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Research Paper

Fixed Point Of Continous Mapping in Closed and Convex Set in Banach Space

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Abstract: *The purpose of this paper is to study for finding the fixed points of continious mappings in closed and convex set in Banach space also we discuss the uniqueness and existence of fixed point for a family of continuous types mappings defined on closed and convex subset of Banach space. We study some fixed point theorem for continuous type mappings in a closed and convex set of Banach space, it is shown that the same algorithm converges to a fixed point of a continuous type mappings under suitable hypotheses on the coefficients. Here the assumptions on the coefficients are different and techniques of the proof are also different.*
Keywords: *Locally compact, contraction mapping, equicontinuous, convex set, orthogonal projection, bounded linear operator.*

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I. INTRODUCTION:

The problem of existence of fixed points of continuous type mappings in closed and compact set in Banach space is now a classical theme. The applications to fixed point of continuous type mapping made it more interesting. A considerable importance has been attached to fixed point theorems in [1] and [2]. In this paper, we proved some fixed point theorems for non-expansive linear mapping of a convex uniformly Banach space.

Preliminaries: In the sequel we shall make use of the following notations, definitions, lemmas and theorems.

Notations: Y will denote the Banach space, D the nonempty subset of Y and B will always denote a closed unit ball centred at origin

Definition: 1. A mapping T: D → Y is called Lipschitzian if and only if

$$\|Tx - Ty\| \leq L\|x - y\|$$
$$\forall\, x,y \in D \text{ some } 0 \leq L < 1$$

Definition: 2. A mapping T: D → Y, is called non expansive if

$$\|Tx - Ty\| \leq L\|x - y\|$$
$$\forall\, x,y \in D \text{ some } 0 \leq L \leq 1$$

Definition: 3. A nonempty subset D of Banach space Y is said to be Convex if it satisfies the following axioms

$$x, y \in D \Rightarrow t\,x+(1-t)y \in D \quad \forall\, x, y \in D \text{ and } \forall\, t \in [0,1]$$

Definition: 4. A normed linear space Y which is complete is called Banach space. In other words each Cauchy sequence converges to a point of Y.

Definition: 5. Let be a mapping T: X → Y and dom(T) be a subspace of x then the mapping from dom(T) to Y is called operator.

Lemma:1. Let Y be a banach space. A operator Q: Y → Y is Known as a projection in Y if $Q^2u = QQu = Qu\ \forall\, u \in Y$. It is easy to check that Q is a identity operator on range of Q, and the relations ker(Q) = ran(I - Q), ran(Q) = ker(I - Q) and ker(Q) ∩ ran(Q) = {0} hold. Moreover every element u ∈ Y admits a unique decomposition u = v + w with v ∈ kerQ) and w ∈ ran(Q).

Lemma: 2. If Y is a Banach space, then a projection Q on Y is continuous iff Y = ker(Q) ⊕ ran(Q) The notation Y = C ⊕ D is used to represent that C and D are closed subspaces of Y such that C ∩ D contains only 0∈ Y and Y = C + D.

Theorem: If f is a non-expansive linear mapping of a convex uniformly Banach space, then all the fixed points of f can be obtained by process of limit.

Our main theorems are as follows:--

Theorem:1. Let Y be locally compact and let as assume that ∀ n ∈ N there is $m_n \geq 1$ such that $g_n^{m_n}$ is a contraction mapping. Let us assume that g be a self map on Y such that g_m is a contraction for some $m \geq 1$. If g_n converges point wise to g, and g_n is an equicontinuous family of functions, then $u_n = g_n(u_n)$ converges uniformly to $\overline{u} = g(\overline{u})$.

Proof: Let $\varepsilon > 0$ be arbitrary however small such that

$$L(\overline{u}, \varepsilon) = \{u \in Y : d(u, \overline{u}) \leq \varepsilon\} \subset Y$$

is compact. By the application of the Ascoli theorem, $\{g_n\}$ converges to g uniformly on $L(\overline{x}, \varepsilon)$, since it is equicontinuous and pointwise convergent therefore for given $\varepsilon > 0$ choose $n_0 = n_0(\varepsilon)$ such that $d(g_n^m(u), g^m(u)) \leq \varepsilon(1-\gamma)$ for all n greater than or equal to n_0 and for all u in $L(\overline{x}, \varepsilon)$ where $\gamma < 1$ is the Lipschitz constant of g^m. Then for all n greater than or equal to n_0 and for all u in $L(\overline{u}, \varepsilon)$ we have

$$\begin{aligned} d(g_n^m(u), \overline{u}) &= d(g_n^m(u), g(\overline{u})) \\ &\leq d(g_n^m(u), g^m(u)) + d(g^m(u)), g^m(\overline{u})) \\ &\leq \varepsilon(1-\gamma) + \gamma\, d(u, \overline{u}) \\ &\leq \varepsilon \end{aligned}$$

Hence $g_n^m(L(\overline{u}, \varepsilon)) \subset L(\overline{u}, \varepsilon)$ for all n greater than or equal to n_0. Since the maps $g_n^{m_n}$ are contractions, it implies that, for $n \geq n_0$, the g_n has a fixed points u_n and belong to $L(\overline{u}, \varepsilon)$ i.e., $d(u_n, \overline{u})) \leq \varepsilon$.

Theorem:2. Let us assume that Y be a convex uniformly Banach space, D be nonempty, convex, bounded and closed subset of Y. If g is a non-expansive map self map on D, then g possesses a fixed point in D.

Proof: Let $u_* \in D$ be fixed, and consider a sequence $\gamma_n \in (0, 1)$ converging to 1. For each n ∈ N, define the map $g_n : D \to D$ as $g_n(u) = \gamma_n g(u) + (1 - \gamma_n) u_*$

Notice that g_n is a contractions on D, hence there is a unique $u_n \in D$ such that $g_n(u_n) = u_n$. Since D is weakly compact therefore u_n has a subsequence weakly convergent to some $\overline{u} \in D$. We shall now prove that $\overline{u}$ is a fixed point of g in set D. We observe first that

$$\lim_{n\to\infty}(\|g(\overline{u}) - u\|^2 - \|\overline{u} - u\|^2) = \|g(\overline{u}) - \overline{u}\|^2.$$

Since g is non-expansive therefore we have

$$\begin{aligned} \|g(\overline{u}) - u_n\| &= \|g(\overline{u}) - g(u_n) + f(u_n) - u_n\| \\ &\leq \|g(\overline{u}) - g(u_n)\| + \|g(u_n) - u_n\| \\ &= \|\overline{u} - u_n\| + \|g(u_n) - u_n\| \quad \because g(\overline{u}) = \overline{u} \\ &\leq \|\overline{u} - u_n\| + (1 - \gamma_n)\|g(u_n) - u_*\| \end{aligned}$$

But $\gamma_n \to 1$ when $n \to \infty$ and D is bounded therefore so we conclude that

$$\lim_{n\to\infty}\sup(\|g(u) - u_n\|^2 - \|\overline{u} - u\|^2) \leq 0$$

Which yields the equality $g(\overline{u}) = \overline{u}$.

Corrolary:1. Within the condition and properties of Theorem 2, let set G be the set of fixed points of g, then G is closed and convex.

Proof: The primary assertion is trivial. Assume then $u_0, u_1 \in G$, with $u_0 \neq u_1$, and Denote $u_s = (1 - s)u_0 + su_1$, with $s \in (0, 1)$. We have

$$\begin{aligned} \|g(u_s) - u_0\| &\leq \|g(u_s) - g(u_0)\| \\ &\leq \|u_s - u_0\| \\ &= s\|u_1 - u_0\| \end{aligned}$$

And

$$\begin{aligned} \|g(u_s) - u_1\| &\leq \|g(u_s) - f(u_1)\| \\ &\leq \|u_s - u_1\| \\ &= (1-s)\|u_1 - u_0\| \end{aligned}$$

The proof is completed if we show that $g(u_s) = (1-s)u_0 + tu_1$.

Theorem:3.. Let us assume that Y be a convex uniformly Banach space, and let β, u, v be in Y such that $\|\beta - u\| = s\|\beta - u\|, \|\beta - v\| = (1-s)\|\beta - u\|$ for any $s \in [0,1]$. Then β is same as $(1 - s)u + sv$.

Proof: Without loss of generality, we will assume s greater than or equal to1/2. We have

$$\begin{aligned} \|(1-s)(\beta - u) - s(\beta - v)\| &= \|(1-2s)(\beta - u) - s(\beta - u) - s(u - v)\| \\ &= \|(1-2s)(\beta - u) - s(u - v)\| \\ &\geq s\|u - v\| - (1-2s)\|\beta - u\| \because \|u - v\| \geq \|u\| - \|v\| \\ &= (1-s)2s\|u - v\| \end{aligned}$$

Also $\|(1-s)(\beta - u) - s(\beta - v)\| \leq (1-s)2s\|u - v\|$

Therefore $\|(1-t)(\beta - x) - t(\beta - y)\| = (1-s)2s\|u - v\|$

Now Y is the uniform convexity therefore we obtain

$\|\beta - (1-s)u - sv\| = \|(1-s)(\beta - u) + s(\beta - v)\| = 0$ and which is the require result.

Theorem:4. Let us suppose that Y be a convex uniformly Banach space. Let f be a self linear operator on Y such that

$$\|fu\| \leq \|u\| \qquad \forall\, u \in Y$$

Then $\forall$ u $\in$ Y the limit

$$q_x = \lim_{n\to\infty} \frac{u+fu+\cdots f^n u}{n+1} \quad \text{exists.}$$

Moreover, the operator Q: Y → Y defined by Qu = q_u is a continuous mapping onto the linear space O = {v ∈ Y : f v = v}.

Proof: Fix u ∈ Y, and set

$$D = \overline{co(\{u, fu, f^2u \ldots\})}$$

D is a nonempty closed and convex set of Y, and from the uniform convexity of Y there is a unique $q_x \in$ D such that

$$\mu = \|q_x\| = \inf\{\|w\| : w \in D\}$$

Select ε > 0. Then, for $q_u \in$ D, ∃ n ∈ N and nonnegative constants $\beta_0, \beta_1, \ldots, \beta_n$ with $\sum_{j=0}^{n} \beta_j = 1$ such that, setting

$$w = \sum_{j=0}^{n} \beta_j f^j u$$

there holds

$$\|q_u - w\| < \epsilon$$

In particular, for every n ∈ N,

$$\left\|\frac{w+fw+\cdots+f^n w}{n+1}\right\| \leq \|w\| \leq \mu$$

Notice that

$$w + fw + \cdots + f^n w = (\beta_0 u + \cdots + \beta_m f^m u) + (\beta_0 u + \cdots + \beta_m f^{m+1} u) + \cdots + (\beta_0 f^n u + \cdots + \beta_m f^{m+n} u).$$

Thus, assuming n≫m, we get

$$w + fw + \cdots + f^n w = u + fu + \ldots + f^n u + \gamma$$

Where

$$\gamma = (\beta_0 - 1)x + \cdots + (\beta_0 + \beta_1 + \cdots + \beta_{m+1} - 1) f^{m-1}u \quad + (1 - \beta_0) f^{1+n} u + \cdots + (1 - \beta_0 - \beta_1 - \cdots - \beta_{m-1}) f^{m+n} u.$$

Therefore

$$\frac{u + fu + \ldots + f^n u}{n+1} = \frac{w+fw+\cdots+f^n w}{n+1} - \frac{\gamma}{n+1}$$

Since $\left\|\frac{\gamma}{n+1}\right\| \leq \frac{2m\|u\|}{n+1}$

upon choosing n enough large such that $2m\|u\| < \varepsilon(n + 1)$ we have

$$\left\|\frac{u + fu + \ldots + f^n u}{n+1}\right\| \leq \left\|\frac{w+fw+\cdots+f^n w}{n+1}\right\| + \left\|\frac{\gamma}{n+1}\right\| \leq \mu + 2\varepsilon.$$

On the other hand, it must be

$$\left\|\frac{u + fu + \ldots + f^n x}{n+1}\right\| \geq \mu$$

Then we conclude that $\lim_{n\to\infty} \left\|\frac{u + fu + \ldots + f^n u}{n+1}\right\| = \mu$

This implies that the above equality is a minimizing sequence in D, and because of the uniform convexity of Y, we gain the convergence

$$\lim_{n\to\infty} \frac{w + fw + \ldots + f^n w}{n+1} = q_x$$

Now remaining to show that the operator Qu = q_u is a continuous projection onto O. virtually, it is apparent that if u ∈ O then Q_u = u. In general,

$$f\, q_u = \lim_{n\to\infty} \frac{fu + \ldots + f^n u + f^{n+1} u}{n+1} = q_u + \lim_{n\to\infty} \frac{f^{n+1} u}{n+1} = q_x$$

At the end, Q^2x = QQu = Q q_u = q_u = Qx. The continuity is ensured by the relation $\|q_u\| \leq \|u\|$.

II. CONCLUSION:

Finding fixed points of nonlinear continuous type mappings (especially, nonexpansive mappings) has received vast investigations due to its extensive applications in, partial differential equations, nonlinear differential equations. In this paper, we devote to construct the methods to finding the fixed points of continuous type mappings in the closed and convex set in Banach space.

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