1960. In 1979, Maddox [3] gave some analogues of Knopp's core theorem.

In 1999, Patterson [4] extended the Knopp core for the double sequences using the convergence of double sequences defined by Pringsheim [5], called it Pringsheim core (shortly, P-core) which is given by $[P - lim\ inf\ x,\ P - lim\ sup\ x]$, and proved some result on them. In 2002, the M-core and σ-core for double sequences were defined and studied by Mursaleen and Edely [6] and Mursaleen and Mohiuddine ([7] and [8]), respectively. The σ-core for single sequences was given by Mishra et al [9]. Kayaduman and Çakan [10] presented the concept of Cesáro core of double sequences.

Mohiuddine and Alotaibi [11] presented a generalization of the notion of almost convergent of double sequence with the help of de la Vallée-Poussin mean and called it $[\lambda, \mu]$ -almost convergent. Using this concept, they defined the notions of regularly of $[\lambda, \mu]$ -almost conservative and $[\lambda, \mu]$ -almost coercive four-dimensional matrices and obtain their necessity and sufficient conditions. Further, they introduced the space $\mathcal{L}_1$ of all absolutely convergent double series and characterize the matrix class $(\mathcal{L}_1, \mathcal{F}_{[\lambda,\mu]})$, where $\mathcal{F}_{[\lambda,\mu]}$ denotes the space of $[\lambda, \mu]$ -almost convergence for double sequences.

Definition 1.1 [5]: A double sequence $x = (x_{jk})$ is said to be convergent to L in the Pringsheim's sense (or P-convergent to L) if for a given $\varepsilon > 0$ there exists an integer N such that $|x_{jk} - L| < \varepsilon$ whenever $j, k > N$. The space of P-convergent sequences is denoted by $\mathcal{C}_P$.

Definitions 1.2 [5]: A double sequence $x = (x_{jk})$ is said to be bounded if $\|x\| = \sup_{j,k \geq 0} |x_{jk}| < \infty$. We denote the space of all bounded double sequences by $\mathfrak{J}_\infty$.

The space of double sequences which are both bounded and P-convergent are denoted by $\mathcal{C}_{BP}$.
Let $\{A = a_{pqmn}, p, q = 0,1,2, \dots\}$ be a doubly infinite matrix of real numbers for all m, n = 0, 1, 2,… . Forming the sums

$$y_{pq} = (Ax)_{pq} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{pqmn}\, x_{mn}\ ,$$

called the A-mean of the sequence $x = \{x_{jk}\}$, yield a method of summability. More exactly, we say that a sequence $x = \{x_{jk}\}$ is A-summable to the limit l if the A-mean exists for all j, k = 0, 1, 2, … in the sense of Pringsheim, i. e.,

$$\lim_{m,n\to\infty} \sum_{j=0}^{m} \sum_{k=0}^{n} a_{pqjk}\, x_{jk} = y_{pq} \; and \; \lim_{pq\to\infty} y_{pq} = l$$

We say that a matrix A is bounded regular if every bounded and convergent sequence $x = \{x_{jk}\}$ is A-summable to the same limit and the A-means are bounded, Başarir [12].

Definition 1.3 [11]**:** A double sequence $x = \{x_{jk}\}$ of real is said to be $[\lambda, \mu]$-almost convergent (briefly, $\mathcal{F}_{[\lambda,\mu]} - convergent$) to some number l if $x \epsilon \mathcal{F}_{[\lambda,\mu]}$, where

$\mathcal{F}_{[\lambda,\mu]} = \{\, x = \{x_{jk}\}: p - \lim_{mn\to\infty} \Omega_{mns,t}\,(x) = L \; exists, uniformly \; in \; s,t; L = \mathcal{F}_{[\lambda,\mu]} - \lim\, x\}$,

$$\Omega_{mn,s,t}(x) = \frac{1}{\lambda_m \mu_n} \sum_{j\in J_m} \sum_{k\in I_n} x_{j+s,k+t}\ ,$$

Denote by $\mathcal{F}_{[\lambda,\mu]}$, the space of all $[\lambda, \mu]$-almost convergent sequence $\{x_{jk}\}$. Note that $\mathcal{C}_{BP} \subset \mathcal{F}_{[\lambda,\mu]} \subset \ell_\infty$.

Definition 1.4 [4] Let $x = \{x_{j,k}\}$ be a double sequence of real numbers and for each n, let $\alpha_n = sup_n\{x_{j,k}: j, k \geq n\}$ The Pringsheim limit superior of $\{x\}$ is defined as follows:

(1) If $\alpha_n = +\infty$ for each n, then $P - lim\, sup\{x\} := +\infty$;

(2) If $\alpha_n < \infty$ for some n, then $P - lim\ sup\{x\} := inf_n\{\alpha_n\}$

Similarly, let $\beta_n = inf_n\{x_{j,k}: j, k \geq n\}$ then the Pringsheim limit inferior of $\{x_{j,k}\}$ is defined as follows:

(3) If $\beta_n = -\infty$ for each n, then $P - lim\, inf\{x\} := -\infty$;

(4) If $\beta_n > -\infty$ for some n, then $P - lim\ inf\{x\} := sup_n\{\beta_n\}$

Let $\lambda = (\lambda_m: m = 0,1,2, \dots)\; and\; \mu = (\mu_n: n = 0,1,2, \dots)$ be two nondecreasing sequences of positive real with each tending to ∞ such that $\lambda_{m+1} \leq \lambda_m + 1, \lambda_1 = 0, \mu_{n+1} \leq \mu_n + 1, \mu_1 = 0$ and define

$$\mathfrak{I}_{mn}(x) = \frac{1}{\lambda_m \mu_n} \sum_{j\in J_m} \sum_{k\in I_n} x_{j,k} \qquad (1.1)$$

called the $double\ generalized\ de\ la\ Vall\acute{e}e - Poussin\ mean$, where $J_m = [m - \lambda_m + 1, m]$ and $I_n = [n - \mu_n + 1, n]$. We denote the set of all $\lambda\ and\ \mu$ type sequence by using the symbol $[\lambda, \mu]$. We wish to study the core of double sequences via the generalized double de la Vallée-Poussin mean.

Define the following sub-linear functional on $\mathcal{L}_\infty$.

$$\Gamma(x) = \limsup_{m,n\to\infty} \frac{1}{\lambda_m \mu_n} \sum_{j\in J_m} \sum_{k\in I_n} x_{j+s,k+t}$$

Then the $\mathcal{F}_{[\lambda,\mu]} - core$ of a real-valued bounded double sequence $\{x_{j,k}\}$ is defined to be the closed interval $[-\Gamma(-x), \Gamma(x)]$. Since BP-convergent double sequence is $\mathcal{F}_{[\lambda,\mu]}$-convergent, we have, $\Gamma(x) \leq L(x)$, where $\mathrm{L}(x) = P - \lim sup\, x$ and hence it follows that $\mathcal{F}_{[\lambda,\mu]} - core\{x\} \subseteq P - core\{x\}$ for all $x \in \mathcal{L}_\infty$.

II. MATERIALS AND METHOD

The following results are used in our work to establish the results in the next sections.

Theorem 2.1 [13]**:** The four-dimensional matrix $A = (a_{pqmn})$ is RH-regular if and only if:

(RH_1) P-$\lim_{p,q\to\infty} a_{p,q,m,n} = 0$, for each m and n

(RH_2) $P - \lim\limits_{p,q\to\infty} \sum_{m=1}^{p} \sum_{n=1}^{q} a_{pqmn} = 1$

(RH_3) $P - \lim_{p,q\to\infty} \sum_{m=1}^{p} |a_{pqmn}| = 0$, for each n,

(RH_4) $\mathrm{P} - \lim_{p,q\to\infty} \sum_{n=1}^{q} |a_{pqmn}| = 0$, for each m,

(RH_5) $\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |a_{mnjk}|$ is P-convergent; and

(RH_6) there exist positive numbers A and B such that $\sum_{j>B}^{\infty} \sum_{k>B}^{\infty} |a_{mnjk}| < A$

Theorem 2.2 [4] If A is a non-negative RH-regular summability matrix, then $P - C\{Ax\} \subseteq P - C\{x\}$ for any bounded sequence $\{x\}$ for which $\{Ax\}$ exists.

Lemma 2.1 [4] If $A = (a_{mnjk})$ is a four-dimensional matrix, such that (RH_1), (RH_3), (RH_4) and

$$\mathrm{P} - \limsup_{m,n} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |a_{mnjk}| = M,$$

holds, then for any bounded double sequence $x = \{x_{jk}\}$,

$$p - \lim\sup\ \{|Ax|\} \leq M(p - \lim\sup\ \{|x|\}),$$

where,

$$y_{mn} = \sum_{j,k=0,0}^{\infty,\infty} a_{mnjk}\, x_{j,k}$$

In addition, there exists a real-valued double sequence {x} such that if a_{mnjk} is real with $0 < P - \lim\sup\, \{|x|\} < \infty$, then

$$limsup\{|y|\} = M(P - limsup\{|x|\}).$$

III. RESULTS

Lemma 3.1. If $A = (a_{mnjk})$ is a four-dimensional matrix, such that (RH$_1$), (RH$_3$), (RH$_4$) and

$$p-\limsup_{m,n} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|a_{mnjk}| = M,$$

holds, then for any bounded double sequence $x = \{x_{jk}\}$, we obtain the following:

$$p-\lim\sup\ \{A\Im\} \le M(p-\lim\sup\ \{\Im\}),$$

where,

$$\Im = \Im_{m,n}(x) = \frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n} x_{j,k}$$

$$A\Im = A\Im_{m,n}(x) = \frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n} a_{mnjk}x_{j,k}$$

In addition, if there exists a real – valued double sequence $x = \{x_{jk}\}$ such that, a_{mnjk} is real with $0 < p-\lim\ sup\{\Im\} < \infty$, then

$$p-\lim\ sup\{|A\ \Im|\} = M(p-lim\ sup\ \{A\Im\}),$$

where $\Im$ is the generalized double de la Vallẻe-Pousin mean.

Proof

Let $x = Sup_{j,k}|x_{j,k}| < \infty$ and let $\beta \coloneqq P-\lim_{m,n} Sup|x_{j,k}| < \infty$, for any $\varepsilon > 0$, there exists a positive integer N such that $|x_{j,k}| < \frac{(\beta+\varepsilon)}{3}$, for each j, k >N.

$$|A\Im_m(x)| \le \frac{1}{\lambda_m\mu_n}\sum_{j=0}^{N}\sum_{k=0}^{N}|a_{mnjk}||x_{,k}| + \frac{1}{\lambda_m\mu_n}\sum_{0\le j\le N}\sum_{N<k<\infty}|a_{mnjk}||x_{,jk}| + \frac{1}{\lambda_m\mu_n}\sum_{N<j\le\infty}\sum_{0\le k\le N}|a_{mnjk}||x_{jk}| + \frac{1}{\lambda_m\mu_n}\sum_{j=N+1}^{\infty}\sum_{k=N+1}^{\infty}|a_{mnjk}||x_{jk}|$$

$$\le \frac{1}{\lambda_m\mu_n}\sum_{j=0}^{N}\sum_{k=0}^{N}|a_{mnjk}||x_{jk}| + \frac{1}{\lambda_m\mu_n}\sum_{0\le j\le N}\sum_{N<k<\infty}|a_{mnjk}|\frac{(\beta+\varepsilon)}{3} + \frac{1}{\lambda_m\mu_n}\sum_{N<j\le\infty}\sum_{0\le k\le N}|a_{mnjk}|\frac{(\beta+\varepsilon)}{3} + \frac{1}{\lambda_m\mu_n}\sum_{j=N+1}^{\infty}\sum_{k=N+1}^{\infty}|a_{mnjk}|\frac{(\beta+\varepsilon)}{3}$$

which yields

$$P-\lim\ sup\{|\ \Im_{m,n}(x)|\} = M(\beta+\varepsilon)$$

Therefore, the following holds:

$$P-\lim\ sup\{|\ \Im_{m,n}(x)|\} = M(p-lim\ \sup\ [|x|])$$

Since P $-\lim\sup_{m,n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|a_{mnjk}| = M$

We may assume that M > 0 without loss of generality. Using RH-regularity conditions, we choose $m_0, n_0, j_0\ and\ k_0$, so large that

$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|a_{m_0n_0j_0k_0}| > M-\frac{1}{4}$, $\sum_{0<j<j_0}\sum_{k_0\le k\le\infty}|a_{m_0n_0j_0k_0}| \le \frac{1}{4}$,

$\sum_{0<j<j_0}\sum_{k_0\le k\le\infty}|a_{m_0n_0j_0k_0}| \le \frac{1}{4}$, $\sum_{j=j_0}^{\infty}\sum_{k=k_0}^{\infty}|a_{m_0n_0j_0k_0}| \le \frac{1}{4}$.

Let $[m_{p-1}], [n_{q-1}], [j_{p-1}]and\ [k_{q-1}]$ be four chosen strictly increasing index sequences with $p, q = 1,2,\dots,i-1,\dots,r-1$ with $j_0 = k_0 > 0$. Using the RH-regularity conditions we now choose $m_i > m_{i-1}$ and $n_r > n_{r-1}$ such that

$\sum_{0\le j\le j_{i-1}}\sum_{0\le k\le\infty}|a_{m_in_rjk}| < \frac{1}{2^{i+r}}$, $\sum_{0\le k\le k_{r-1}}\sum_{k_{r-1}<k\le\infty}|a_{m_in_rjk}| < \frac{1}{2^{i+r}}$,

$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|a_{m_in_rjk}| > M-\frac{1}{2^{i+r}}$

Let us define {x} as follows:

$$x_{jk} \coloneqq \begin{cases} \dfrac{\hat{a}_{m_in_r,jk}}{a_{m_in_rjk}}, & if\ j_{i-1} < j < j_i, k_{r-1} < k < k_r\ and\ a_{m_in_rjk} \ne 0; \\ 0, & otherwise. \end{cases}$$

Consider the following:

$$\left|A\Im_{m_i n_r}(x)\right| = \left|\frac{1}{\lambda_{m_i}\mu_{n_r}}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{m_i n_r jk}\, x_{jk}\right|$$
$$\geq -\frac{1}{\lambda_{m_i}\mu_{n_r}}\sum_{0\leq j\leq j_{i-1}}\sum_{0\leq k\leq\infty}\left|a_{m_i n_r jk}\right| - \frac{1}{\lambda_{m_i}\mu_{n_r}}\sum_{0\leq j\leq j_{i-1}}\sum_{k_{r-1}<k\leq\infty}\left|a_{m_i n_r jk}\right|$$
$$-\frac{1}{\lambda_{m_i}\mu_{n_r}}\sum_{k_{r-1}<k<k_r}\sum_{j_i\leq j\leq\infty}\left|a_{m_i n_r jk}\right| - \frac{1}{\lambda_{m_i}\mu_{n_r}}\sum_{j_{i-1}<j<\infty}\sum_{k_r\leq k\leq\infty}\left|a_{m_i n_r jk}\right| < \frac{1}{2^{i+r}}$$
$$+\frac{1}{\lambda_{m_i}\mu_{n_r}}\sum_{j_{i-1}<j<j_i}\sum_{k_{r-1}<k<k_r} a_{m_i n_r jk}\, sgn(a_{m_i n_r jk})$$
$$\geq -\frac{1}{2^{i+r}} - \frac{1}{2^{i+r}} - \frac{1}{2^{i+r}} - \frac{1}{2^{i+r}} + M - 5\left(\frac{1}{2^{i+r}}\right) = M - 9\frac{1}{2^{i+r}}$$

This implies that

$$P-\lim\ sup\{|\,\Im_{m,n}(x)|\} \geq M = M(p-lim\ \sup\,[|x|])$$

Thus, if A is real-valued then so is [x] with $0 < limSup[x] < \infty$

$$P-\lim\ sup\{|\,\Im_{m,n}(x)|\} = M(p-lim\ \sup\,[|x|])$$

This completes the proof.

We use the above lemma to prove the following theorem.

Theorem 3.2

If $A = (a_{mnjk})$ is a four–dimensional matrix, then the following are equivalent

(i) For all real – valued double sequences $x = \{x_{jk}\}$

$$p-\lim sup\,\{A\,\Im\} \leq p-\lim sup\,\{x\}$$

(ii) A is an RH – regular summability matrix with

$$p-\lim_{m,n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left|a_{mnjk}\right| = 1 \qquad (3.1)$$

__Proof__ (i) ⇒ (ii)

Let $x = \{x_{jk}\}$ be a bounded p–convergent double sequence.

Then $p-\lim\inf\{\Im\} \leq p-\lim\sup\{\Im\} = p-\lim\{\Im\}$

And also,

$$p-\lim\sup\{|A(-\Im)|\} \leq -(p-\lim inf\,\{\Im\})$$

These imply that

$$p-\lim\inf\{\Im\} \leq P-\lim\inf\,\{A\Im\} \leq p-\lim sup\,\{A\Im\} \leq p-lim\ sup\,\{\Im\}$$

Hence $\{A\Im\}$ is p– convergent and $p-\lim\{A\Im\} = p-\lim\{\Im\}$.

Therefore, A is an RH – regular summability matrix. By Lemma 3.1, there exists a bounded double sequence $x = \{x_{jk}\}$ such that $\lim\sup\{|\Im|\} = 1$ and $p-\lim\sup\{A\Im\} = A,$ where A is defined by (RH_6). This implies that

$$1 \leq p-\lim inf_{m,n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{mnj,k} \leq p-\lim\ sup_{m,n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{mnjk} \leq 1$$

whence

$$p-\lim_{m,n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left|a_{mnjk}\right| = 1$$

(ii) ⇒ (i)

Here we show that if $\{\Im\}$ is a p– convergent sequence and A is an RH – regular matrix satisfying (3.1), then

$$p-\lim\{A\Im\} \leq p-\lim\sup\{\Im\}$$

For p, q > 1, we obtain the following

$$A\Im \leq \left|\frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n} a_{mnjk}\, x_{jk}\right|$$
$$= \frac{1}{\lambda_m\mu_n}\left|\sum_{j\in J_m}\sum_{k\in I_n}\frac{|a_{mnjk}\,x_{jk}| - a_{mnjk}\,x_{jk}}{2} + \sum_{j\in J_m}\sum_{k\in J_n}\frac{|a_{mnjk}\,x_{jk}| + a_{mnjk}\,x_{jk}}{2}\right|$$
$$\leq \frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n}\left|a_{mnjk}\right|\left|x_{jk}\right| + \frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in J_n}(\left|a_{mnjk}\right| - a_{mnjk})|x_{jk}|$$
$$\leq$$
$$\frac{||x||}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n}\left|a_{mnjk}\right| + \frac{||x||}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in J_n}\left|a_{mnjk}\right| + \frac{||x||}{\lambda_m N_n}\sum_{j\in J_m}\sum_{k\in I_n}\left|a_{mmjk}\right| +$$
$$\sup_{j,k>p,q}\frac{||x||}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n}\left|a_{mmjk}\right| + \frac{|x|}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n}(\left|a_{mnjk}\right| - a_{mnjk}).$$

Using (RH_1)- (RH_4) and (3.6), we take the Pringsheim limit to get the required result.

Theorem 3.3:

If $A = (a_{mn,jk})$ is a non-negative RH-regular summability matrix, then

$$\mathcal{F}_{[\lambda,\mu]} - core\{A\mathfrak{J}\} \subseteq \mathcal{F}_{[\lambda,\mu]} - core\{\mathfrak{J}\}$$

For any bounded $\mathcal{F}_{[\lambda,\mu]}$-double sequence $\{x\}$ for which $A\mathfrak{J}$ exist.

Proof:

We have

$$\mathfrak{J} = \mathfrak{J}_{m,n}(x) = \frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n} x_{j,k}$$

$$A\mathfrak{J} = A\mathfrak{J}_{m,n}(x) = \frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n} a_{mnjk}x_{j,k}$$

If $\mathcal{F}_{[\lambda,\mu]} - core\{\mathfrak{J}\}$ is the complex plane, then the result is trivial. Now we consider the case where {x} is bounded or unbounded and establish the required result. In both cases, the result will be established by proving the following:

If there exists a q such that $\omega \notin \mathcal{F}_{[\lambda,\mu]} - core_q\{\mathfrak{J}\}$, then there exist a p such that $\omega \notin \mathcal{F}_{[\lambda,\mu]} - core_p\{\mathfrak{J}\}$. When {x} is bounded $\omega \notin \mathcal{F}_{[\lambda,\mu]} - core\{\mathfrak{J}\}$ is not in the complex plane, thus there exists an $\omega \notin \mathcal{F}_{[\lambda,\mu]} - core\{\mathfrak{J}\}$. This implies that there exists a q for which $\omega \notin \mathcal{F}_{[\lambda,\mu]} - core_q\{\mathfrak{J}\}$. Since ω is finite, we may assume that ω=0 by linearity of A. Since we are also given that $\omega \notin \mathcal{F}_{[\lambda,\mu]} - core_q\{\mathfrak{J}\}$ is a convex set, we can rotate $\omega \notin \mathcal{F}_{[\lambda,\mu]} - core_q\{\mathfrak{J}\}$ so that the distance from zero to $\omega \notin \mathcal{F}_{[\lambda,\mu]} - core_q\{\mathfrak{J}\}$ is the minimum of $\{|\mathfrak{J}|: \mathfrak{J} \in \omega \notin \mathcal{F}_{[\lambda,\mu]} - coreq\mathfrak{J},$ and is on positive real axis; say that this minimum is 3d. Since $\omega \notin \mathcal{F}[\lambda,\mu] - coreq\{\mathfrak{J}\}$ is convex, all points on $\omega \notin \mathcal{F}_{[\lambda,\mu]} - core_q\{\mathfrak{J}\}$ have real part which is at least 3d. Let $M = max\left\{\frac{|x_{j,k}|}{\lambda_m\mu_n}\right\}$. By regularity conditions (RH_1) - (RH_4) and assumption $a_{mnjk} \geq 0$, there exists an N such that for m,n>N, the following hold:

$$\sum_{j,k\in\alpha_1} a_{mnjk} < \frac{d}{3M}, \sum_{j,k\in\alpha_2} a_{mnjk} < \frac{d}{3M}$$

$$\sum_{j,k\in\alpha_3} a_{mnjk} < \frac{d}{3M}, \sum_{j,k\in\alpha_4} a_{mnjk} < \frac{d}{3M}$$

where,

$$\alpha_1 = \{(j,k): 0 \leq j \leq j_0 \ and\ 0 \leq k \leq k_0\},$$
$$\alpha_2 = \{(j,k): j_0 \leq j < \infty \ and\ 0 \leq k \leq k_0\},$$
$$\alpha_3 = \{(j,k): 0 < j \leq j_0 \ and\ k_0 < k < \infty\},$$
$$\alpha_4 = \{(j,k): j_0 < j < \infty \ and\ k_0 < k < \infty\}.$$

Therefore, for m, n > N,

$$R\left\{\frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n} a_{mnjk}x_{j,k}\right\} = R\left\{\frac{1}{\lambda_m\mu_n}\sum_{j\in\alpha_1}\sum_{k\in\alpha_1} a_{mnjk}x_{j,k}\right\} + R\left\{\frac{1}{\lambda_m\mu_n}\sum_{j\in\alpha_2}\sum_{k\in\alpha_2} a_{mnjk}x_{j,k}\right\} + R\left\{\frac{1}{\lambda_m\mu_n}\sum_{j\in\alpha_3}\sum_{k\in\alpha_3} a_{mnjk}x_{j,k}\right\} + R\left\{\frac{1}{\lambda_m\mu_n}\sum_{j\in\alpha_4}\sum_{k\in\alpha_4} a_{mnjk}x_{j,k}\right\}$$

$$> -M\left\{\sum_{j,k\in\alpha_1} a_{mnjk}\right\} - M\left\{\sum_{j,k\in\alpha_2} a_{mnjk}\right\} - M\left\{\sum_{j,k\in\alpha_3} a_{mnjk}\right\} + 3d\left\{\sum_{j,k\in\alpha_4} a_{mnjk}\right\}$$

$$> -M\frac{3d}{3M} + 3d\frac{2}{3} = d.$$

Therefore, $R\{A\mathfrak{J}\} > d$, which implies that there exists a p for which ω=0 is also outside $\omega \notin \mathcal{F}_{[\lambda,\mu]} - core_p\{\mathfrak{J}\}$. Now suppose that {x} is unbounded. Then ω may be the point at infinity or not. If ω is not the point at infinity, then choose N such that for m, n > N, the following hold:

$$\left\{\sum_{j,k\in\alpha_1} a_{mnjk}\right\} < \frac{d}{3M}, \sum_{j,k\in\alpha_2\cup\alpha_3\cup\alpha_4} a_{mnjk} > \frac{2}{3}$$

In a manner similar to the first part, we obtain $R\{A\mathfrak{J}\} > d$. In the case when ω is the point at infinity, $\omega \notin \mathcal{F}[\lambda,\mu] - coreq\{\mathfrak{J}\}$ is bounded for j, k > q. We may assume that [|x|]<A for some positive number A without loss of generality. Thus for m and n large, we obtain the following:

$$\left|\frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n} a_{mnjk}x_{j,k}\right| \leq \frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n} a_{mnjk}|x_{j,k}| \leq \frac{|x_{j,k}|}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n} a_{mnjk} \leq A\sum_{j\in J_m}\sum_{k\in I_n} a_{mnjk} < \infty.$$

Hence there exists a p such that the point at infinity is outside of $\omega \notin \mathcal{F}_{[\lambda,\mu]} - core_q\{\mathfrak{J}\}$.

This completes the proof of the theorem.

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