Quest Journals
Journal of Research in Applied Mathematics
Volume 7 ~ Issue 9 (2021) pp: 17-27
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

Block Formulation of Linear Multistep Methods for Initial Value Problems in Ordinary Differential Equations

I. M. ESUABANA*[1], F. O. TOM[2],U. A. ABASIEKWERE[3]
[1,2]Department of Mathematics, University of Calabar, P.M.B. 1115, Calabar, Cross River State, Nigeria
[3]Department of Mathematics, University of Uyo, P.M.B. 1017, Uyo, Akwa Ibom State, Nigeria

Abstract: *This work seeks to obtain more accurate approximate solutions to mathematical modelsin the form of ordinary differential equations which do not have analytical solutions. This method is derived using Taylor's series approach with the help of MATHEMATICA software and it satisfies the basic requirements for convergence such as zero-stability, consistency and A-stability and can be applied in solving initial value problems in ordinary differential equations of order two or a system of second order initial value problems in ordinary differential equations with less emphasis on the choice of the step size used. The results obtained show that our proposed method is more efficient than some methods in literature in terms of accuracy.*
Keywords: *Initial value problem, Multi-step, Ordinary Differential Equation, numerical.*
Subject Classification codes: *65L05*

Received 01 September, 2021; Revised: 12 September, 2021; Accepted 14 September, 2021

I. INTRODUCTION

Modeling of physical phenomenon has result to complex equations and some appear in the form of ordinary differential equations.The solution of these equations required suitable numerical method to achieved faster convergence of the solution component. In spite of the existing methods for solving initial value problems for ordinary differential equations [7], [8], [9], [10], etc, there is need to search for a better method that would increase the accuracy of solutions, reduce the errors of the existing ones.

The aim in this work is to formulate a block method from linear multistep methods for initial value problems in ordinary differential equations which provides an alternative method for solution of IVPs in stiff ODEs. To investigate the stability and convergence of the block methods.

This study concentrates on the development of new block method and efficient codes that are based on implicit block methods for numerical solution of IVPs. The proposed method cannot be used in solving partial differential equations and the analysis is limited to the order, consistency, zero stability and convergence of the method. We only concentrate on achieving a family of block methods that is *A*-stable.

II. PRELIMINARIES

Here, we consider the tools needed for the derivation of a block formulation of linear multistep methods for initial value problems in ordinary differential equations.
Taylor's series approach is used in this process and the values of the unknown variables, the order of the methods and the error constants are obtained using *Mathematica*software.
We shall also consider a block method formulated by [11]and defined as:

$$y_m = \sum_{i=1}^{k} A_i\, y_{m-i} + h^3 \sum_{i=0}^{k} B_i\, f_{m-i}, \qquad (1)$$

where A_i's and B_i's are properly chosen rxr matrices and m = 0, 1, 2, … represent the block number in the mth block, r being the proposed block size. Where

$$y_m = \begin{bmatrix} y_n \\ . \\ . \\ . \\ y_{n+r-1} \end{bmatrix}, \quad f_m = \begin{bmatrix} f_n \\ . \\ . \\ . \\ f_{n+r-1} \end{bmatrix},$$

respectively.

Convergence: According to [1], [2], [3] the necessary and sufficient condition for a LMM to be convergent is to be consistence and zero stable. Since the method satisfies these conditions hence, it is convergent.

Consistency: According to [4],[5] a linear multistep method is said to be consistent if it has order at least one (i.e. $v \geq 1$). Owing to this definition, the block methods constructed are of order two, three, four and five. Thus the schemes are consistent.

Zero Stability: Block is said to be zero stable if the roots of the first characteristics polynomial p(r) defined by P(r) = det $[rA_0 - B]$ have modulus less than or equal to unity and those of modulus unity are simple (i.e. satisfies $|r| \leq 1$ and every root with $|r_0|$ = 1 has multiplicity not exceeding two in the limit as h → 0) [6].

Order and Error Constant: The linear operator associated with Eq. (1) can be defined as

$\mathcal{L}\{y(x):h\} = \sum_{i=1}^{k} A_i\, Y_{m-i} + h\sum_{i=0}^{k} B_i F_{m-i}$(2)

Eq. (2) is expanded in Taylor series, which gives

$\mathcal{L}\{y(x):h\} = C_0 y(x) + C_1 h y'(x) + \cdots + C_v h^{(v)} y^{(v)}(x) + C_{v+1} h^{(v+1)} y^{(v+1)}(x) + \cdots$

where

$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \ldots \alpha_k$,

$C_1 = \alpha_1 + 2\alpha_2 + \ldots + K\alpha_k - (\beta_1 + \beta_1 + \cdots + \beta_k)$,

.

.

.

$C_v = \frac{1}{v!}(\alpha_1 + 2^v\alpha_2 + \cdots + k^v\alpha_k) - \frac{1}{(v-1)!}(\beta_1 + 2^{v-1}\beta_2 + \cdots + k^{v-1}\beta_k)$, v =0,1, 2 , 3, … k

III. MAIN RESULTS

In this chapter the derivation of the block formulation of linear multistep methods for initial value problems in ordinary differential equations will be carried out and the result will also be discussed.

Derivation of the proposed methods: Consider the block method defined in (1) as:

$$y_m = \sum_{i=1}^{k} A_i\, y_{m-i} + h^3 \sum_{i=0}^{k} B_i\, f_{m-i}$$

Adjusting the highest power of h to 2 for second order initial value problems in ordinary differential equations and introduced F'_{m-1} *and* F'_m the derivative points then we have

$A_0 Y_m = BY_{m-1} + hGF_m + h^2 DF'_{m-1} + h^2 EF'_m$,(3)

where

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & . & . & 0 \\ 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & 0 & . & 0 \\ . & . & 0 & 1 & 0 & 0 \\ . & . & . & 0 & 1 & 0 \\ 0 & 0 & 0 & . & . & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & . & . & 1 \\ 0 & 0 & 0 & . & . & 1 \\ 0 & 0 & 0 & 0 & . & 1 \\ . & . & 0 & 0 & 0 & 1 \\ . & . & . & 0 & 0 & 1 \\ 0 & 0 & 0 & . & . & 1 \end{bmatrix},$$

$$G = \begin{bmatrix} g_1 & g_2 & . & . & . & g_r \\ 0 & g_{r+1} & . & . & . & g_{2r-1} \\ 0 & 0 & g_{2r} & . & . & g_{2r} \\ . & . & 0 & g_{2r+2} & . & g_{2r+1} \\ . & . & 0 & 0 & g_{3r} & g_{2r+2} \\ 0 & 0 & 0 & 0 & 0 & g_{3r+3} \end{bmatrix}, \quad E = \begin{bmatrix} e_1 & e_2 & . & . & . & e_r \\ e_{r+1} & e_{r+2} & . & . & . & e_{2r} \\ e_{2r+1} & e_{2r+2} & . & . & . & e_{3r} \\ e_{3r+1} & e_{3r+2} & . & . & . & e_{4r} \\ . & . & . & . & . & . \\ e_{nr+1} & e_{nr+2} & . & . & . & e_{(n+1)r} \end{bmatrix},$$

$$D = \begin{bmatrix} d_1 & 0 & 0 & 0 & 0 & 0 \\ d_2 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ d_r & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, F_m = [f_{m+1}, f_{m+2}, \dots, f_{m+r}]^T, \ F_m^1 = [f_{m+1}^1, f_{m+2}^1, \dots, f_{m+r}^1]^T,$$

$F_{m-1}^1 = [f_{m-r+1}^1, \dots, f_m^1]^T$, $Y_m = [y_{m+1}, y_{m+2}, \dots, y_{m+r}]$, $Y_{m-1} = [y_{m-r+1}, \dots, y_m]$.

When r = 2,

$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, B = $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, G = $\begin{bmatrix} g_1 & g_2 \\ 0 & g_3 \end{bmatrix}$, D = $\begin{bmatrix} d_1 & 0 \\ d_2 & 0 \end{bmatrix}$, E = $\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}$, $Y_m = \begin{bmatrix} y_{m+1} \\ y_{m+2} \end{bmatrix}$, $Y_{m-1} = \begin{bmatrix} y_{m-1} \\ y_m \end{bmatrix}$,
$F_m = \begin{bmatrix} f_{m+1} \\ f_{m+2} \end{bmatrix}$, $F'_{m-1} = \begin{bmatrix} f'_{m-1} \\ f'_m \end{bmatrix}$, $F'_m = \begin{bmatrix} f'_{m+1} \\ f'_{m+2} \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{m+1} \\ y_{m+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{m-1} \\ y_m \end{bmatrix} + h \begin{bmatrix} g_1 & g_2 \\ 0 & g_3 \end{bmatrix} \begin{bmatrix} f_{m+1} \\ f_{m+2} \end{bmatrix} + h^2 \begin{bmatrix} d_1 & 0 \\ d_2 & 0 \end{bmatrix} \begin{bmatrix} f'_{m-1} \\ f'_m \end{bmatrix} + h^2 \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix} \begin{bmatrix} f'_{m+1} \\ f'_{m+2} \end{bmatrix}$$

$$\begin{aligned} y_{m+1} &= y_m + hg_1 f_{m+1} + hg_2 f_{m+2} + h^2 d_1 f'_{m-1} + h^2 e_1 f'_{m+1} + h^2 e_2 f'_{m+2} \\ y_{m+2} &= y_m + hg_3 f_{m+2} + h^2 d_2 f'_{m-1} + h^2 e_3 f'_{m+1} + h^2 e_4 f'_{m+2} \end{aligned} \quad (4)$$

The Taylor series of equation (4) using *Mathematica* software, the values of the constants are obtained as
$g_1 = \frac{127}{135}$, $g_2 = \frac{8}{135}$, $g_3 = 2$, G = $\begin{bmatrix} \frac{127}{135} & \frac{8}{135} \\ 0 & 2 \end{bmatrix}$, $d_1 = \frac{11}{648}$, $d_2 = 0$, D = $\begin{bmatrix} \frac{11}{648} & 0 \\ 0 & 0 \end{bmatrix}$, $e_1 = \frac{-1313}{3240}$, $e_2 = \frac{-277}{1620}$, $e_3 = \frac{-4}{3}$,
$e_4 = \frac{-2}{3}$, E = $\begin{bmatrix} \frac{-1313}{3240} & \frac{-277}{1620} \\ \frac{-4}{3} & \frac{-2}{3} \end{bmatrix}$.

Substituting those constants into equation (4) gives us the block

$$\begin{aligned} y_{m+1} &= y_m + \frac{127}{135} hf_{m+1} + \frac{8}{135} hf_{m+2} + \frac{11}{648} h^2 f'_{m-1} - \frac{1313}{3240} h^2 f'_{m+1} - \frac{277}{1620} h^2 e_2 f'_{m+2} \\ y_{m+2} &= y_m + 2hf_{m+2} - \frac{4}{3} h^2 f'_{m+1} - \frac{2}{3} h^2 e_4 f'_{m+2}. \end{aligned} \quad (5)$$

When r=3

$A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, B = $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, G = $\begin{bmatrix} g_1 & g_2 & g_3 \\ 0 & g_4 & g_5 \\ 0 & 0 & g_6 \end{bmatrix}$, E = $\begin{bmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix}$, D = $\begin{bmatrix} d_1 & 0 & 0 \\ d_2 & 0 & 0 \\ d_3 & 0 & 0 \end{bmatrix}$, $Y_m = \begin{bmatrix} y_{m+1} \\ y_{m+2} \\ y_{m+3} \end{bmatrix}$, Y_{m-1}
$= \begin{bmatrix} y_{m-2} \\ y_{m-1} \\ y_m \end{bmatrix}$, $F_m = \begin{bmatrix} f_{m+1} \\ f_{m+2} \\ f_{m+3} \end{bmatrix}$, $F'_{m-1} = \begin{bmatrix} f'_{m-2} \\ f'_{m-1} \\ f'_m \end{bmatrix}$, $F'_m = \begin{bmatrix} f'_{m+1} \\ f'_{m+2} \\ f'_{m+3} \end{bmatrix}$.

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}y_{m+1}\\y_{m+2}\\y_{m+3}\end{bmatrix} = \begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}\begin{bmatrix}y_{m-2}\\y_{m-1}\\y_m\end{bmatrix} + h\begin{bmatrix}g_1&g_2&g_3\\0&g_4&g_5\\0&0&g_6\end{bmatrix}\begin{bmatrix}f_{m+1}\\f_{m+2}\\f_{m+3}\end{bmatrix} + h^2\begin{bmatrix}d_1&0&0\\d_2&0&0\\d_3&0&0\end{bmatrix}\begin{bmatrix}f'_{m-2}\\f'_{m-1}\\f'_m\end{bmatrix} +$$

$$h^2\begin{bmatrix}e_1&e_2&e_3\\e_4&e_5&e_6\\e_7&e_8&e_9\end{bmatrix}\begin{bmatrix}f'_{m+1}\\f'_{m+2}\\f'_{m+3}\end{bmatrix}$$ which yields

$$\begin{aligned}y_{m+1} &= y_m + hg_1f_{m+1} + hg_2f_{m+2} + hg_3f_{m+3} + h^2d_1f'_{m-2} + h^2e_1f'_{m+1} + h^2e_2f'_{m+2} + h^2e_3f'_{m+3}\\ y_{m+2} &= y_m + hg_4f_{m+2} + hg_5f_{m+3} + h^2d_2f'_{m-2} + h^2e_4f'_{m+1} + h^2e_5f'_{m+2} + h^2e_6f'_{m+3} \qquad (6)\\ y_{m+3} &= y_m + hg_6f_{m+3} + h^2d_3f'_{m-2} + h^2e_7f'_{m+1} + h^2e_8f'_{m+2} + h^2e_9f'_{m+3}\end{aligned}$$

The Taylor series of equation (6) using *Mathematica*, the values of the constants are obtained as

$g_1 = \frac{-65631}{26320}$, $g_2 = \frac{3602}{1645}$, $g_3 = \frac{34319}{26320}$, $g_4 = \frac{38}{17}$, $g_5 = \frac{-4}{17}$, $g_6 = 3$, G= $\begin{bmatrix}\frac{-65631}{26320}&\frac{3602}{1645}&\frac{34319}{26320}\\0&\frac{38}{17}&\frac{-4}{17}\\0&0&3\end{bmatrix}$, $d_1 = \frac{-106}{74025}$, $d_2 = \frac{-53}{15300}$, $d_3 = \frac{-3}{200}$,

D= $\begin{bmatrix}\frac{-106}{74025}&0&0\\\frac{-53}{15300}&0&0\\\frac{-3}{200}&0&0\end{bmatrix}$, $e_1 = \frac{-100643}{47376}$, $e_2 = \frac{-27667}{9870}$, $e_3 = \frac{-145513}{394800}$, $e_4 = \frac{-2017}{1530}$, $e_5 = \frac{-191}{340}$, $e_6 = \frac{101}{850}$, $e_7 = \frac{-39}{40}$, $e_8 = \frac{-99}{40}$, $e_9 = \frac{-207}{200}$,

E= $\begin{bmatrix}\frac{-100643}{47376}&\frac{-27667}{9870}&\frac{-145513}{394800}\\\frac{-2017}{1530}&\frac{-191}{340}&\frac{101}{850}\\\frac{-39}{40}&\frac{-99}{40}&\frac{-207}{200}\end{bmatrix}$.

Substituting those constants into equation (6) gives us the block

$$\begin{aligned}y_{m+1} &= y_m - \frac{65631}{26320}hf_{m+1} + \frac{3602}{1645}hf_{m+2} + \frac{34319}{26320}hf_{m+3} - \frac{106}{74025}h^2f'_{m-2} - \frac{100643}{47376}h^2f'_{m+1} - \frac{27667}{9870}h^2f'_{m+2} - \frac{145513}{394800}h^2f'_{m+3}\\ y_{m+2} &= y_m + \frac{38}{17}hf_{m+2} + \frac{4}{17}hf_{m+3} - \frac{53}{15300}h^2f'_{m-2} - \frac{2017}{1530}h^2f'_{m+1} - \frac{191}{340}h^2f'_{m+2} + \frac{101}{850}h^2f'_{m+3} \qquad (7)\\ y_{m+3} &= y_m + 3hf_{m+3} - \frac{3}{200}h^2f'_{m-2} - \frac{39}{40}h^2f'_{m+1} - \frac{99}{40}h^2f'_{m+2} - \frac{207}{200}h^2f'_{m+3}\end{aligned}$$

This continue till when r = 6, the result obtained in the same way.

Zero stability

$$A_0 = \begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}, \quad B = \begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}$$

$$P(r) = \left[\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right]$$

$$P(r) = \begin{bmatrix} r & 0 & 0 & 0 & 0 & -1 \\ 0 & r & 0 & 0 & 0 & -1 \\ 0 & 0 & r & 0 & 0 & -1 \\ 0 & 0 & 0 & r & 0 & -1 \\ 0 & 0 & 0 & 0 & r & -1 \\ 0 & 0 & 0 & 0 & 0 & r-1 \end{bmatrix}$$

$r^5(r-1) = 0$

$r^5 = 0$ or $r - 1 = 0$

$r = 0$ or 1.

Region of stability of the methods: Given the block method below

$$A_0 Y_m = BY_{m-1} + hGF_m + h^2 DF'_{m-1} + h^2 EF'_m$$

Using the test equation

$$\boldsymbol{y'_m = \lambda y_m \,,\ y'_{m+1} = \lambda y_{m+1} \,,\ y''_m = \lambda^2 y_m \,,\ y''_{m+1} = \lambda^2 y_{m+1}}$$

Hence the block method becomes

$A_0 Y_m = BY_{m-1} + hG\lambda Y_m + h^2 D\lambda^2 Y_{m-1} + h^2 E\lambda^2 Y_m$ (8)

Rearranging yield

$$A_0 Y_m - hG\lambda Y_m - h^2 E\lambda^2 Y_m = BY_{m-1} + h^2 D\lambda^2 Y_{m-1}$$

Let $Z = h\lambda$

$(A_0 - ZG - Z^2E)Y_m = (B + Z^2D)Y_{m-1}$

$Y_m = \frac{(B+Z^2D)Y_{m-1}}{(A_0 - ZG - Z^2E)}$

$Y_m = (A_0 - ZG - Z^2E)^{-1}(B + Z^2D)Y_{m-1}$ (9)

Equation (9) is the stability polynomial or the amplification matrix.

Applying the boundary locus techniques on the application matrix of (3) we have

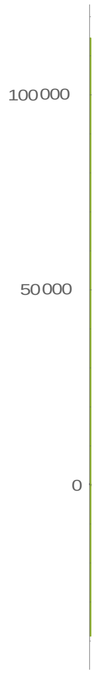

FIG.1 Region of stability: r = 2

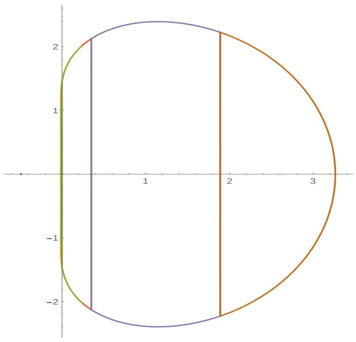

FIG. 2: Region of stability for r = 3

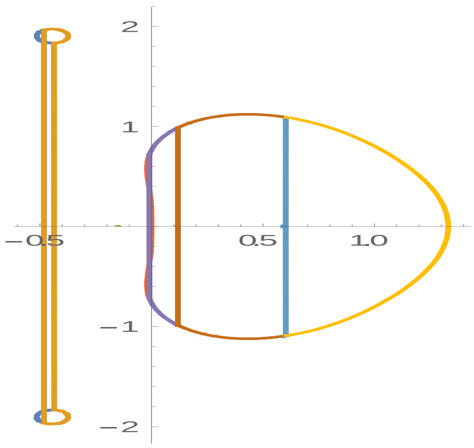

FIG. 3 : Region of stability for r = 4

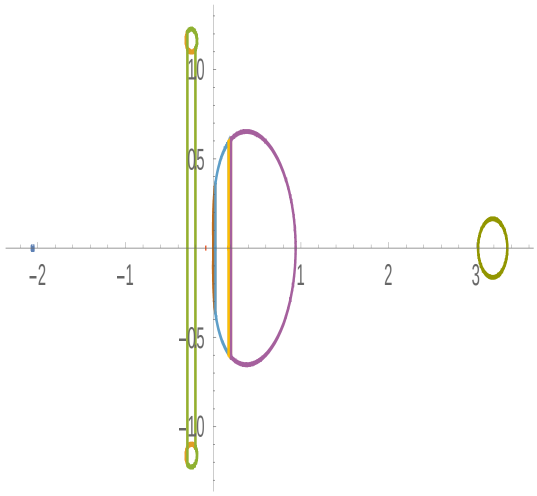

FIG. 4: Region of stability for r = 5

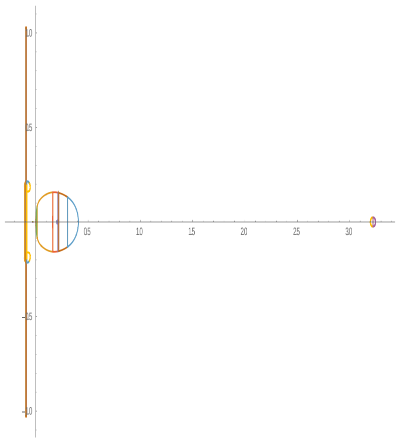
FIG. 5: Region of stability for r = 6

Order and error constant

With the help of *Mathematica* software on the Taylor series expansion (1)

When r = 2 we have the truncated Taylor series expansion as:

$$\{-\frac{1181y^{(6)}[x]h^6}{194400}-\frac{1}{350}y^{(7)}[x]h^7-\frac{18413y^{(8)}[x]h^8}{16329600}-\frac{71y^{(9)}[x]h^9}{226800}-\frac{14951y^{(10)}[x]h^{10}}{195955200}-\frac{197y^{(11)}[x]h^{11}}{12474000}-\frac{95539y^{(12)}[x]h^{12}}{32332608000}-\frac{323y^{(13)}[x]h^{13}}{648648000}-\frac{5457643y^{(14)}[x]h^{14}}{70614415872000}-\frac{1811y^{(15)}[x]h^{15}}{163459296000}+O[h]^{16}\}$$

$$\{\frac{2}{45}y^{(5)}[x]h^5+\frac{1}{18}y^{(6)}[x]h^6+\frac{23}{630}y^{(7)}[x]h^7+\frac{1}{60}y^{(8)}[x]h^8+\frac{67y^{(9)}[x]h^9}{11340}+\frac{29y^{(10)}[x]h^{10}}{16800}+\frac{1291y^{(11)}[x]h^{11}}{2993760}+\frac{257y^{(12)}[x]h^{12}}{2721600}+\frac{7181y^{(13)}[x]h^{13}}{389188800}+\frac{911y^{(14)}[x]h^{14}}{279417600}+\frac{12293y^{(15)}[x]h^{15}}{23351328000}+O[h]^{16}\}$$

Therefore $C_0=\begin{bmatrix}0\\0\end{bmatrix}$, $C_1=\begin{bmatrix}0\\0\end{bmatrix}$, $C_2=\begin{bmatrix}0\\0\end{bmatrix}$, $C_3=\begin{bmatrix}0\\0\end{bmatrix}$, $C_4=\begin{bmatrix}0\\0\end{bmatrix}$, $C_5=\begin{bmatrix}0\\\frac{2}{45}\end{bmatrix}\neq\begin{bmatrix}0\\0\end{bmatrix}$.

Therefore the method is of order 4 with error constant $\begin{bmatrix}0\\\frac{2}{45}\end{bmatrix}$.

Also When r = 3 we have the truncated Taylor series expansion as:

$$\{\frac{901517y^{(8)}[x]h^8}{198979200}+\frac{328063y^{(9)}[x]h^9}{59693760}+\frac{1602599y^{(10)}[x]h^{10}}{397958400}+\frac{69466631y^{(11)}[x]h^{11}}{32831568000}+\frac{350888737y^{(12)}[x]h^{12}}{393978816000}+\frac{73128059y^{(13)}[x]h^{13}}{232805664000}+\frac{2955819931y^{(14)}[x]h^{14}}{30730347648000}+\frac{2807601913y^{(15)}[x]h^{15}}{107556216768000}+O[h]^{16}$$

$$\{-\frac{1129y^{(7)}[x]h^7}{71400}-\frac{197y^{(8)}[x]h^8}{11900}-\frac{89557y^{(9)}[x]h^9}{7711200}-\frac{22451y^{(10)}[x]h^{10}}{3855600}-\frac{815809y^{(11)}[x]h^{11}}{339292800}-\frac{2126329y^{(12)}[x]h^{12}}{2544696000}-\frac{16897499y^{(13)}[x]h^{13}}{66162096000}-\frac{16095197y^{(14)}[x]h^{14}}{231567336000}-\frac{190891523y^{(15)}[x]h^{15}}{1111523212800 0}+O[h]^{16}\}$$

$$\{\frac{21}{160}y^{(6)}[x]h^6+\frac{219y^{(7)}[x]h^7}{1400}+\frac{421y^{(8)}[x]h^8}{3200}+\frac{109y^{(9)}[x]h^9}{1400}+\frac{3387y^{(10)}[x]h^{10}}{89600}+\frac{11261y^{(11)}[x]h^{11}}{739200}+\frac{21533y^{(12)}[x]h^{12}}{4032000}+\frac{29677y^{(13)}[x]h^{13}}{18018000}+\frac{3400457y^{(14)}[x]h^{14}}{7451136000}+\frac{396491y^{(15)}[x]h^{15}}{3459456000}+O[h]^{16}\}.$$

Therefore $C_0=\begin{bmatrix}0\\0\\0\end{bmatrix}, C_1=\begin{bmatrix}0\\0\\0\end{bmatrix}, C_2=\begin{bmatrix}0\\0\\0\end{bmatrix}, C_3=\begin{bmatrix}0\\0\\0\end{bmatrix}, C_4=\begin{bmatrix}0\\0\\0\end{bmatrix}, C_5=\begin{bmatrix}0\\0\\0\end{bmatrix}, C_6=\begin{bmatrix}0\\0\\\frac{21}{160}\end{bmatrix}\neq\begin{bmatrix}0\\0\\0\end{bmatrix}$.

Thus the method is of order 5 with error constant $\begin{bmatrix}0\\0\\\frac{21}{160}\end{bmatrix}$.

Also When r = 4 we have the truncated Taylor series expansion as:

$$-\frac{202603579}{57879360}, e_2\to-\frac{178961201}{16077600}, e_3\to-\frac{11721865}{1929312}, e_4\to-\frac{87618659}{202577760}\}\}$$

$$\{\frac{181079951y^{(10)}[x]h^{10}}{145855987200}+\frac{136935347y^{(11)}[x]h^{11}}{66298176000}+\frac{63167091331y^{(12)}[x]h^{12}}{32088317184000}+\frac{1671891911251y^{(13)}[x]h^{13}}{1251444370176000}+\frac{9171505571y^{(14)}[x]h^{14}}{12769840512000}+\frac{940507248169y^{(15)}[x]h^{15}}{2920036863744000}+O[h]^{16}\}$$

$$\{-\frac{142663y^{(9)}[x]h^9}{25401600}-\frac{12731y^{(10)}[x]h^{10}}{1270080}-\frac{13993877y^{(11)}[x]h^{11}}{1397088000}-\frac{3738211y^{(12)}[x]h^{12}}{523908000}-\frac{1750602961y^{(13)}[x]h^{13}}{435891456000}-\frac{205486871y^{(14)}[x]h^{14}}{108972864000}-\frac{3497977291y^{(15)}[x]h^{15}}{4576860288000}+O[h]^{16}\}$$

$$\{\frac{75087y^{(8)}[x]h^8}{2602880}+\frac{595521y^{(9)}[x]h^9}{13014400}+\frac{1185339y^{(10)}[x]h^{10}}{26028800}+\frac{1860301y^{(11)}[x]h^{11}}{57263360}+\frac{10713981y^{(12)}[x]h^{12}}{572633600}+\frac{335814233y^{(13)}[x]h^{13}}{37221184000}+\frac{56316801y^{(14)}[x]h^{14}}{14888473600}+\frac{313398431y^{(15)}[x]h^{15}}{223327104000}+O[h]^{16}\}$$

$$\{\frac{16}{315}y^{(7)}[x]h^7+\frac{16}{135}y^{(8)}[x]h^8+\frac{2056y^{(9)}[x]h^9}{14175}+\frac{194y^{(10)}[x]h^{10}}{1575}+\frac{2531y^{(11)}[x]h^{11}}{31185}+\frac{1873y^{(12)}[x]h^{12}}{42525}+\frac{1242167y^{(13)}[x]h^{13}}{60810750}+\frac{326173y^{(14)}[x]h^{14}}{39293100}+\frac{2196287y^{(15)}[x]h^{15}}{729729000}+O[h]^{16}\}.$$

Therefore,

$$C_0=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, C_1=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, C_2=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, C_3=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, C_4=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, C_5=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, C_6=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, C_7=\begin{bmatrix}0\\0\\0\\16/315\end{bmatrix}\neq\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Thus, the order will be 5 with the error constant $= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 16/315 \end{bmatrix}$.

Also When r = 5 we have the truncated Taylor series expansion as:

$$\{\frac{3940475569y^{(12)}[x]h^{12}}{11880580884480} + \frac{57358442105387y^{(13)}[x]h^{13}}{81084964536576000} + \frac{67698110145761y^{(14)}[x]h^{14}}{81084964536576000} + \frac{197518696719887y^{(15)}[x]h^{15}}{283797375878016000} + O[h]^{16}\}$$

$$\{-\frac{\mathbf{18250377533y^{(11)}[x]h^{11}}}{\mathbf{6133652910000}} - \frac{\mathbf{39984384883y^{(12)}[x]h^{12}}}{\mathbf{6133652910000}} - \frac{\mathbf{283403470327y^{(13)}[x]h^{13}}}{\mathbf{35438883480000}} - \frac{\mathbf{6641614130663y^{(14)}[x]h^{14}}}{\mathbf{956849853960000}} - \frac{\mathbf{31748041022717y^{(15)}[x]h^{15}}}{\mathbf{6697948977720000}} + \mathbf{O[h]^{16}}\}$$

$$\{\frac{2043511069y^{(10)}[x]h^{10}}{164687040000} + \frac{24259866913y^{(11)}[x]h^{11}}{905778720000} + \frac{26774457851y^{(12)}[x]h^{12}}{805136640000} + \frac{4151135065591y^{(13)}[x]h^{13}}{141301480320000} + \frac{69778144379y^{(14)}[x]h^{14}}{3396670200000} + \frac{1972075682261y^{(15)}[x]h^{15}}{164851727040000} + O[h]^{16}\}$$

$$\{-\frac{4426012y^{(9)}[x]h^{9}}{160645275} - \frac{2856766y^{(10)}[x]h^{10}}{53548425} - \frac{12365828y^{(11)}[x]h^{11}}{196344225} - \frac{56427766y^{(12)}[x]h^{12}}{1060258815} - \frac{1185958639y^{(13)}[x]h^{13}}{32817534750} - \frac{3146523179y^{(14)}[x]h^{14}}{153148495500} - \frac{20992911941y^{(15)}[x]h^{15}}{2067504689250} + O[h]^{16}\}$$

$$\{\frac{3575\,y^{(8)}[x]h^{8}}{24192} + \frac{38275\,y^{(9)}[x]h^{9}}{127008} + \frac{21925\,y^{(10)}[x]h^{10}}{56448} + \frac{665725\,y^{(11)}[x]h^{11}}{1862784} + \frac{3235465\,y^{(12)}[x]h^{12}}{12192768} + \frac{6827935\,y^{(13)}[x]h^{13}}{41513472} + \frac{3561483\,5y^{(14)}[x]h^{14}}{402361344} + \frac{27496895\,y^{(15)}[x]h^{15}}{653837184} + O[h]^{16}\}.$$

$$C_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C_8 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3576/24192 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

Thus, the order will be 6 with the error constant $= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3576/24192 \end{bmatrix}$.

Also When r = 6 we have the truncated Taylor series expansion as:

$$\{\frac{1334407206904451y^{(14)}[x]h^{14}}{15271706490872832000} + \frac{18296159180905987y^{(15)}[x]h^{15}}{80176459077082368000} + O[h]^{16}\}$$

$$\{-\frac{88032972600553y^{(13)}[x]h^{13}}{89965319872428000} - \frac{59520925938937y^{(14)}[x]h^{14}}{22491329968107000} - \frac{525119010240823y^{(15)}[x]h^{15}}{134947979808642000} + O[h]^{16}\}$$

$$\{\frac{214618661707y^{(12)}[x]h^{12}}{42759378816000} + \frac{182318223357139y^{(13)}[x]h^{13}}{13340926190592000} + \frac{584197733109y^{(14)}[x]h^{14}}{28506252544000} + \frac{1010472267882089y^{(15)}[x]h^{15}}{46693241667072000} + O[h]^{16}\}$$

$$\{-\frac{59639355037y^{(11)}[x]h^{11}}{3850754205375} - \frac{31466557951y^{(12)}[x]h^{12}}{770150841075} - \frac{255728872942909y^{(13)}[x]h^{13}}{4205023592269500} - \frac{89969553666191y^{(14)}[x]h^{14}}{1401674530756500} - \frac{673681616438053y^{(15)}[x]h^{15}}{12615070776808500} + O[h]^{16}\}$$

$$\{\frac{110375525y^{(10)}[x]h^{10}}{3179699712} + \frac{496967825y^{(11)}[x]h^{11}}{5829449472} + \frac{17222939875y^{(12)}[x]h^{12}}{139906787328} + \frac{1213752218225y^{(13)}[x]h^{13}}{9548638235136} + \frac{333000846655y^{(14)}[x]h^{14}}{3182879411712} + \frac{28054268075y^{(15)}[x]h^{15}}{389740336128} + O[h]^{16}\}$$

$$\{\frac{9}{175}y^{(9)}[x]h^{9} + \frac{243y^{(10)}[x]h^{10}}{1400} + \frac{67}{220}y^{(11)}[x]h^{11} + \frac{129}{350}y^{(12)}[x]h^{12} + \frac{4831041y^{(13)}[x]h^{13}}{14014000} + \frac{228453y^{(14)}[x]h^{14}}{862400} + \frac{521681y^{(15)}[x]h^{15}}{3003000} + O[h]^{16}\}$$

Therefore, $C_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $C_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $C_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $C_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $C_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $C_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $C_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $C_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$,

$C_8 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $C_9 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 9/175 \end{bmatrix}$.

Thus, the order is 7 with the error constant $= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 9/175 \\ \end{bmatrix}$.

IV. CONCLUSION

In this study, a new block method approach which is capable of solving higher order initial value problems of ordinary differential equations is presented. With the block approach, the non self-starting nature associated with the predictor-corrector method and the Runge Kutta method is being eliminated. As opposed to what is obtainable in the predictor-corrector method where additional equations are provided from a different formulation, in this work, all the required additional equations are obtained from the same formulation. Some basic properties of the method are investigated and the method is found to be zero-stable, consistent and convergent. The stability region of the block method is examined and it shows that the proposed constructed block method is *A*-stable as can be seen in figure 1,2,3,4 and 5. This method is effective for a wide-range of ordinary differential equations.

Conflict of interest

The authors declare that there is no conflict of interest

REFERENCES

[1]. Dahlquist, G. (1959). Stability and Error Bound in the Numerical Integration of ODEs. *Transcript, 130. Royal institute of Technology, Stockholm*.

[2]. Kamoh, N., Abada, A. & Soomiyol, C., (2018). A block procedure with continuous coefficients for the direct solution of general second order initial value problems of (ODEs) using shifted Legendre polynomials as basis function. *International Journal of Multidisciplinary Research and Development, Vol. 5*, Issue 4. 236-241.

[3]. Kayode S. (2008). An order six zero-stable numerical methods for direct solution of fourth-order ordinary differential equations. *American journal of applied sciences, 5*(11): 1461-1466.

[4]. Lambert, J. (1973), computational method in ordinary differential equation, *John Wiley and Sons, London*, U.K,.

[5]. Lambert, J. (1991). Numerical Methods for Ordinary Differential Equations *New York*, 293.

[6]. Kayode, S. (2008). An efficient zero stable numerical method for fourth order differential equation. I*nternational Journal of mathematical sciences and mathematics,10*.1|55|364021.

[7]. Esuabana, I. M., Ekoro, S. E., Ikhile, M. N. O. (2018). Implicit Second Derivative Hybrid Linear Multistep Method with Nested Predictors for Ordinary Differential Equations. *American Scientific Research Journal for Engineering Technology and Sciences, 42*(1): 297-308.

[8]. Esuabana, I. M., Ekoro, S. E. (2017). Hybrid Linear Multistep Methods with Nested Hybrid Predictors for Solving Linear and Non-linear Initial Value Problems in Ordinary Differential Equations,*Journal of Mathematical Theory and Modelling, 7*(11): 77-88.

[9]. Esuabana, I. M., Ekoro, S. E. (2018). Derivation and Implementation of new Family of Second Derivative Hybrid Linear Multistep Methods for Stiff Ordinary Differential Equations,*Global Journal of Mathematics, 8*(2): 821-828.

[10]. Esuabana, I. M., Ekoro, S. E. (2018). Adam's block with first and second derivative future points for initial value problems in ordinary differential equations,*J. Math. Comput. Sci., 11*(2): 1470-1485.

[11]. Rauf, K., Aniki, S., Ibrahim, S., Omolehin, J. (2015). A Zero-Stable Block Method for the Solution of Third Order Ordinary Differential Equations, *Pac. J. Sci. Tech*, 16, 91-103.