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**Research Paper**

# MILDLY (1, 2)*-η-NORMAL SPACES AND SOME (1, 2)*-η-FUNCTIONS IN BITOPOLOGICAL SPACES

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***Abstract**: In this paper, we introduce and study a new class of spaces called mildly (1, 2)*-η-normal spaces and some new functions called (1, 2)*-η-continuous, (1, 2)*-gη-continuous, (1, 2)*-rgη-continuous, almost (1, 2)*-η-continuous, almost (1, 2)*-gη-continuous, almost (1, 2)*-rgη-continuous, (1, 2)*-η-closed, (1, 2)*-gη-closed, (1, 2)*-rgη-closed, almost (1, 2)*-η-closed, almost (1, 2)*-gη-closed and almost (1, 2)*-rgη-closed functions in bitopological spaces. Moreover, we obtain characterizations and preservation theorems for mildly (1, 2)*-η-normal spaces.*

***Keywords**: (1, 2)*-η-open,(1, 2)*-rgη-closed sets; (1, 2)*-η-continuous, almost (1, 2)*-rgη-continuous, (1, 2)*-rgη-closed functions; mildly (1, 2) *-η-normal space.*
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## I. Introduction

The study of bitopological spaces was first intiated by Kelly [**4**] in 1963. By using the topological notions, namely, semi-open, α-open and pre-open sets, many new bitopological sets are defined and studied by many topologists. Levine [**8**] initiated the study of generalized closed sets in topological spaces. The notion of regular g-closed sets as a generalization of g-closed sets due to Levine [**8**] was defined and investigated by Palaniappan and Rao [**9**]. In particular, the notion of mildly normal spaces and almost continuous functions were introduced by M. K. Singal and A. R. Singal [**14, 15**]. In 2008, Ravi et al. [**12**] studied the notion of (1, 2)*-sets in bitopological spaces. In 2004, Ravi and Thivagar [**10**] studied the concept of stronger from of (1, 2)*-quotient mapping in bitopological spaces and also introduced the concepts of (1, 2)*-semi-open and (1, 2)*-α-open sets in bitopological spaces. In 2010, K. Kayathri et al. [**3**] introduced and studied a new class of sets called regular (1, 2)*-g-closed sets and used it to obtain a new class of functions called (1, 2)*-rg-continuous, (1, 2)*-R-map, almost (1, 2)*-continuous and almost (1, 2)*-rg-closed functions in bitopological spacesand also obtained characterizations and preservation theoremsfor mildly (1, 2)*-normal spaces. In 2022, H. Kumar [**5**] introduced the concept of (1, 2)*-η-open sets and (1, 2)*-η-neighbourhood and; studied their properties. H. Kumar [**6**] introduced the concept of (1, 2)*-generalized η-closed sets and studied some basic properties (1, 2)*-gη-closed sets.Recently, H. Kumar [7] introduced the concept of regular (1, 2)*-generalized η-closed sets and (1, 2)*-gη-neighbourhood and; discussed their properties.

In this paper, we introduce and investigate a new class of spaces called mildly (1, 2)*-η-normal spaces and also a new class of functions called (1, 2)*-η-continuous, (1, 2)*-gη-continuous, (1, 2)*-rgη-continuous, almost (1, 2)*-η-continuous, almost (1, 2)*-gη-continuous, almost (1, 2)*-rgη-continuous, (1, 2)*-η-closed, (1, 2)*-gη-closed, (1, 2)*-rgη-closed, almost (1, 2)*-η-closed, almost (1, 2)*-gη-closed and almost (1, 2)*-rgη-closed functions in bitopological spaces. Moreover, we obtain characterizations and preservation theorems for mildly (1, 2)*-η-normal spaces.

## II. Preliminaries

Throughout the paper $(X, \Im_1, \Im_2)$, $(Y, \sigma_1, \sigma_2)$ and $(Z, \wp_1, \wp_2)$ (or simply X, Y and Z) denote bitopological spaces.

**Definition 2.1**. Let S be a subset of X. Then S is said to be **$\Im_{1,2}$-open [10]** if $S = A \cup B$ where $A \in \Im_1$ and $B \in \Im_2$. The complement of a $\Im_{1,2}$-open set is **$\Im_{1,2}$-closed**.

**Definition 2.2** [**10**]. Let S be a subset of X. Then
(i) the **$\Im_{1,2}$-closure** of S, denoted by $\Im_{1,2}$-cl(S), is defined as $\cap$ {F : S $\subset$ F and F is $\Im_{1,2}$-closed}; (ii) the **$\Im_{1,2}$-interior** of S, denoted by $\Im_{1,2}$-int(S), is defined as $\cup$ {F : F $\subset$ S and F is $\Im_{1,2}$-open}.

**Note 2.3** [**10**]. Notice that $\Im_{1,2}$-open sets need not necessarily form a topology.

**Definition 2.4.** A subset A of a bitopological space $(X, \Im_1, \Im_2)$ is called
(i) **regular (1 , 2 )*-open** [**12**] if A = $\Im_{1,2}$-int($\Im_{1,2}$-cl((A)).
(iv) **(1, 2)*-η-open** [**5**] if A $\subset \Im_{1,2}$-int($\Im_{1,2}$-cl($\Im_{1,2}$-int)(A)) $\cup \Im_{1,2}$-cl($\Im_{1,2}$-int)(A)).

The complement of a regular (1, 2)*-open (resp. (1, 2)*-η-open) set is called **regular (1 , 2)*-closed** (resp. **(1, 2)*-η-closed**).

The **(1, 2)*-η-closure** of a subset A of X is denoted by **(1, 2)*-η-cl(A)**, defined as the intersection of all (1, 2)*-η-closed sets containing A.

The family of all regular (1 , 2)*-open (resp. regular (1 , 2)*-closed, (1, 2)*-η-open, (1, 2)*-η-closed) sets in X is denoted by (1, 2)*-RO(X) (resp. (1, 2)*-RC(X),(1, 2)*-η-O(X), (1, 2)*-η-C(X).

**Remark 2.5.** We have the following implications for the properties of subsets [**5**]:

**regular (1 , 2)*-open $\Rightarrow \Im_{1,2}$-open$\Rightarrow$ (1, 2)*-η-open**

Where none of the implications is reversible.

## III. (1, 2)*-rgη-closed sets in bitopological spaces

**Definition 3.1**. A subset A of a bitopological space $(X, \Im_1, \Im_2)$ is called
(i) **(1, 2)*-generalized closed** (briefly **(1, 2)*-g-closed**) [**13**] if $\Im_{1,2}$-cl(A) $\subset$ U whenever A $\subset$ U and U is $\Im_{1,2}$-open in X.
(ii) **regular (1, 2)*-generalized closed** (briefly **(1, 2)*-rg-closed**) [**3**] if $\Im_{1,2}$-cl(A) $\subset$ U whenever A $\subset$ U and U $\in$ (1, 2)*-RO(X).
(iii) **(1, 2)*-generalized η-closed** (briefly **(1, 2)*-gη-closed**) [**6**] if (1, 2)*-η-cl(A) $\subset$ U whenever A $\subset$ U and U is $\Im_{1,2}$-open in X.
(iv) **regular (1, 2)*-generalized η-closed** (briefly **(1, 2)*-rgη-closed**) [7] if (1, 2)*-η-cl(A) $\subset$ U whenever A $\subset$ U and U $\in$ (1, 2)*-RO(X).

The complement of a (1, 2)*-g-closed (resp. (1, 2)*-rg-closed,(1, 2)*-gη-closed, (1, 2)*-rgη-closed) set is called **(1, 2)*-g-open** (resp. **(1, 2)*-rg-open**,**(1, 2)*-gη-open**, **(1, 2)*-rgη-open**).

We denote the set of all (1, 2)*-rgη-closed (resp. (1, 2)*-rgη-open) sets in $(X, \Im_1, \Im_2)$ by (1, 2)***-rgη-C(X)** (resp. **rgη-O(X)**).

**Remark 3.2.** We have the following implications for the properties of subsets:

regular (1, 2)*-closed $\Rightarrow \Im_{1,2}$-closed$\Rightarrow$(1, 2)*-g-closed$\Rightarrow$(1, 2)*-rg-closed

⇓⇓⇓

(1, 2)*-η-closed $\Rightarrow$(1, 2)*-gη-closed$\Rightarrow$ (1, 2)*-rgη-closed

**Where none of the implications is reversible as can be seen from the following examples:**

**Example 3.3**. Let X = {a, b, c, d} with $\Im_1$ = {ϕ, X, {a}, {b}, {a, b}, {b, c, d}} and $\Im_2$ = {ϕ, X, {c}, {a, c, d}}. Then
(i) regular $(1, 2)^*$-closed : ϕ, X, {a}, {b}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(ii) $\Im_{1,2}$-closed sets : ϕ, X, {a}, {b}, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(iii) $(1, 2)^*$-g-closed sets :ϕ, X, {a}, {b}, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(iv) $(1, 2)^*$-rg-closed sets :ϕ, X, {a}, {b}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.
(v) $(1, 2)^*$-η-closed sets :ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(vi) $(1, 2)^*$-gη-closed sets :ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(vii) $(1, 2)^*$-rgη-closed sets :ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

**Example 3.4**. Let X = {a, b, c} with $\Im_1$ = {ϕ, X, {b}} and $\Im_2$ = {ϕ, X, {c}}. Then
(i) regular $(1, 2)^*$-closed : ϕ, X, {a, b}, {a, c}.
(ii) $\Im_{1,2}$-closed sets : ϕ, X, {a}, {a, b}, {a, c}.
(iii) $(1, 2)^*$-g-closed sets :ϕ, X, {a}, {a, b}, {a, c}.
(iv) $(1, 2)^*$-rg-closed sets :ϕ, X, {a}, {a, b}, {a, c}, {b, c}.
(v) $(1, 2)^*$-η-closed sets :ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}.
(vi) $(1, 2)^*$-gη-closed sets :ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}.
(vii) $(1, 2)^*$-rgη-closed sets :ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}.

**Example 3.5**. Let X = {a, b, c, d} with $\Im_1$ = {ϕ, X, {a}} and $\Im_2$ = {ϕ, X, {b}, {a, b, c}}. Then
(i) regular $(1, 2)^*$-closed : ϕ, X, {a, c}, {b, c}.
(ii) $\Im_{1,2}$-closed sets : ϕ, X, {d}, {c, d}, {a, c, d}, {b, c, d}.
(iii) $(1, 2)^*$-g-closed sets :ϕ, X, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(iv) $(1, 2)^*$-rg-closed sets :ϕ, X, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.
(v) $(1, 2)^*$-η-closed sets :ϕ, X, {a}, {b}, {c}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.
(vi) $(1, 2)^*$-gη-closed sets :ϕ, X, {a}, {b}, {c}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}.
(vii) $(1, 2)^*$-rgη-closed sets :ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

**Example 3.6**. Let X = {a, b, c, d} with $\Im_1$ = {ϕ, X, {a}, {b}, {a, b}, {a, b, c}} and $\Im_2$ = {ϕ, X, {a, b, d}}. Then
(i) regular $(1, 2)^*$-closed : ϕ, X, {a, c, d}, {b, c, d}.
(ii) $\Im_{1,2}$-closed sets : ϕ, X, {c}, {d}, {c, d}, {a, c, d}, {b, c, d}.
(iii) $(1, 2)^*$-g-closed sets :ϕ, X, {c}, {d}, {c, d}, {a, c, d}, {b, c, d}.

(iv) $(1, 2)^*$-rg-closed sets :ϕ, X, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.
(v) $(1, 2)^*$-η-closed sets :ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.
(vi) $(1, 2)^*$-gη-closed sets :ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}.
(vii) $(1, 2)^*$-rgη-closed sets :ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

**Example 3.7**. Let X = {a, b, c} with $\Im_1$ = {ϕ, X, {a}, {a, c}} and $\Im_2$ = {ϕ, X, {c}}. Then
(i) regular $(1, 2)^*$-closed : ϕ, X, {a, b}, {b, c}.
(ii) $\Im_{1,2}$-closed sets : ϕ, X, {b}, {a, b},{b, c}.
(iii) $(1, 2)^*$-g-closed sets :ϕ, X, {a}, {b}, {a, b}, {b, c}.
(iv) $(1, 2)^*$-rg-closed sets :ϕ, X, {a}, {b}, {a, b}, {a, c}, {b, c}.
(v) $(1, 2)^*$-η-closed sets :ϕ, X, {a}, {b}, {c}, {a, b}, {b, c}.

(vi) $(1, 2)^*$-gη-closed sets :ϕ, X, {a}, {b}, {c}, {a, b}, {b, c}.
(vii) $(1, 2)^*$-rgη-closed sets :ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}.

**Theorem 3.8.** A set A is $(1, 2)^*$-rgη-open if and only if the following condition holds:

$F \subset (1, 2)^*$-η-int(A) whenever F is regular $(1, 2)^*$-closed and $F \subset A$.

## IV. Mildly (1, 2)*-η-normal spaces in bitopological spaces

In this section, we introduce mildly $(1, 2)^*$-η-normal spacesin bitopological spaces and study some basic properties of mildly $(1, 2)^*$-η-normal spaces.

**Definition 4.1.** A space X is said to be **mildly (1, 2)*-η-normal** (resp. **mildly (1, 2)*-normal** [**3**]) if for every pair of disjoint H, K $\in (1, 2)^*$-RC(X), there exist disjoint $(1, 2)^*$-η-open (resp. $\Im_{1,2}$-open) sets U, V of X such that $H \subset U$ and $K \subset V$

**Definition 4.2.** A space X is said to be **(1, 2)*-η-normal** if for every pair of disjoint $\Im_{1,2}$-closed sets H and K, there exist disjoint $(1, 2)^*$-η-open sets U, V of X such that $H \subset U$ and $K \subset V$.

**Remark 4.3.** From the definitions stated above, we obtain the following diagram.

$(1, 2)^*$-normal ⇒ mildly $(1, 2)^*$-normal

⇓⇓

$(1, 2)^*$-η-normal ⇒ mildly $(1, 2)^*$-η-normal

**Where none of the implications is reversible as can be seen from the following example:**

**Example 4.4**. Let X = {a, b, c, d} with $\Im_1$ = {ϕ, X, {a}, {b}, {a, b}, {b, c, d}} and $\Im_2$ = {ϕ, X, {c}, {a, c, d}}. Then the pair of disjoint regular $(1, 2)^*$-closed sets H = {a} and K = {c, d}, there exist disjoint $(1, 2)^*$-η-open sets U = {a} and V = {c, d} such that $H \subset U$ and $K \subset V$. Hence $(X, \Im_1, \Im_2)$ is mildly $(1, 2)^*$-η-normal but not mildly $(1, 2)^*$-normal, since V = {c, d} is not $\Im_{1,2}$-open set.

**Theorem 4.5.** The following are equivalent for a space X.
(i) X is mildly $(1, 2)^*$-η-normal;
(ii) for any disjoint H, K $\in (1, 2)^*$-RC(X), there exist disjoint $(1, 2)^*$-gη-open sets U, V such that $H \subset U$ and $K \subset V$ ;
(iii) for any disjoint H, K $\in (1, 2)^*$-RC(X), there exist disjoint $(1, 2)^*$-rgη-open sets U, V such that $H \subset U$ and $K \subset V$ ;
(iv) for any disjoint H $\in (1, 2)^*$-RC(X) and any V $\in (1, 2)^*$-RO(X) containing H, there exists a $(1, 2)^*$-rgη-open set U of X such that $H \subset U \subset (1, 2)^*$-η-cl(U) $\subset V$ .

**Proof.**
(i) ⇒ (ii).Let X be mildly $(1, 2)^*$-η-normal space. Let H, K$\in (1, 2)^*$-RC(X). By assumption, there exist disjoint $(1, 2)^*$-η-open sets U, V such that $H \subset U$ and $K \subset V$. Since every $(1, 2)^*$-η-open set is $(1, 2)^*$-gη-open, so, U and V are $(1, 2)^*$-gη-open sets such that $H \subset U$ and $K \subset V$.

(ii) ⇒ (iii). Let H, K $\in (1, 2)^*$-RC(X). By assumption, there exist disjoint $(1, 2)^*$-gη-open sets U, V such that $H \subset U$ and $K \subset V$. Since every $(1, 2)^*$-gη-open set is $(1, 2)^*$-rgη-open, so, U and V are $(1, 2)^*$-rgη-open sets such that $H \subset U$ and $K \subset V$.

(iii) ⇒ (iv). Let H $\in (1, 2)^*$-RC(X) and $H \subset V \in (1, 2)^*$-RO(X). There exist disjoint $(1, 2)^*$ -rgη-open sets U, W such that $H \subset U$ and $(X - V) \subset W$. By **Theorem 3.8**, we have $(X - V) \subset (1, 2)^*$-η-int(W) and [U $\cap (1, 2)^*$-η-int(W)] = ϕ. Therefore, we obtain [$(1, 2)^*$-η-cl(U) $\cap$ $(1, 2)^*$-η-int(W)] = ϕ and hence $H \subset U \subset (1, 2)^*$-η-cl(U) $\subset$ [X− $(1, 2)^*$-η-int(W)] $\subset V$ .

(iv) $\Rightarrow$ (i). Let H, K be disjoint regular $(1, 2)^*$-closed sets of X. Then $H \subset (X - K) \in (1, 2)^*$-RO(X) and there exists a $(1, 2)^*$-rgη-open set G of X such that $H \subset G \subset (1, 2)^*$-η-cl(G) $\subset (X - K)$. Put U = $(1, 2)^*$-η-int(G) and V = X –$(1, 2)^*$-η-cl(G). Then U and V are disjoint $(1, 2)^*$-η-open sets of X such that $H \subset U$ and $K \subset V$. Therefore, X is mildly $(1, 2)^*$-η-normal.

## V. Some $(1, 2)^*$-η-bitopological functions

In this section, we shall recall the definitions of some functions used in the sequel. Further we introduce some $(1, 2)^*$-η-functions in bitopological spaces.

**Definition 5.1.** A function $f : X \to Y$ is said to be
(i) **$(1, 2)^*$-η-continuous** if $f^{-1}(F)$ is $(1, 2)^*$-η-closed in X for every $\Im_{1,2}$-closed set F of Y;
(ii) **$(1, 2)^*$-gη-continuous** if $f^{-1}(F)$ is $(1, 2)^*$-gη-closed in X for every $\Im_{1,2}$-closed set F of Y;
(iii) **$(1, 2)^*$-rgη-continuous** if $f^{-1}(F)$ is $(1, 2)^*$-rgη-closed in X for every $\Im_{1,2}$-closed set F of Y;
(iv) **$(1, 2)^*$-R-map** [3] if $f^{-1}(F) \in (1, 2)^*$-RO(X) for every $F \in (1, 2)^*$ -RO(Y);
(v) **completely $(1, 2)^*$-continuous** [3] if $f^{-1}(F) \in (1, 2)^*$-RO(X) for every $\Im_{1,2}$-open set F of Y.

**Definition 5.2.** A function $f : X \to Y$ is said to be
(i) **almost $(1, 2)^*$-continuous**[3] if $f^{-1}(F)$ is $\Im_{1,2}$-open in X for every $F \in (1, 2)^*$-RO(Y);
(ii) **almost $(1, 2)^*$-η-continuous** if $f^{-1}(F)$ is $(1, 2)^*$-η-closed in X for every $F \in (1, 2)^*$-RC(Y);
(iii) **almost $(1, 2)^*$-gη-continuous** if $f^{-1}(F)$ is $(1, 2)^*$-gη-closed in X for every $F \in (1, 2)^*$-RC(Y );
(iv) **almost $(1, 2)^*$-rgη-continuous** if $f^{-1}(F)$ is $(1, 2)^*$-rgη-closed in X for every $F \in (1, 2)^*$-RC(Y);

**Example 5.3.** Let X = {a, b, c}, $\Im_1$ = {ϕ, X, {a}, {a, c}} and $\Im_2$ = {ϕ, X, {c}}. Let Y = {a, b, c}, $\psi_1$ = {ϕ, Y, {a}} and $\psi_2$ = {ϕ, Y, {b}}. Define $f : X \to Y$ as f(a) = c; f(b) = b; f(c) = a. Clearly f is almost $(1, 2)^*$-rgη-continuous.

**Example 5.4.** Let X = {a, b, c}, $\Im_1$ = {ϕ, X, {a}, {a, c}} and $\Im_2$ = {ϕ, X, {c}}. Let Y = {a, b, c}, $\psi_1$ = {ϕ, Y, {a}} and $\psi_2$ = {ϕ, Y, {b}}. Define $f : X \to Y$ as f(a) = a, f(b) = c, f(c) = b. Clearly f is $(1, 2)^*$-continuous as well as $(1, 2)^*$-R-map but it is not completely $(1, 2)^*$-continuous.

**Example 5.5.**Let X = Y = {a, b, c}, $\Im_1$ = {ϕ, X, {a}} and $\Im_2$ = {ϕ, X, {a, b}}. Let $\psi_1$ = {ϕ, Y, {a}} and $\psi_1$ = {ϕ, Y, {a, c}}. Define $f : X \to Y$ as f(a) = b; f(b) = c; f(c) = a. Clearly f is both $(1, 2)^*$-rgη-continuous and almost $(1, 2)^*$-gη-continuous but it is neither $(1, 2)^*$-η-continuous nor $(1, 2)^*$-gη-continuous. It is not $(1, 2)^*$-continuous

**Example 5.6.**Let X = Y = {a, b, c}, $\Im_1$ = {ϕ, X, {a}} and $\Im_2$ = {ϕ, X, {a, b}}. Let $\psi_1$ = {ϕ, Y, {a}} and $\psi_1$ = {ϕ, Y, {a, c}}. Define $f : X \to Y$ as f(a) = c, f(b) = a, f(c) = b. Clearly f is both $(1, 2)^*$-gη-continuous and almost $(1, 2)^*$-η-continuous. But it is neither $(1, 2)^*$-continuous nor $(1, 2)^*$-η-continuous.

**Example 5.7.** Let X = Y = {a, b, c}, $\Im_1$ = {ϕ, X, {a}} and $\Im_2$ = {ϕ, X, {b}, {a, b}}. Let $\psi_1$ = {ϕ, Y, {a}} and $\psi_2$ = {ϕ, Y, {b}}. Define $f : X \to Y$ as f(a) = b; f(b) = a; f(c) = c. Clearly f is almost $(1, 2)^*$-continuous as well asalmost $(1, 2)^*$-η-continuous.

**Example 5.8.** Let X = Y = {a, b, c}, $\Im_1$ = {ϕ, X, {a}} and $\Im_2$ = {ϕ, X, {b, c}}. Let $\psi_1$ = {ϕ, Y, {b}, {c}, {b, c}} and $\psi_2$ = {ϕ, Y, {a, b}}. Define $f : X \to Y$ as f(a) = a; f(b) = c; f(c) = b. Clearly f is almost $(1, 2)^*$-gη-continuous as well as almost $(1, 2)^*$-rgη-continuous but it is neither almost $(1, 2)^*$-continuous nor almost $(1, 2)^*$-η-continuous. It is neither $(1, 2)^*$-continuous nor $(1, 2)^*$-η-continuous.

**Remark 5.9.** From the definitions stated above and the examples given above, we obtain the following diagram:

complete $(1, 2)^*$-continuity $\Rightarrow (1, 2)^*$-R-map
$\Downarrow\Downarrow$
$(1, 2)^*$-continuity $\Rightarrow$ almost $(1, 2)^*$-continuity
$\Downarrow\Downarrow$
$(1, 2)^*$-η-continuity $\Rightarrow$ almost $(1, 2)^*$-η-continuity
$\Downarrow\Downarrow$

$(1, 2)^*$-gη-continuity⇒almost $(1, 2)^*$-gη-continuity
⇓⇓
$(1, 2)^*$-rgη -continuity⇒ almost $(1, 2)^*$-rgη-continuity

The above examples enable us to realize that none of the implications in the above diagram is reversible.

**Definition 5.10.** A space X is said to be **$(1, 2)^*$-rgη-$T_{1/2}$** if every $(1, 2)^*$-rgη-closed set of X is regular $(1, 2)^*$-closed in X.
**Proposition 5.11.** If a function f : X → Y is $(1, 2)^*$-rgη-continuous and X is $(1, 2)^*$-rgη-$T_{1/2}$, then f is completely $(1, 2)^*$-continuous.
**Proof**. Let F be any $\Im_{1,2}$-closed set of Y . Since f is $(1, 2)^*$-rgη-continuous, $f^{-1}(F)$ is $(1, 2)^*$-rgη-closed in X. Since X is $(1, 2)^*$-rgη-$T_{1/2,}$ so $f^{-1}(F) \in (1, 2)^*$-RC(X). Therefore, f is completely $(1, 2)^*$-continuous.

**Definition 5.12**. A function f : X → Y is said to be **$(1, 2)^*$-rgη-irresolute** if $f^{-1}(F)$ is $(1, 2)^*$-rgη-closed in X for every $(1, 2)^*$-rgη-closed set F of Y .

**Remark 5.13**. Every $(1, 2)^*$-rgη-irresolute function is $(1, 2)^*$-rgη-continuous but not conversely.

**Proposition 5.14**. If f : X → Y is almost $(1, 2)^*$-rgη-continuous and X is $(1, 2)^*$-rgη-$T_{1/2}$, then f is a $(1, 2)^*$-R-map.
**Proof.** Let $V \in (1, 2)^*$-RC(Y). Since f is almost $(1, 2)^*$-rgη-continuous, $f^{-1}(V)$ is $(1, 2)^*$-rgη-closed in X. But X is $(1, 2)^*$-rgη-$T_{1/2}$. Therefore $f^{-1}(V) \in (1, 2)^*$-RC(X). Hence f is a $(1, 2)^*$-R-map.

**Definition 5.15**. A function f : X → Y is said to be
(i) **regular $(1, 2)^*$-closed [3]** if f(F) is regular $(1, 2)^*$-closed in Y for every $\Im_{1,2}$-closed set F of X;
(ii) **$(1, 2)^*$-η-closed** if f(F) is $(1, 2)^*$-η-closed in Y for every $\Im_{1,2}$-closed set F of X;
(iii) **$(1, 2)^*$-gη-closed** if f(F) is $(1, 2)^*$-gη-closed in Y for every $\Im_{1,2}$-closed set F of X;
(iv) **$(1, 2)^*$-rgη-closed** if f(F) is $(1, 2)^*$-rgη-closed in Y for every $\Im_{1,2}$-closed set F of X.

**Definition 5.16**. A function f : X → Y is said to be
(i) **$(1, 2)^*$-rc-preserving**[3] if f(F) is regular $(1, 2)^*$-closed in Y for every $F \in (1, 2)^*$-RC(X);
(ii) **almost $(1, 2)^*$-closed** if f(F) is $\psi_{1,2}$-closed in Y for every $F \in (1, 2)^*$-RC(X);
(iii) **almost $(1, 2)^*$-η-closed** if f(F) is $(1, 2)^*$-η-closed in Y for every $F \in (1, 2)^*$-RC(X);
(iv) **almost $(1, 2)^*$-gη-closed** if f(F) is $(1, 2)^*$-gη-closed in Y for every $F \in (1, 2)^*$-RC(X);
(v) **almost $(1, 2)^*$-rgη-closed** if f(F) is $(1, 2)^*$-rgη-closed in Y for every $F \in (1, 2)^*$-RC(X);

**Remark 5.17**. From the definitions stated above, we obtain the following diagram.

regular $(1, 2)^*$-closed ⇒$(1, 2)^*$-rc-preserving
⇓⇓
$(1, 2)^*$-closed ⇒ almost $(1, 2)^*$-closed
⇓⇓
$(1, 2)^*$-η-closed ⇒ almost $(1, 2)^*$-η-closed
⇓⇓
$(1, 2)^*$-gη-closed⇒almost $(1, 2)^*$-gη-closed
⇓⇓
$(1, 2)^*$-rgη-closed⇒almost $(1, 2)^*$-rgη-closed

The following examples enable us to realize that none of the implications in the above diagram is reversible.

**Example 5.18.** Let X = Y = {a, b, c}, $\Im_1$ = {ϕ, X, {a}}, $\Im_2$ = {ϕ, X, {b}}, $\psi_1$ = {ϕ, Y, {a, b}} and $\psi_2$ = {ϕ, Y, {a}}. Define f : X → Y as f(a) = b; f(b) = a; f(c) = c. Clearly f is $(1, 2)^*$-gη-closed as well as almost $(1, 2)^*$-gη-closed. It is also almost $(1, 2)^*$-rgη-closed. But it is neither $(1, 2)^*$-closed nor almost $(1, 2)^*$-closed. It is neither $(1, 2)^*$-η-closed nor almost $(1, 2)^*$-η-closed.

**Example 5.19.** Let X = Y = {a, b, c}, $\Im_1$ = {ϕ, X, {a}}, $\Im_2$ = {ϕ, X, {b}}, $\psi_1$ = {ϕ, Y, {a, b}} and $\psi_2$ = {ϕ, Y, {a}}. Define f : X → Y as f(a) = b, f(b) = c, f(c) = a. Clearly f is $(1, 2)^*$-rgη-closed as well as almost $(1, 2)^*$-rgη-closed. But it is neither $(1, 2)^*$-η-closed nor almost $(1, 2)^*$-η-closed. It is neither $(1, 2)^*$-gη-closed nor almost $(1, 2)^*$-gη-closed.

**Example 5.20**. Let $X = Y = \{a, b, c\}$, $\Im_1 = \{\phi, X, \{a\}\}$, $\Im_2 = \{\phi, X, \{a, c\}\}$, $\psi_1 = \{\phi, Y, \{a, b\}\}$ and $\psi_2 = \{\phi, Y, \{a\}\}$. Define $f : X \to Y$ as $f(a) = b$; $f(b) = a$; $f(c) = c$. Clearly f is almost $(1, 2)^*$-closed as well as almost $(1, 2)^*$-gη-closed, but it is not $(1, 2)^*$-closed.

**Example 5.21**. Let $X = Y = \{a, b, c\}$, $\Im_1 = \{\phi, X, \{a\}\}$, $\Im_2 = \{\phi, X, \{b\}\}$, $\psi_1 = \{\phi, Y, \{b\}, \{c\}, \{b, c\}\}$ and $\psi_2 = \{\phi, Y, \{a, b\}\}$. Define $f : X \to Y$ as $f(a) = c$, $f(b) = b$, $f(c) = a$. Clearly f is $(1, 2)^*$-closed as well as almost $(1, 2)^*$-closed. It is $(1, 2)^*$-η-closed as well as almost $(1, 2)^*$-gη-closed. But it is neither regular $(1, 2)^*$-closed nor $(1, 2)^*$-rc-preserving.

**Example 5.22**. Let $X = Y = \{a, b, c\}$, $\Im_1 = \{\phi, X, \{a\}\}$, $\Im_2 = \{\phi, X, \{b\}\}$, $\psi_1 = \{\phi, Y, \{a\}, \{a, c\}\}$ and $\psi_2 = \{\phi, Y, \{c\}\}$. Define $f : X \to Y$ as $f(a) = a$; $f(b) = c$; $f(c) = b$. Clearly f is $(1, 2)^*$-rc-preserving as well as almost $(1, 2)^*$-η-closed but it is not regular $(1, 2)^*$-closed.

**Example 5.23**. Let $X = Y = \{a, b, c\}$, $\Im_1 = \{\phi, X, \{a\}\}$, $\Im_2 = \{\phi, X, \{b\}\}$, $\psi_1 = \{\phi, Y, \{a\}, \{a, c\}\}$ and $\psi_2 = \{\phi, Y, \{c\}\}$. Define $f : X \to Y$ as $f(a) = b$, $f(b) = c$, $f(c) = a$. Clearly f is almost $(1, 2)^*$-rgη-closed.

**Proposition 5.24**. Let $f : X \to Y$ be a function. Then
(i) if f is $(1, 2)^*$-rgη-continuous, $(1, 2)^*$-rc-preserving, then it is $(1, 2)^*$-rgη-irresolute;
(ii) if f is a $(1, 2)^*$-R-map and $(1, 2)^*$-rgη-closed, then f(A) is $(1, 2)^*$-rgη-closed in Y for every $(1, 2)^*$-rgη-closed set A of X.
**Proof**.
(i) Let B be any $(1, 2)^*$-rgη-closed set of Y and let $U \in (1, 2)^*$-RO(X) contain $f^{-1}(B)$. Put $V = Y - f(X - U)$, then we have $B \subset V$, $f^{-1}(V) \subset U$ and $V \in (1, 2)^*$-RO(Y ) since f is $(1, 2)^*$-rc-preserving. Hence we obtain $\psi_{1,2}$-cl(B) $\subset V$ and hence $f^{-1}(\psi_{1,2}\text{-cl}(B)) \subset U$. By the $(1, 2)^*$-rgη-continuity of f we have $\Im_{1,2}\text{-cl}(f^{-1}(B)) \subset \Im_{1,2}\text{-cl}(f^{-1}(\psi_{1,2}\text{-cl}(B))) \subset U$. This shows that $f^{-1}(B)$ is $(1, 2)^*$-rgη-closed in X. Therefore f is $(1, 2)^*$-rgη-irresolute.
(ii) Let A be any $(1, 2)^*$-rgη-closed set of X and let $V \in (1, 2)^*$-RO(X) contain f(A). Since f is a $(1, 2)^*$-R-map, $f^{-1}(V) \in (1, 2)^*$-RO(X) and $A \subset f^{-1}(V)$. Therefore, we have $\Im_{1,2}\text{-cl}(A) \subset f^{-1}(V)$ and hence $f(\Im_{1,2}\text{-cl}(A)) \subset V$ . Since f is $(1, 2)^*$-rgη-closed, $f(\Im_{1,2}\text{-cl}(A))$ is $(1, 2)^*$-rgη-closed in Y and hence we obtain $\psi_{1,2}\text{-cl}(f(A)) \subset \psi_{1,2}\text{- cl}(f(\Im_{1,2}\text{-cl}(A))) \subset V$ . This shows that f(A) is $(1, 2)^*$-rgη-closed in Y.

**Corollary 5.25**. Let $f : X \to Y$ be a function.
(i) If f is $(1, 2)^*$-continuous, regular $(1, 2)^*$-closed, then $f^{-1}(B)$ is $(1, 2)^*$-rgη-closed in X for every $(1, 2)^*$-rgη-closed set B of Y .
(ii) If f is a $(1, 2)^*$-R-map and $(1, 2)^*$-closed, then f(A) is $(1, 2)^*$-rgη-closed in Y for every $(1, 2)^*$-rgη-closed set A of X.
**Proof**. This is an immediate consequence of **Proposition 5.24**.
**Proposition 5.26**. A surjection $f : X \to Y$ is almost $(1, 2)^*$-rgη-closed (resp. almost $(1, 2)^*$-gη-closed) if and only if for each subset S of Y and each $U \in (1, 2)^*$-RO(X) containing $f^{-1}(S)$ there exists an $(1, 2)^*$-rgη-open (resp. $(1, 2)^*$-gη-open) set V of Y such that $S \subset V$ and $f^{-1}(V ) \subset U$.
**Proof**. We prove only the first case, the proof of the other being entirely analogous.
**Necessity**. Suppose that f is almost $(1, 2)^*$-rgη-closed. Let S be a subset of Y and let $U \in (1, 2)^*$-RO(X) contain $f^{-1}(S)$. Put $V = Y - f(X - U)$, then V is a $(1, 2)^*$-rgη-open set of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.
**Sufficient**. Let F be any regular $(1, 2)^*$-η-closed set of X. Then $f^{-1}(Y - f(F)) \subset (X - F)$ and $(X - F) \in (1, 2)^*$-RO(X). There exists a $(1, 2)^*$-rgη-open set V of Y such that $(Y - f(F)) \subset V$ and $f^{-1}(V) \subset (X - F)$. Therefore, we have $f(F) \supset Y - V$ and $F \subset f^{-1}(Y - V)$. Hence we obtain $f(F) = Y - V$ , and f(F) is $(1, 2)^*$-rgη-closed in Y . This shows that f is almost $(1, 2)^*$-rgη-closed.

## VI. Preservation theorems

In this section, we investigate preservation theorems concerning mildly $(1, 2)^*$-η-normal spaces in bitopological spaces.

**Theorem 6.1.** If $f : X \to Y$ is an almost $(1, 2)^*$-rgη-continous $(1, 2)^*$-rc-preserving (resp. almost $(1, 2)^*$-η-closed) injection and Y is mildly $(1, 2)^*$-η-normal (resp $(1, 2)^*$-η-normal), then X is mildly $(1, 2)^*$-η-normal.

**Proof**. Let A and B be any disjoint regular $(1, 2)^*$-closed sets of X. Since f is an $(1, 2)^*$-rc-preserving (resp. almost $(1, 2)^*$-η-closed) injection, f(A) and f(B) are disjoint regular $(1, 2)^*$-closed ($(1, 2)^*$-η-closed) sets of Y . By the mild $(1, 2)^*$-η-normality (resp.$(1, 2)^*$-η-normality) of Y, there exist disjoint $(1, 2)^*$-η-open sets U and V of Y such that $f(A) \subset U$ and $f(B) \subset V$ . Now, put $G = \psi_{1,2}\text{-int}(\psi_{1,2}\text{-cl}(U))$ and $H = \psi_{1,2}\text{-int}(\psi_{1,2}\text{-cl}(V))$, then G and H are disjoint regular $(1, 2)^*$-open sets such that $f(A) \subset G$ and $f(B) \subset H$. Since f is almost $(1, 2)^*$-rgη-continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are disjoint $(1, 2)^*$-rgη-open sets containing A and B, respectively. It follows from **Theorem 4.5** that X is mildly $(1, 2)^*$-η-normal.

**Theorem 6.2**. If $f : X \to Y$ is a completely $(1, 2)^*$-continuous almost $(1, 2)^*$-gη-closed surjection and X is mildly $(1, 2)^*$-η-normal, then Y is $(1, 2)^*$-η-normal.

**Proof**.Let A and B be any disjoint $\psi_{1,2}$-closed sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint regular $(1, 2)^*$-closed sets of X. Since X is mildly $(1, 2)^*$-η-normal, there exist disjoint $(1, 2)^*$-η-open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Let $G = \Im_{1,2}\text{-int}(\Im_{1,2}\text{-cl}(U))$ and $H = \Im_{1,2}\text{-int}(\Im_{1,2}\text{-cl}(V))$, then G and H are disjoint regular $(1, 2)^*$-open sets such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. By **Proposition 5.26**, there exist $(1, 2)^*$-gη-open sets K and L of Y such that $A \subset K$, $B \subset L$, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint, so are K and L. Since K and L are $(1, 2)^*$-gη-open, we obtain $A \subset \psi_{1,2}\text{-int}(K)$, $B \subset \psi_{1,2}\text{-int}(L)$ and $[\psi_{1,2}\text{-int}(K) \cap \psi_{1,2}\text{-int}(L)] = \phi$. This shows that Y is $(1, 2)^*$-η-normal.

**Corollary 6.3**. If $f : X \to Y$ is a completely $(1, 2)^*$-continuous $(1, 2)^*$-η-closed surjection and X is mildly $(1, 2)^*$-η-normal, then Y is $(1, 2)^*$-η-normal.

**Corollary 6.4**. If $f : X \to Y$ is a completely $(1, 2)^*$-continuous $(1, 2)^*$-closed surjection and X is mildly $(1, 2)^*$-η-normal, then Y is $(1, 2)^*$-η-normal.

**Theorem 6.5**. Let $f : X \to Y$ be an $(1, 2)^*$-R-map (resp.almost $(1, 2)^*$-η-continuous) and almost $(1, 2)^*$-rgη-closed surjection. If X is mildly $(1, 2)^*$-η-normal (resp.$(1, 2)^*$-η-normal), then Y is mildly $(1, 2)^*$-η-normal.

**Proof**. Let A and B be any disjoint regular $(1, 2)^*$-closed sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint regular $(1, 2)^*$-closed (resp. $\Im_{1,2}$-closed) sets of X. Since X is mildly $(1, 2)^*$-η-normal (resp. $(1, 2)^*$-η-normal), there exist disjoint $(1, 2)^*$-η-open sets U and V of X such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$ . Put $G = \Im_{1,2}\text{-int}(\Im_{1,2}\text{-cl}(U))$ and $H = \Im_{1,2}\text{-int}(\Im_{1,2}\text{-cl}(V))$, then G and H are disjoint regular $(1, 2)^*$-open sets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. By **Proposition 5.26**, there exist $(1, 2)^*$-rgη-open sets K and L of Y such that $A \subset K$, $B \subset L$, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint, so are K and L. It follows from **Theorem 4.5** that Y is mildly $(1, 2)^*$-η-normal.

**Corollary 6.6**. Let $f : X \to Y$ be an $(1, 2)^*$-R-map (resp. almost $(1, 2)^*$-continuous) and almost $(1, 2)^*$-rgη-closed surjection. If X is mildly $(1, 2)^*$-η-normal (resp. $(1, 2)^*$-η-normal), then Y is mildly $(1, 2)^*$-η-normal.

## VII. Conclusion

In this paper, we introduce and study a new class of spaces called mildly $(1, 2)^*$-η-normal spaces and some new types of functions called $(1, 2)^*$-η-continuous, $(1, 2)^*$-gη-continuous, $(1, 2)^*$-rgη-continuous, almost $(1, 2)^*$-η-continuous, almost $(1, 2)^*$-gη-continuous, almost $(1, 2)^*$-rgη-continuous, $(1, 2)^*$-η-closed, $(1, 2)^*$-gη-closed, $(1, 2)^*$-rgη-closed, almost $(1, 2)^*$-η-closed, almost $(1, 2)^*$-gη-closed and almost $(1, 2)^*$-rgη-closed functions in bitopological spaces. Moreover, we obtain characterizations and preservation theorems for mildly $(1, 2)^*$-η-normal spaces in bitopological spaces. The regular $(1, 2)^*$-generalized η-closed sets can be used to derive a new decomposition of unity, closed map and open map, homeomorphism, closure and interior and new separation axioms. This idea can be extended to ordered topological, ordered bitopological and fuzzy topological spaces etc.

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