Quest Journals
Journal of Research in Applied Mathematics
Volume 8 ~ Issue 10 (2022) pp: 01-04
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

Examining the Effect of a Small Perturbation from Its Steady-State in a Mutualistic Interaction

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ABSTRACT
It is well known that a mutualistic interaction can be described through the process of a mathematical modelling. In this paper, we will study the stability of system of nonlinear first order ordinary differential equation of Lotka-Volterra type, by using the method of perturbation from its steady state. The results which we obtained are in consistent with the dominant linearization technique of characterizing the stability or instability of a mutualistic interaction.
Keywords: *Mutualistic Interaction, Mathematical Modelling, Stability, Lotka-Volterra, Perturbation, Steady-state.*

***R**eceived 19 Sep., 2022; **R**evised 01 Oct., 2022; **A**ccepted 03 Oct., 2022 © The author(s) 2022.*
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I. INTRODUCTION

The concepts of steady state and its stability can be found in the works of Glendinning (1994), Murray (2002), May and Leonard (1975), Halanay (1966), Gopalsamy (1992). According to these experts, it is unanimously agreed that a system is said to reach a state of equilibrium or steady state when it shows no further tendency to change its properties with time.

Stability theory occupies a central theme in mathematics. There are several concepts of stability such as asymptotic stability, absolute stability, Lyapunov stability, and stability of periodic solutions. These stability concepts have extensive literatures. Readers who are interested in a detailed and comprehensive mathematical treatment of stability theory are referred to references (Halanay, 1996; Glendinning, 1994; Gopalsamy, 1992) and several other references which are also cited by these authors.

We know that many systems in ecological theory (Morin, 2002; Murray, 2002; May, 1974; Damgaard, 2004; Mutsaers, 1991).

$$\dot{x} = F(x,t) \tag{1.1}$$

Here the function $F(.)$ is a function of only $x$and t. If $F(x,t)$ does not explicitly depend on t, then the system is called autonomous, otherwise is nonautonomous.

In a state space $\mathcal{R}^n$, there is a special set that corresponds to equation (1.1) with a given function $F(x,t)$. If we start at any point xe in this special set such that for all t that belongs to the interval $[t_0, \infty)$

$$F(x_e, t) = 0 \tag{1.2}$$

Such a point, x_e, is called an equilibrium point or a steady state solution.

A standard method of defining the concept of linearization in the neighbourhood of a steady state can be seen in the work of Glendinning (1994). Linearization around a steady state is an important analytical method for checking if the steady state is either stable or unstable on the assumption that the interaction functions are both continuous and partially differentiable.

II. MATHEMATICAL FORMULATION

The effect of (+/+) interaction between plant species N_1 and N_2 can modeled by the addition of the term $+cN_1, N_2$ which represents the enhancement due to the ecological mutualistic interaction between these two plant species populations.

Therefore, the appropriate model equation is

$$\frac{dN_1}{dt} = aN_1(t) - bN(t)_1^2 + cN_1(t)N_2(t) \qquad (2.1)$$

where:

We note that the term $+cN_1(t)N_2(t)$ is an empirical correction that is similar to the law of mass action (which states that the rate of change of a population over time is proportional to the product of the two interacting populations).

Similarly, the effect of (+/+) interaction between plant species N_2 and N_1 can also be modeled by the addition of the term $+eN_1(t)N_2(t)$. Therefore, another similar model equation is: later

$$\frac{dN_2}{dt} = dN_2(t) + eN_1(t)N_2(t) - fN_2^2(t) \qquad (2.2)$$

Therefore, the dynamics of two competing plant species can be modeled by the following coupled Lotka-Volterra logistic nonlinear ordinary differential equations of first order (Kot, 2001; Murray, 2002).

$$\frac{dN_2(t)}{dt} = N_1(t)\big(a - bN_1(t) + cN_2(t)\big) \qquad (2.3)$$

$$\frac{dN_2(t)}{dt} = N_2(t)\big(d + eN_1(t) - fN_2(t)\big) \qquad (2.4)$$

with initial conditions $N_1(0) = \alpha > 0$ and $N_2(0) = \beta > 0$

III. METHOD OF SOLUTION

It is a well known fact that the stability of the steady-state solutions for a mutualistic interaction can be studied using the technique of linearization in the neighbourhood of an arbitrary steady-state solution on the assumption that the interaction functions are both continuous and partially differentiable at its steady-state solution. Following Ekaka-a (2009), Glendinning (1994), Kot (2001), the trivial steady-state solution is unstable having its eigenvalues as $\lambda_1 = a$ and $\lambda_2 = d$ the first border steady-state solution $\left(\frac{a}{b}, 0\right)$is unstable having its eigenvalues as $\lambda_1 = -a$ and $\lambda_2 = \left(d + \frac{ae}{b}\right)$; the second border steady-state solution $\left(0, \frac{d}{f}\right)$ is unstable having its eigenvalues as $\lambda_1 = -d$ and $\lambda_2 = \left(a + \frac{cd}{f}\right)$.

The stability and instability of a steady state can also be studied by the method of a small perturbation from the steady state apart from the method of linearization just discussed in the last section.

The defined model equations of competition as formulated by equation 1 and equation 2 can be rewritten in the following two-dimensional systems.

$$\frac{dN_1(t)}{dt} = F\big(N_1(t), N_2(t)\big) \qquad (3.1)$$

$$\frac{dN_2(t)}{dt} = G\big(N_1(t), N_2(t)\big) \qquad (3.2)$$

Suppose that (N_{1s}, N_{2s}) is a steady state, that is, $F(N_{1s}, N_{2s}) = 0$and $G(N_{1s}, N_{2s}) = 0$. We consider a small perturbation from the steady state, that is,

$$N_1 = N_{1s} + u \qquad (3.3)$$

$$N_2 = N_{2s} + v \qquad (3.4)$$

where u << 1 and v << 1

What are we looking for? We want to find whether the perturbation grows or decays. The starting point to achieving this at all is to derive the differential equations for u and v. we would conduct this analysis as follows: since N_{1s} and N_{2s} are positive constants independent of the time variable, it follows from elementary calculus that

$$\frac{du}{dt} = \frac{dN_1(t)}{dt} \qquad (3.5)$$

$$\frac{dv}{dt} = \frac{dN_2(t)}{dt} \qquad (3.6)$$

By using our earlier definitions, we would obtain

$$\frac{du}{dt} = F\big(N_1(t), N_2(t)\big) \qquad (3.7)$$

$$\frac{du}{dt} = F(N_{1s} + u, N_{2s} + v) \qquad (3.8)$$

By using the Taylor series expansion on the right hand side of this equation, we obtain

$$\frac{du}{dt} = F(N_{1s}, N_{2s}) + \frac{\partial f}{\partial N}(N_{1s}, N_{2s})v + O(u^2, v^2, uv) \qquad (3.9)$$

Since $F(N_{1s}, N_{2s}) = 0$, it follows that

$$\frac{du}{dt} = \frac{\partial f}{\partial N_1}(N_{1s}, N_{2s})u + \frac{\partial f}{\partial N_2}(N_{1s}, N_{2s})v + O(u^2, v^2, uv) \qquad (3.10)$$

By a similar line of analysis, we can also obtain

$$\frac{dv}{dt} = \frac{\partial G}{\partial N_1}(N_{1s}, N_{2s})u + \frac{\partial G}{\partial N_2}(N_{1s}, N_{2s})v + O(u^2, v^2, uv) \qquad (3.11)$$

We learn that the higher order terms will be extremely small because u and v are assumed to be small. In summary, knowing whether the perturbation is growing or decaying canbe studied by using these equations.

$$\frac{du}{dt} = \frac{\partial f}{\partial N_1}(N_{1s}, N_{2s})u + \frac{\partial f}{\partial N_2}(N_{1s}, N_{2s})v \qquad (3.12)$$

$$\frac{dv}{dt} = \frac{\partial G}{\partial N_1}(N_{1s}, N_{2s})u + \frac{\partial G}{\partial N_2}(N_{1s}, N_{2s})v \qquad (3.13)$$

IV. CHARACTERIZATION OF STABILITY PROPERTIES OF MUTUALISM: THE METHOD OF A SMALL PERTURBATION FROM THE STEADY STATE

In this section, we shall use the method of a small perturbation from the steady state which we have defined and discussed in the previous section to investigate the stability and instability of each steady state as a sort of a reality check to see whether we would obtain the same conclusions about the qualitative behaviour of steady state solutions as those obtained by using the linearization about each steady state.

We consider the following interaction functions

$$F\big(N_1(t), N_2(t)\big) = aN_1(t) - bN_1^2(t) + cN_1(t)N_2(t) \qquad (4.1)$$

$$G\big(N_1(t), N_2(t)\big) = aN_2(t) - bN_2(t) + eN_1(t)N_2(t) - fN_2^2(t) \qquad (4.2)$$

First, does a small perturbation from the trivial steady state grows or decays?

For the purpose of clarity, we can see that at the trivial steady state, $F(0,0) = 0$and $G(0,0) = 0$. By partial differentiation with respect to N_1, we know that

$$\frac{\partial F}{\partial N_1} = a - 2bN_1(t) + cN_2(t) \qquad (4.3)$$

where $\frac{\partial F}{\partial N_1}(N_{1s}, N_{2s}) = \frac{\partial F}{\partial N_1}(0,0) = a$

By partial differentiation with respect to N_2, we obtain

$$\frac{\partial F}{\partial N_2} = cN_1(t) \qquad (4.4)$$

where $\frac{\partial F}{\partial N_2}(0,0) = 0$

Similarly, we obtain

$$\frac{\partial G}{\partial N_1} = cN_2(t) \qquad (4.5)$$

where $\frac{\partial G}{\partial N_1}(0,0) = 0$

and

$$\frac{\partial G}{\partial N_2} = d + cN_1(t) - 2fN_2(t) \qquad (4.6)$$

where $\frac{\partial G}{\partial N_2}(0,0) = d$

Hence, the qualitative behaviour of a small perturbation from the trivial steady state over time is described by $\frac{du}{dt} = au$ and $\frac{dv}{dt} = dv$

For a mutualistic system of equations, since a and d are positive constants, it follows that the perturbations from the trivial steady state will grow unboundedly.

In this case, both $N_1(t)$ and $N_2(t)$ will move away from the steady state. Therefore, the trivial steady state is unstable which is consistent with the qualitative behaviour of solutions over time when the method of linearization is applied to characterize the qualitative behaviour of the trivial steady-state solution (Ekaka-a, 2009).

V. RESULT AND DISCUSSION

For other steady state solutions, we would simply summarise our findings as

1. The qualitative behaviour of a small perturbation from the steady state $\left(a, \frac{d}{f}\right)$ over time is described by $\frac{du}{dt} = \left(a + \frac{cd}{f}\right)u$ and $\frac{dv}{dt} = -d\left(v - \frac{eu}{f}\right)$ indicating that u will grow over time provided $\left(a + \frac{cd}{f}\right) > 0$ and v will decay provided $\left(v - \frac{eu}{f}\right) > 0$. In this case, $N_1(t)$ will move away from the steady state whereas $N_2(t)$ will

move towards the steady state. Therefore, the steady state $\left(a, \frac{d}{f}\right)$ is unstable. This conclusion is also consistent with the qualitative behaviour of solutions over time when we used the method of linearization.

2. The qualitative behaviour of a small perturbation from the steady state $\left(\frac{a}{b}, 0\right)$ over time is described by $\frac{du}{dt} = -a\left(u - \frac{c}{b}v\right)$ and $\frac{dv}{dt} = \left(d + \frac{ae}{b}\right)v$ indicating that u will decay over time provided $\left(u - \frac{c}{b}v\right) > 0$ and v will grow unboundedly over time provided $d + \frac{ae}{b} > 0$. In this case, $N_1(t)$ will move towards the steady state whereas $N_2(t)$ will move away from the steady state. Therefore, the steady state solution $\left(\frac{a}{b}, 0\right)$ is unstable. Similarly, this conclusion is consistent with the qualitative behaviour of solutions over time when we used the method of linearization.

VI. CONCLUSION

In this work, we have obtained key characterizations of the stability of three steady-state solutions for a mutualistic interaction by using the method of perturbation from a steady state solution.
The results which we have achieved in this work are consistent with the dominant linearization technique of characterizing the stability or instability of a mutualistic interaction.

Therefore, the perturbation method and the linearization technique for the three steady-state solutions can be considered as robust mathematical techniques of characterizing the stability or instability properties of a system of first order ordinary differential equations which describe the mutualistic ecological interaction between two plant species for a limited resource in an environment.

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