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Research Paper

Analytical Solutions of Fractional Boussinesq Saint-Venant Shallow Water Equations Based On the Decomposition Method

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Abstract

The objective of this paper is to elaborate analytical solutions of fractional Boussinesq and Saint-Venant shallow water equations based on the Adomian decomposition method. The Caputo fractional time derivative is used and the coupled Boussinesq Saint-Venant equation is investigated. The decomposition series convergence is demonstrated and explicite formulations are given in a general form. Parametric analysis is investigated based on the obtained series. Various particular cases are considered and well compared with available results and with the elaborated solution based on the Kudryashov procedure. The developed procedure is extremely simple and effective and can used to various types of Boussinesq and Saint-Venant shallow water models.

Keywords: Boussinesq-Saint-Venant, Caputo fractional derivative, Adomian decomposition method, Kudryashov procedure, shallow water

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I. INTRODUCTION

The classical Boussinesq equation was derived from incompressible Navier-Stokes equations neglecting density dependence in all terms except one involving gravity by Boussinesq[1] in 1872. It was firstly a model for describing shallow water waves, and presently found in many geophysical applications [2]. It is also used to describe nonlinear string oscillations and irrotational flows of a non-viscous liquid in a uniform rectangular channel see [3, 4, 5]. Because of its many real-world applications, the Cauchy problem corresponding to the Boussinesq equation has attracted the attention of many researchers see [6, 7, 8, 9, 10].

The used derivatives have long time been limited to integers. However, some phenomena reveal the integration of a half order in the heat equations for example. Since then, there has been a lot of research and work on partial differential equations with fractional derivatives. These derivatives are also used in the mechanical modelling of various mechanical phenomina such as rubbers and all kinds of materials that retain the memory of past deformations. This is because in the discretization of fractional equations to find the nth value, it is necessary to know all the values before 0 to (n-1)[11] and this reflects the memory effect. The transition to fractional equations begins with the replacement of integer derivatives by fractional ones and then solutions of the original natural equations become a special case of the solution of fractional equations.

In recent years, there has been a particular interest in fractional differential equations that are used to model and stimulate new applications in fluid mechanics, viscoelasticity,...etc. For example, the nonlinear oscillation of an earthquake can be modelled with fractional derivatives[12], the fluidodynamic model of traffic [13] as well as in new advances in composite materials[14]. An approximate solution for a fractional diffusion wave equation using the decomposition method has been presented in[15], and analytical and approximate solutions of space-time telegraph equations in[16], as well as the diffusion and advection-dispersion problem in[17], Hamzah et al used the variable fractional derivative for a pulsatile blood flow in constricted tapered artery problem in [18].

In general, there is no procedure for obtaining exact solutions of all nonlinear partial differential equations. The Adomian Decomposition Method (ADM)[19, 20, 21, 22, 23, 24] is an efficient alternative that allows finding analytical solutions for several linear or nonlinear, deterministic or stochastic models, as well as approximate numerical solutions. This method solves the problems of initial values and limit values[25] and the approximate analytical solutions obtained can be verified by direct substitution. By implementing the method ADM , it also allows obtaining an efficient algorithm to get approximate analytical solutions without discretization or rounding, easy to implement as a program in different languages. These programs do not consume much memory as in the case of finite elements or volume methods that require even computing centers to program parallel codes for large and complex domains.

Our concern in this work is the investigation of analytical solutions of nonlinear fractional partial differential equations of Boussinesq and Saint-Venant types as well as the coupling of these equations. The Adomian decomposition terms are explicitly given for each nonlinear equation. The solution is obtained in the form of series and the convergence of the solutions are demonstrated. Various Boussinesq and Saint-Venant shallow water models are considered. The elaborated solution are well compared to available and elaborated solutions based on the Kudryashov algorithm.

As the considered PDEs are fractional ones some properties of the used fractional derivative are firstly recalled.

2. Basic definitions

a. Definition

A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number p $(p > \mu)$, such that $f(x) = x^p.f_1(x)$, where $f_1(x) \in C[0,\infty)$, and it is said to be in the space C_μ if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

b. Definition

The Riemann-Liouville fractional integral operator of order $\beta \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$ is defined as [26]

$$I^\beta f(x) = \begin{cases} \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^x (x-t)^{\beta-1} f(t)dt, & \beta > 0, x > 0, \\ I^0 f(x) = f(x), & \beta = 0, \end{cases}$$

where $\Gamma(.)$ is the well-known Gamma function. Properties of the operator I^β can be found in [27, 28] and here we mention only the followings :

For $f \in C_\mu$, $\mu \geq -1$, γ, $\beta \geq 0$ and $\gamma > -1$

- $I^\beta I^\gamma f(x) = I^{\gamma+\beta} f(x)$

- $I^\gamma I^\beta f(x) = I^\beta I^\gamma f(x)$

- $I^{\beta}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+1)}x^{\beta+\gamma}$

c. Definition

The fractional derivative D^{β} of f(x) in the Caputo sense is defined as [29, 30]

$$D^{\beta}f(x) = I^{n-\beta}D^{n}f(x) = \frac{1}{\Gamma(n-\beta)}\int_0^x (x-t)^{n-\beta-1}f^{n}(t)dt \tag{1}$$

where $n-1 < \beta \leqslant n$, $n \in \mathbb{N}$, $x > 0$, $f \in C^{n}_{-1}$.
The basic properties of the operator D^{β} are given by the following lemma.

d. Lemma

If $m-1 < \beta \leqslant m$, $m \in \mathbb{N}$ and $f \in C_{\mu}$,$\mu \geq -1$ then

- $D^{\beta}I^{\beta}f(x) = f(x)$

- $I^{\beta}D^{\beta}f(x) = f(x) - \Sigma_{k=0}^{m-1}f^{(k)}(0^{+}).\frac{x^k}{k!},\ x > 0$

3. Fractional Boussinesq equation

Let us consider the following time fractional Boussinesq equation with general initial conditions.

$$(FB)_{sw} = \begin{cases} D_t^{\alpha}(D_t^{\alpha}U(x,t)) - aU_{xx}(x,t) - b(U^2(x,t))_{xx} - \varepsilon U_{xxxx}(x,t) = f(x,t) \\ U_t(x,0) = g_1(x),\ U(x,0) = g_2(x), \end{cases} \tag{2}$$

The classical Boussinesq equation is obtained as a paticular case ($\alpha = 1$).

$$(B)_{sw} = \begin{cases} U_{tt}(x,t) - a.U_{xx}(x,t) - b.(U^2(x,t))_{xx} - \varepsilon U_{xxxx}(x,t) = f(x,t) \\ U_t(x,0) = g_1(x),\ U(x,0) = g_2(x), \end{cases} \tag{3}$$

where a, b and ε are constant coefficients and f(x,t), $g_1(x)$ and $g_2(x)$ are arbitrary given functions.

3.1. Uniqueness of the solution and convergence of the decomposition

Firstly, the uniqueness of the solution of the Boussinesq equation (2) is considered. For that aim we assume that $F(U) = (U^2)_{xx}$, $D^{i}U = \dfrac{\partial^{i}U(x,t)}{\partial x^{i}}$, $\quad i = 2, 4$ are Lipschitz continuous functions with

$$\begin{aligned} \|F(U) - F(U^*)\|_{\infty} &\leq L_1\|U - U^*\|_{\infty}, \\ \|D^2U - D^2U^*\|_{\infty} &\leq L_2\|U - U^*\|_{\infty}, \\ \|D^4U - D^4U^*\|_{\infty} &\leq L_3\|U - U^*\|_{\infty}, \end{aligned} \tag{4}$$

We also assume that $z(x,t) = U(x,0) + tU(x,0)$ is bounded for all x, $t \in [0,T]$ and $|t-s| \leq M$,

$$\forall\ 0\ \leq s \leq t \leq T, \quad M \in \mathbb{R}^{+}.$$

We denote by α_1 the following constant :

$$\alpha_1 = \frac{M^{2\alpha-1}}{\Gamma(2\alpha)} T(|a|L_2 + |b|L_1 + |\varepsilon|L_3) \tag{5}$$

Theorem 3.1. *Let $0 < \alpha_1 < 1$, then the fractional Boussinesq equation (2) has a unique solution.*

Proof. Let U, U^* be two solutions of fractional Boussinesq equation (2).

$$U(x,t) = z(x,t) + \frac{a}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} D^2U(x,s)ds + \frac{b}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} F(U(x,s))ds + \frac{\varepsilon}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1} D^4U(x,s)ds.$$

$$\begin{aligned}
|U - U^*| &= |\frac{a}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1}(D^2U(x,s) - D^2U^*(x,s))ds \\
&+ \frac{b}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1}(F(U(x,s)) - F(U^*(x,s)))ds \\
&+ \frac{\varepsilon}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1}(D^4U(x,s) - D^4u^*(x,s))ds| \\
&\leq |\frac{a}{\Gamma(2\alpha)}| \int_0^t |t-s|^{2\alpha-1}|D^2U(x,s) - D^2U^*(x,s)|ds \\
&+ |\frac{b}{\Gamma(2\alpha)}| \int_0^t |t-s|^{2\alpha-1}|F(U(x,s)) - F(U^*(x,s))|ds \\
&+ |\frac{\varepsilon}{\Gamma(2\alpha)}| \int_0^t |t-s|^{2\alpha-1}|D^4U(x,s) - D^4U^*(x,s)|ds \\
|U - U^*| &\leq |\frac{a}{\Gamma(2\alpha)}| M^{2\alpha-1} t L_2 Max_{0\leqslant t\leqslant T}|U - U^*| \\
&+ |\frac{b}{\Gamma(2\alpha)}| M^{2\alpha-1} t L_1 Max_{0\leqslant t\leqslant T}|U - U^*| \\
&+ \frac{\varepsilon}{|\Gamma(2\alpha)|} M^{2\alpha-1} t L_3 Max_{0\leqslant t\leqslant T}|U - U^*| \\
\|U - U^*\|_\infty &\leq \frac{M^{2\alpha-1}}{\Gamma(2\alpha)}(T(|a|L_2 + |b|L_1 + |\varepsilon|L_3)\|U - U^*\|_\infty \\
\|U - U^*\|_\infty &\leq \alpha_1 \|U - U^*\|_\infty
\end{aligned}$$

So $(1-\alpha_1)\|U - U^*\|_\infty \leq 0$. Since $0 < \alpha_1 < 1$, then $\|U - U^*\|_\infty = 0 \quad \Rightarrow U = U^*$. □

Based on the Adomiam decomposition method the unknown function U(x,t), the solution of (2), is given by the following decomposition series.

$$U(x,t) = \sum_{i=0}^{\infty} U_i(x,t) \tag{6}$$

$$D^2U(x,t) = \sum_{i=0}^{\infty} D^2U_i(x,t), \tag{7}$$

$$D^4U(x,t) = \sum_{i=0}^{\infty} D^4U_i(x,t), \tag{8}$$

$$(U^2)_{xx}(x,t) = \sum_{i=0}^{\infty} A_i(x,t). \tag{9}$$

Theorem 3.2. *The series solution $U(x,t) = \sum_{i=0}^{\infty} U_i(x,t)$ of the fractional Boussinesq equation (2) using ADM is convergent when*

$$0 < \alpha_1 < 1, \quad |U_1(x,t)| < \infty.$$

Proof. Denote as $(\mathcal{C}([0,T]),||.||)$ the Banach space of all continuous functions on $[0,T]$ with the norm $||f|| = \max_{0\le t\le T}|f(t)|$. $s_n = \sum_{i=0}^{n} U_i(x,t)$ is sequence of partial sums. We prove that $(s_n)_n$ is a Cauchy sequence in this Banach space.

For $n,m \in \mathbb{N}^*$ such that $n>m$.

$$\begin{aligned}
||s_n - s_m|| &= \max_{0\le t\le T}|\sum_{i=m+1}^{n} U_i(x,t)| \\
&= \max_{0\le t\le T}|\frac{a}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}(\sum_{i=m}^{n-1} D^2U_i(x,s))ds \\
&+ \frac{b}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}(\sum_{i=m}^{n-1} A_i)ds \\
&+ \frac{\varepsilon}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}(\sum_{i=m}^{n-1} D^4U_i(x,s))ds|
\end{aligned}$$

We have

$$\begin{aligned}
\sum_{i=m}^{n-1} D^2U_i &= D^2(s_{n-1}) - D^2(s_{m-1}), \\
\sum_{i=m}^{n-1} A_i &= F(s_{n-1}) - F(s_{m-1}), \\
\sum_{i=m}^{n-1} D^4U_i &= D^4(s_{n-1}) - D^4(s_{m-1}),
\end{aligned}$$

We make

$$\begin{aligned}
B(t) &= |\frac{a}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}(\sum_{i=m}^{n-1} D^2U_i(x,s))ds \\
&+ \frac{b}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}(\sum_{i=m}^{n-1} A_i)ds \\
&+ \frac{\varepsilon}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}(\sum_{i=m}^{n-1} D^4U_i(x,s))ds| \\
&So, for\ t\in[0,T] \\
B(t) &\le |\frac{a}{\Gamma(2\alpha)}|\int_0^t |t-s|^{2\alpha-1}|D^2s_{n-1} - D^2s_{m-1}|ds \\
&+ |\frac{b}{\Gamma(2\alpha)}|\int_0^t |t-s|^{2\alpha-1}|F(s_{n-1}) - F(s_{m-1})|ds \\
&+ |\frac{\varepsilon}{\Gamma(2\alpha)}|\int_0^t |t-s|^{2\alpha-1}|D^4s_{n-1} - D^4s_{m-1}|ds \\
&\le \alpha_1||s_{n-1} - s_{m-1}||, \\
&so, \quad ||s_n - s_m|| \le \alpha_1||s_{n-1} - s_{m-1}||.
\end{aligned}$$

For $n = m+1$

$$||s_{m+1} - s_m|| \le \alpha_1||s_m - s_{m-1}|| \le \alpha_1^2||s_{m-1} - s_m||$$

$$||s_{m+1} - s_m|| \le \alpha_1^m||s_1 - s_0||$$

Recurrently one gets :

$$\begin{aligned}||s_n - s_m|| &\leq ||s_{m+1} - s_m|| + ||s_{m+2} - s_{m-1}|| + ... + ||s_n - s_{n-1}|| \\ &\leq [\alpha_1^m + \alpha_1^{m+1} + ... + \alpha_1^{n-1}]||s_1 - s_0|| \\ &\leq \alpha_1^m[1 + \alpha_1 + ... + \alpha_1^{n-m-1}]||s_1 - s_0|| \\ &\leq \alpha_1^m[\frac{1-\alpha_1^{n-m}}{1-\alpha_1}]||U_1(x,t)||\end{aligned}$$

Since $0 < \alpha_1 < 1, \ n > m \quad 1 - \alpha_1^{n-m} < 1$

$$||s_n - s_m|| \leq \frac{\alpha_1^m}{1-\alpha_1}||U_1(x,t)|| \tag{10}$$

$||U_1(x,t)|| < \infty$, so, as $n \longrightarrow \infty$, then $||s_n - s_m|| \longrightarrow 0$.

$(s_n)_n$ is a Cauchy sequence in $\mathcal{C}([0,T])$, and then $(s_n)_n$ is convergent. □

Theorem 3.3. *If $||U(x,t)|| \leqslant K$ then the maximum error of the series (6) of the problem (FB) is :*

$$||U(x,t) - \sum_{i=0}^{m} U_i(x,t)|| \leq \frac{\alpha_1^m}{1-\alpha_1}.K \tag{11}$$

Proof. From Theorem 3.2 and the inequality (10) we have :

$$||s_n - s_m|| \leq \frac{\alpha_1^m}{1-\alpha_1}||U_1(x,t)||$$

As $n \longrightarrow \infty$ $s_n \longrightarrow U(x,t)$ and $||U_1(x,t)|| \leqslant K$ the maximum absolute truncation error is finally :

$$||U(x,t) - \sum_{i=0}^{m} U_i(x,t)|| \leq \frac{\alpha_1^m}{1-\alpha_1}.K$$

□

3.2. Fractional Boussinesq decomposition

The main part of this subsection is the elaboration of explicite decomposition for the fractional Boussinesq nonlinear differential equation (2). For this aim, we apply the inverse operator I^α in both parts of the equation partial (2) :

$$(FB)_{sw} : D_t^\alpha U(x,t) = \Sigma_{k=0}^{m-1} U^{(k)}(x,0^+).\frac{t^k}{k!} + I^\alpha(aU_{xx}) + bI^\alpha\left((U^2)_{xx}\right) + \varepsilon I^\alpha(U_{xxxx}) + I^\alpha(f) \tag{12}$$

By reapplying the inverse operator I^α in both parts of the first equation and using integral's propriety one gets :

$$(FB)_{sw} : U(x,t) = \Sigma_{k=0}^{m-1} U^{(k)}(x,0^+).\frac{t^k}{k!} + I^{2\alpha}(aU_{xx}) + bI^{2\alpha}\left((U^2)_{xx}\right) + \varepsilon I^{2\alpha}(U_{xxxx}) + I^{2\alpha}(f) \tag{13}$$

Inserting of the decompositions (6)(9) in (2) and using some mathematical developments and identifications, the following sequences are obtained :

- $A_0 = U_0^2$
- $A_1 = 2.U_0U_1$
- $A_2 = 2.U_0U_2 + U_1^2$
- $A_3 = 2.U_0U_3 + 2.U_1U_2$
- ...

- $A_n = \Sigma_{i=0}^{n} U_i U_{n-i}$

Using the previous decompositions and inversed formulation (13), one gets :

$$(FB)_{sw} : \Sigma_{n=0}^{\infty} U_n = \Sigma_{k=0}^{m-1} U_t^{(k)}(x, 0^+).\frac{t^k}{k!} + I^{2\alpha}(\Sigma_{n=0}^{\infty}(a(U_n)_{xx} + \varepsilon(U_n)_{xxxx})) + bI^{2\alpha}(\Sigma_{n=0}^{\infty}(A_n)_{xx}) + I^{2\alpha}(f) \tag{14}$$

To lighten the writing we omit (x,t). The identification of these decompositions leads to :

$$(FB)_0 : U_0 = \Sigma_{k=0}^{m-1} U_t^{(k)}(x, 0^+).\frac{t^k}{k!} \tag{15}$$

$$(FB)_1 : U_1 = I^{2\alpha}(a(U_0)_{xx} + \varepsilon(U_0)_{xxxx}) + bI^{2\alpha}((A_0)_{xx}) + I^{2\alpha}(f) \tag{16}$$

The recursivity leads to :

$$(FB)_{n+1} : U_{n+1} = I^{2\alpha}(a(U_n)_{xx} + \varepsilon(U_n)_{xxxx}) + bI^{2\alpha}(A_n)_{xx} + I^{2\alpha}(f) \tag{17}$$

This recurrent relation can be rewritten as

$$(FB)_{n+1} : U_{n+1} = I^{2\alpha}(a(U_n)_{xx} + \varepsilon(U_n)_{xxxx}) + bI^{2\alpha}(\Sigma_{i=0}^{n} U_i U_{n-i})_{xx} + I^{2\alpha}(f) \tag{18}$$

Finally, the presented recurrent relationships allow one to get analytical solution of the fractional nonlinear equation (2) with an arbitrary value of the fractional order α. Higher orders can be computed in the presented straithforward maner specifying the coefficients, the source and initial conditions.

3.3. Application

For the sake of validation, the obtained formulations are tested in the following fractional Boussinesq equation.

$$(FB)_{sw} = \begin{cases} D_t^{\alpha}(D_t^{\alpha}U(x,t)) - aU_{xx}(x,t) - b(U^2(x,t))_{xx} + \varepsilon U_{xxxx}(x,t) = 0 \\ U_t(x,0) = g_1(x), \ U(x,0) = g_2(x). \end{cases} \tag{19}$$

with the following initial conditions :

$$g_2(x) = 1 + K \sec h^2(\sqrt{\frac{K}{6}}x) \tag{20}$$

$$g_1(x) = \sqrt{\frac{2}{3}} K^{\frac{3}{2}} c \sec h^2(\sqrt{\frac{K}{6}}x) \tanh(\sqrt{\frac{K}{6}}x) \tag{21}$$

K is a constant that can represent the amplitude of the pulse.

$$(FB)_0 : U_0 = 1 + K \sec h^2(\sqrt{\frac{K}{6}}x) \tag{22}$$

$$(FB)_1 : U_1 = tg_1(x) + I^{2\alpha}((U_0)_{xx} + \varepsilon(U_0)_{xxxx} + (A_0)_{xx}) \tag{23}$$

$$(FB)_1 : U_1 = tg_1(x) + \frac{1}{9}K^2 \sec h^2(\sqrt{\frac{K}{6}}x)\{30K \sec h^2(\sqrt{\frac{K}{6}}x) tanh^2(\sqrt{\frac{K}{6}}x) + 30K tanh^4(\sqrt{\frac{K}{6}}x) - 6K \sec h^2(\sqrt{\frac{K}{6}}x) - 30K tanh^2(\sqrt{\frac{K}{6}}x) + 27 tanh^2(\sqrt{\frac{K}{6}}x) + 4K - 9\}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{24}$$

$$(FB)_{n+1} : U_{n+1} = I^{2\alpha}((U_n)_{xx} + \varepsilon(U_n)_{xxxx} + (A_n)_{xx}) \tag{25}$$

To test the accuracy of the presented decomposition exact solution of equation (19) is elaborated here for $\alpha = 1$ based on the Kurdyashov algorithm. Mathematical details of the associated exact solution is given in the appendix here in. The obtained solution is :

$$U(x,t) = \varepsilon \left(c_0 - \frac{2K}{3b} + \frac{K}{b} \sec h^2 (\sqrt{\frac{K}{6}}(x - ct) - \frac{\ln(A)}{2}) \right) \tag{26}$$

with :

$$c = \sqrt{2(\frac{a}{2} + \frac{\varepsilon K}{3})},\ K = \frac{3k^2}{2}$$

If a=b=1, f=0, A=1, $\varepsilon = \pm 1$ the exact solution elaborated by (Manoranjan et al. [31] is :

$$U(x,t) = \varepsilon \left(\frac{a}{2\varepsilon} + \frac{\varepsilon}{2} - \frac{2K}{3b} + \frac{K}{b} \sec h^2 (\sqrt{\frac{K}{6}}(x - ct)) \right) \tag{27}$$

$$U(x,t) = \varepsilon \left(\beta + \frac{\varepsilon}{2} + K \sec h^2 (\sqrt{\frac{K}{6}}(x - ct)) \right)$$

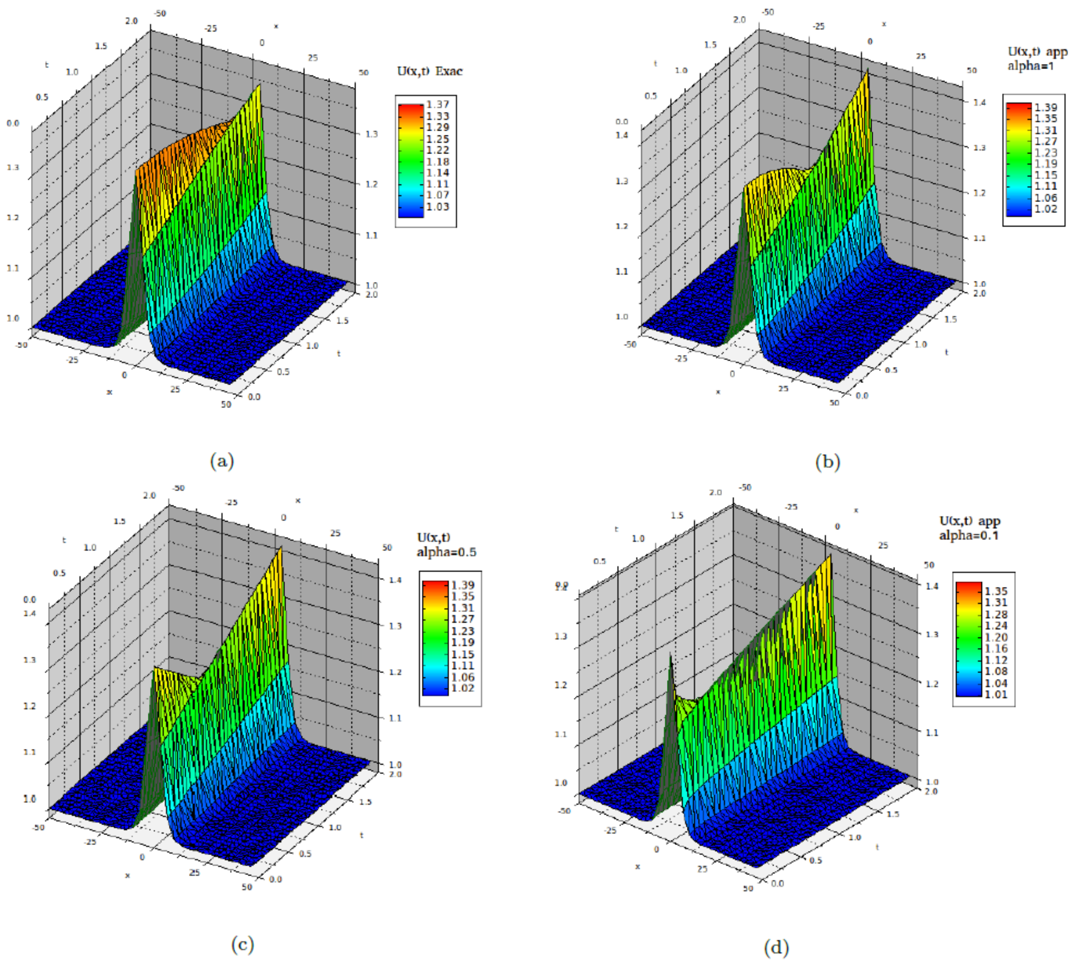


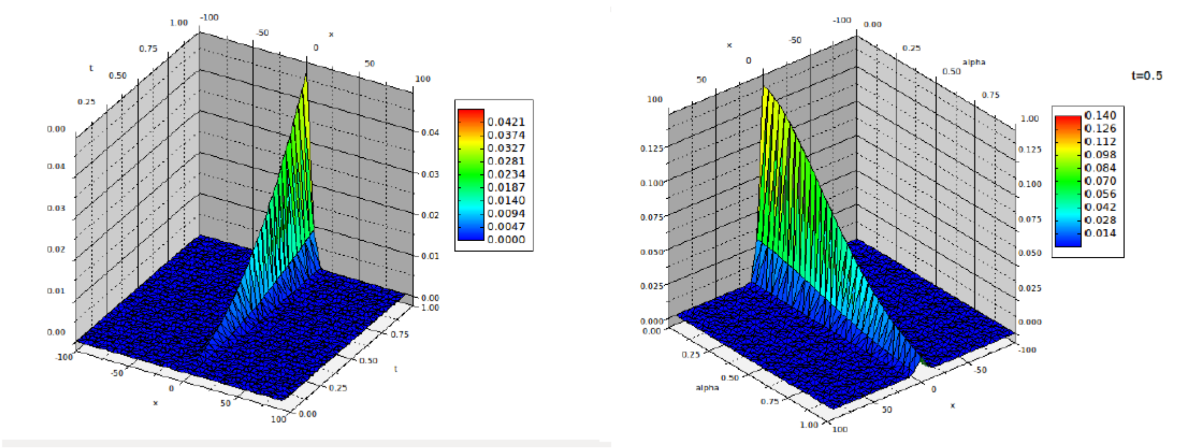
Figure 1: Exact and approached solutions for K=0.369, c=1.116432, $0 \leqslant t \leqslant 2$:(a) exact (b) approached solution for $\alpha = 1$, (c) approached solution for $\alpha = 0.5$,(d) approached solution for $\alpha = 0.1$

with : $\beta = \frac{a}{2\varepsilon}$, $c = \pm\sqrt{2\varepsilon(\beta + \frac{K}{3})}$ the velocity and K is the amplitude of the pulse, β is an arbitrary parameter, x_0 is the initial position of the pulse, where $\varepsilon = -1$ gives the Good Boussinesq (GB) and $\varepsilon = 1$ gives the Bad Boussinesq.

For numerical representation we take $\varepsilon = 1$, $\beta = 0.5$, K=0.369, c=1.116243701

Figure 1 presents the exacte solution in Fig 1-a and approached solution at order 1 in Fig 1-b for $\alpha = 1$. The accuracy is observed and the approached solution can be improved by higher orders. For $\alpha \neq 0$, there is no available exact solution. Thus, the presented analytical approximate solution can be used to analyse the corresponding solutions behaviour. For $\alpha = 0.5$ and $\alpha = 0.1$, the approached solutions are presented in Fig 1-c and Fig 1-d.

Absolut error in (x,t) for $\alpha = 1$ is presented in figure 2-a and in (α, x) in figure 2-b for t=0.5. These figures show clearly the accuracy of the approached solution The presented solution can be used for parameters



(err1) (err2)

Figure 2: Absolut error in (x,t) for $\alpha = 1$ in (err1) and in (α, x) for t=0.5 (err2)

effects analysis on the Boussinesq shallow water system. The effects of parameters a, b and ε are presented in figures-3-a-b-c.

4. Saint-Venant shallow water system

In this subsection, the following nonlinear fractional Saint-Venant system is considered.

$$(FSV)_{sw} = \begin{cases} D_t^\alpha H(x,t) + (H.U)_x(x,t) = 0 \\ D_t^\alpha U + (\frac{1}{2}U^2 + g.H)_x + g.Fd_x = 0 \\ U(x,0) = g_2(x),\ H(x,0) = g_3(x) \end{cases} \tag{28}$$

where the velocity U(x,t)and heigh H(x,t) of the wave are the unknowns. The acceleration g and the initial conditions g_2 and g_3 are given functions as well as the bed topography function Fd. The following classical Saint-Venant partial differential system is widely used

$$(SV)_{sw} = \begin{cases} H_t(x,t) + (H.U)_x(x,t) = 0 \\ (H.U)_t(x,t) + ((H.U^2)(x,t) + \frac{gH^2(x,t)}{2})_x + gH(x,t)(Fd(x))_x = 0 \\ U(x,0) = g_2(x),\ H(x,0) = g_3(x) \end{cases} \tag{29}$$

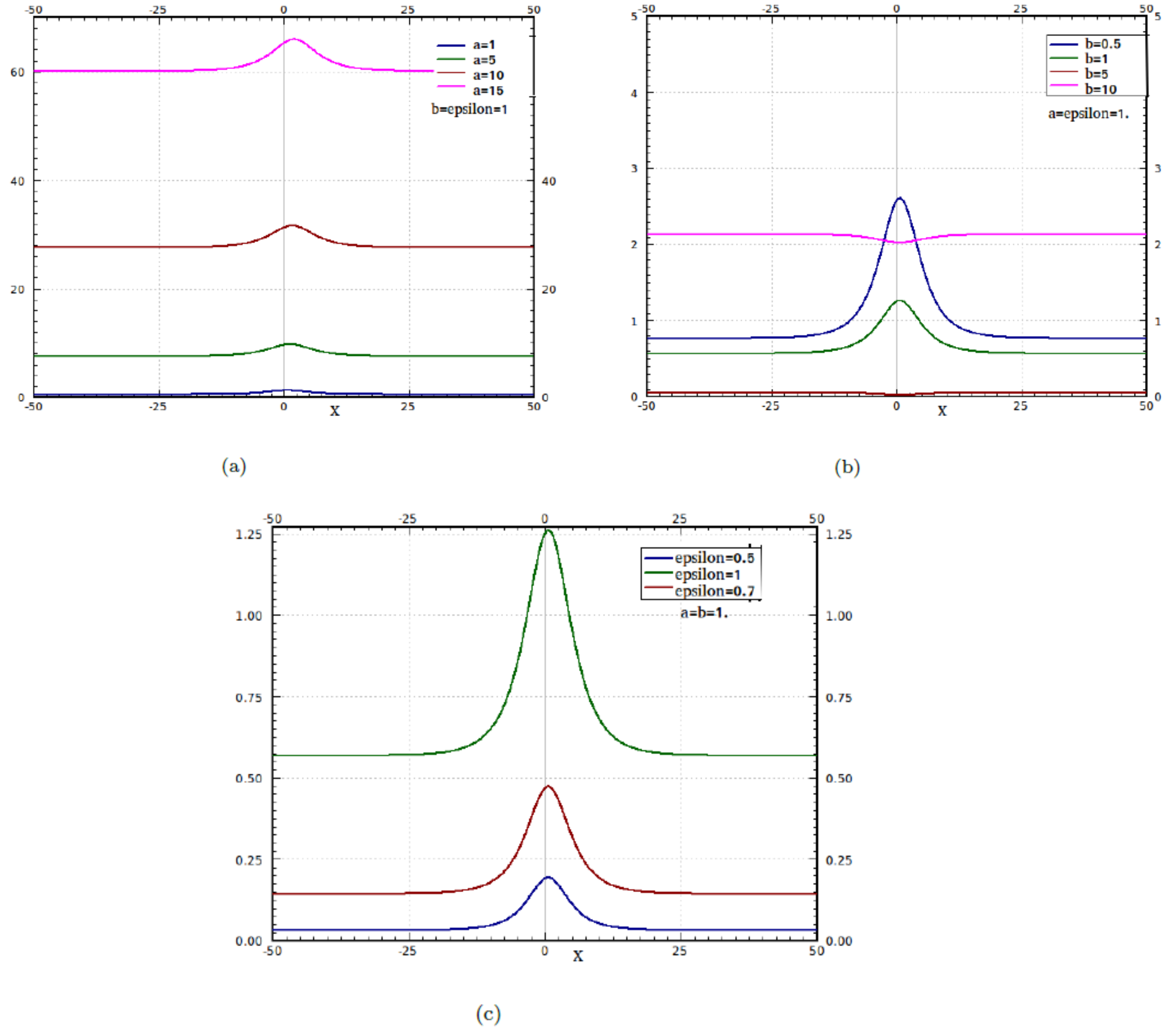


Figure 3: Variation of U(x,t) for different values of a, b and ϵ at t=0.5, K=0.369

This equation can be rearanged as

$$H_t(x,t) + (H.U)_x(x,t) = 0 \Rightarrow H_t = -H_x U - HU_x$$

$$(H.U)_t(x,t) + ((H.U^2)(x,t) + \frac{gH^2(x,t)}{2})_x + gH(x,t)(Fd(x))_x = 0$$

$$U(-H_x U - HU_x) + H.U_t + (H.U^2)_x + gH.H_x + gHFd_x = 0$$

$$H(U_t - UU_x) - H_x.U^2 + H_x.U^2 + H.(U^2)_x + gH.H_x + gHFd_x = 0$$

$$H(U_t - UU_x) + 2H.UU_x + gH.H_x + gHFd_x = 0$$

$$H\left(U_t + UU_x + g.H_x + g.Fd_x\right) = 0$$

$$U_t + UU_x + g.H_x + g.Fd_x = 0$$

$$U_t + (\frac{1}{2}U^2 + g.H)_x + g.Fd_x = 0$$

Thus,the following compact system is resulted

$$(SV)_{sw} = \begin{cases} H_t(x,t) + (H.U)_x(x,t) = 0 \\ U_t + (\frac{1}{2}U^2 + g.H)_x + g.Fd_x = 0 \\ U(x,0) = g_2(x),\ H(x,0) = g_3(x) \end{cases} \tag{30}$$

This equation is a particular case of (26) for $\alpha = 1$ and widely used for shallow water analysis

4.1. Uniqueness of the solution and comvergence of the decomposition

For the uniqueness of the solution of the considered fractional Saint-Venant system (26), let us assume that :

$F(U) = \frac{1}{2}(U^2)_x,\ \ G(H) = (H.U)_x$ are Lipschitz continuous functions with

$$\|F(U) - F(U^*)\|_\infty \leq L_1\|U - U^*\|_\infty, \tag{31}$$

$$\|G(H) - G(H^*)\|_\infty \leq L_2\|H - H^*\|_\infty, \tag{32}$$

$$\tag{33}$$

$U(x,0)$ and $H(x,0)$ are bounded for all x, $t \in [0,T]$ and $|t-s| \leq M, \quad \forall 0 \leq s \leq t \leq T, \quad M \in \mathbb{R}^+$.

We denote by α_2 the following constant :

$$\alpha_2 = M^{\alpha-1}T|\frac{1}{\Gamma(\alpha)}|Max(L_1, L_2) \tag{34}$$

Theorem 4.1. *Let $0 < \alpha_2 < 1$, then the fractional Saint-Venant system (26) has a unique couple of solution (U, H).*

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Proof. Assuming that H is unique, let us demonstrate that U is unique too.

Let $U, \quad U^*$ be two solutions of fractional Saint-Venant equation (26). Thus,

$$U(x,t) = U(x,0) + \frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(U)ds - \frac{g}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}H_x ds - \frac{g}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Fd_x ds.$$

$$\begin{aligned} |U - U^*| &= |\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(F(U) - F(U^*)(x,s))ds| \\ &\leq |\frac{1}{\Gamma(\alpha)}|\int_0^t |t-s|^{\alpha-1}|F(U) - F(U^*)(x,s))|ds \\ Max_{0\leqslant t\leqslant T}|U - U^*| &\leq |\frac{1}{\Gamma(\alpha)}|M^{\alpha-1}TL_1 Max_{0\leqslant t\leqslant T}|U - U^*| \\ \|U - U^*\|_\infty &\leq \alpha_2\|U - U^*\|_\infty \end{aligned}$$

So $(1-\alpha_2)\|U - U^*\|_\infty \leq 0$. Since $0 < \alpha_2 < 1$, then $\|U - U^*\|_\infty = 0 \quad \Rightarrow U = U^*$.

Let $H, \quad H^*$ be two solutions second equation of fractional Saint-Venant system.

Assuming that U is unique, let us demonstrate that H is unique too.

$$H(x,t) = H(x,0) + \frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}G(H)ds.$$

$$|H - H^*| = |\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(G(H) - G(H^*)(x,s))ds|$$

$$Max_{0\leqslant t\leqslant T}|H - H^*| \leq |\frac{1}{\Gamma(\alpha)}|M^{\alpha-1}TL_2 Max_{0\leqslant t\leqslant T}|H - H^*|$$

$$\|H - H^*\|_\infty \leq \alpha_2 \|H - H^*\|_\infty$$

So, $(1-\alpha_2)\|H - H^*\|_\infty \leq 0$. Since $0 \ < \alpha_2 < \ 1$, then $\|H - H^*\|_\infty = 0 \quad \Rightarrow H = H^*$. □

The Adomiam decomposition of the solution of (26) is given by the series

$$U(x,t) = \sum_{i=0}^{\infty} U_i(x,t) \tag{35}$$

$$H(x,t) = \sum_{i=0}^{\infty} H_i(x,t) \tag{36}$$

Theorem 4.2. *The series solution of $U(x,t) = \sum_{i=0}^{\infty} U_i(x,t)$, $H(x,t) = \sum_{i=0}^{\infty} H_i(x,t)$ of fractional Saint-Venant system (26) using ADM is convergent when*

$$0 < \alpha_2 < 1, \quad |H_1(x,t)| < \infty \ and \ |U_1(x,t)| < \infty$$

Proof. $s_n = \sum_{i=0}^{n} U_i(x,t)$ is a sequence of partial sums. We prove that $(s_n)_n$ is a Cauchy sequence in the Banach space.$(C[0,T], \|.\|)$

For $n, m \in \mathbb{N}^*$ such that $n > m$.

$$||s_n - s_m|| = \max_{0\leq t\leq T} |\sum_{i=m+1}^{n} H_i(x,t)|$$

$$= \max_{0\leq t\leq T} |\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{2\alpha-1}(\sum_{i=m}^{n-1} B_i|$$

We have

$$\sum_{i=m}^{n-1} B_i = G(s_{n-1}) - G(s_{m-1}),$$

We take

$$C(t) = |\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(\sum_{i=m}^{n-1} B_i)ds|$$

So, for $t \in [0,T]$

$$C(t) \leq |\frac{1}{\Gamma(\alpha)}| \int_0^t |t-s|^{\alpha-1}|G(s_{n-1}) - G(s_{m-1})|ds$$

$$+ |\frac{L_2}{\Gamma(\alpha)}| \int_0^t |t-s|^{2\alpha-1}|s_{n-1} - s_{m-1}|ds$$

$$\leq \alpha_2 ||s_{n-1} - s_{m-1}||,$$

so, $||s_n - s_m|| \leq \alpha_2 ||s_{n-1} - s_{m-1}||$.

For $n = m+1$

$$||s_{m+1} - s_m|| \leq \alpha_2 ||s_m - s_{m-1}|| \leq \alpha_2^2 ||s_{m-1} - s_{m-2}||$$

$$||s_{m+1} - s_m|| \leq \alpha_2^m ||s_1 - s_0||$$

Recurrently, one gets :

$$\begin{aligned}||s_n - s_m|| &\leq ||s_{m+1} - s_m|| + ||s_{m+2} - s_{m-1}|| + ... + ||s_n - s_{n-1}||\\ &\leq [\alpha_2^m + \alpha_2^{m+1} + ... + \alpha_2^{n-1}]||s_1 - s_0||\\ &\leq \alpha_2^m[1 + \alpha_2 + ... + \alpha_2^{n-m-1}]||s_1 - s_0||\\ &\leq \alpha_2^m[\frac{1 - \alpha_2^{n-m}}{1 - \alpha_0}]||H_1(x,t)||\end{aligned}$$

Since $0 < \alpha_2 < 1, \; n > m \qquad 1 - \alpha_2^{n-m} < 1$

$$||s_n - s_m|| \leq \frac{\alpha_2^m}{1 - \alpha_2}||H_1(x,t)|| \tag{37}$$

$||H_1(x,t)|| < \infty$, so, as $n \longrightarrow \infty$, then $||s_n - s_m|| \longrightarrow 0$.

$(s_n)_n$ is a Cauchy sequence in $\mathcal{C}([0,T])$, so $(s_n)_n$ is convergent. □

Theorem 4.3. *If $||H(x,t)|| \leqslant K$ then the maximum error of the series (34) of the problem (FSV) is :*

$$||H(x,t) - \sum_{i=0}^{m} H_i(x,t)|| \leq \frac{\alpha_2^m}{1 - \alpha_2}.K \tag{38}$$

Proof. From Theorem 4.2 and the inequality (35) we have :

$$||s_n - s_m|| \leq \frac{\alpha_2^m}{1 - \alpha_2}||H_1(x,t)||$$

As $n \longrightarrow \infty$ $s_n \longrightarrow H(x,t)$ and $||H_1(x,t)|| \leqslant K$, the maximum absolute truncation error is finally :

$$||H(x,t) - \sum_{i=0}^{m} H_i(x,t)|| \leq \frac{\alpha_2^m}{1 - \alpha_2}.K$$

□

4.2. Fractional Saint-Venant decomposition

This subsection focusses on the elaboration of the decomposition method for Saint-Venant nonlinear fractional differential system (26). Applying the inverse operator I^α in both parts of the equations (26) leads to :

$$(FSV)_{sw} = \begin{cases} H(x,t) = \Sigma_{k=0}^{m-1} H^{(k)}(x,0^+).\dfrac{t^k}{k!} - I^\alpha(H.U)_x \\ U(x,t) = \Sigma_{k=0}^{m-1} U^{(k)}(x,0^+).\dfrac{t^k}{k!} - \dfrac{1}{2}I^\alpha(U^2)_x - gI^\alpha(H_x + Fd_x) \end{cases} \tag{39}$$

The following decompositions are used :

$$U^2 = \Sigma_{n=0}^{\infty} A_n(x,t), \tag{40}$$

$$H.U = \Sigma_{n=0}^{\infty} B_n(x,t), \tag{41}$$

The insertion of these series decomposition in (26) and using some mathematical developments and identifications, the following sequences are resulted

- $A_0 = U_0^2$
- $A_1 = 2.U_0U_1$
- $A_2 = 2.U_0U_2 + U_1^2$
- $A_3 = 2.U_0U_3 + 2.U_1U_2$
- ...
- $A_n = \Sigma_{i=0}^{n} U_i U_{n-i}$

Sequence of B_n

- $B_0 = H_0.U_0$
- $B_1 = H_0U_1 + H_1U_0$
- $B_2 = H_0U_2 + H_1U_1 + H_2U_0$
- $B_3 = H_0U_3 + H_1U_2 + H_2U_1 + H_3U_0$
- ...
- $B_n = \Sigma_{i=0}^{n} H_i U_{n-i}$

Assume that g is constant (gravity) and that Fd(.) does not depend on time, the previous decompositions and inversed equation (37), lead to :

$$(FSV)_{sw} = \begin{cases} \Sigma_{n=0}^{\infty} H_n(x,t) = \Sigma_{k=0}^{m-1} H^{(k)}(x,0^+).\dfrac{t^k}{k!} - I^{\alpha}\Sigma_{n=0}^{\infty}(B_n)_x \\ \Sigma_{n=0}^{\infty} U_n(x,t) = \Sigma_{k=0}^{m-1} U^{(k)}(x,0^+).\dfrac{t^k}{k!} - \dfrac{1}{2} I^{\alpha}\Sigma_{n=0}^{\infty}(A_n)_x - gI^{\alpha}(\Sigma_{n=0}^{\infty}(H_n)_x + Fd_x) \end{cases} \quad (42)$$

The identification of these decompositions leads to :

$$(FSV)_0 = \begin{cases} H_0(x,t) = \Sigma_{k=0}^{m-1} H^{(k)}(x,0^+).\dfrac{t^k}{k!} = g_2(x) \\ U_0(x,t) = \Sigma_{k=0}^{m-1} U^{(k)}(x,0^+).\dfrac{t^k}{k!} = g_3(x) \end{cases} \quad (43)$$

$$(FSV)_1 = \begin{cases} H_1 = -I^{\alpha}(B_0)_x \\ U_1 = -I^{\alpha}(\dfrac{1}{2}.A_0 + g.H_0 + gFd)_x \end{cases} \quad (44)$$

The recursivity leads to :

$$(FSV)_{n+1} = \begin{cases} H_{n+1} = -I^{\alpha}(B_n)_x \\ U_{n+1} = -I^{\alpha}(\dfrac{1}{2}.A_n + g.H_n + gFd)_x \end{cases} \quad (45)$$

135 The presented recurrent relationships allow one to get analytical solution of the fractional Saint-Venant system. The higher orders can be computed in the presented straithforward maner by specifying the coefficients and the source and initial conditions. The use of symbolic softwares such as Maple or Mathematica can be very helpful to get easily higher orders.

4.3. Application 1

For the sake of validation, the obtained formulations are tested in the following fractional Saint-Venant equations. Consider the initial condition for the depth and velocity and the gravity is unity (g=1) as follows :

$$(FSV)_{sw} = \begin{cases} D_t^\alpha H(x,t) + (H.U)_x(x,t) = 0 \\ D_t^\alpha U + (\frac{1}{2}U^2 + H)_x + Fd_x = 0 \\ U(x,0) = 0,\ H(x,0) = \frac{1}{10} + \frac{1}{4}sech(x) + \frac{e^{-x^2}}{1+e^{-x^2}} \end{cases} \tag{46}$$

with the bed topography function $Fd = -\frac{e^{-x^2}}{1+e^{-x^2}}$. This problem has been investigated in [32, 33] for ($\alpha = 1$) using Adomian decomposition.
Based on the previous formulations we obtain :

$$(FSV)_0 = \begin{cases} H_0(x,t) = \frac{1}{10} + \frac{1}{4}sech(x) + \frac{e^{-x^2}}{1+e^{-x^2}} \\ U_0(x,t) = 0 \end{cases} \tag{47}$$

The MAPLE software, is used here to get the following H_k, and U_k.

$$(FSV)_1 = \begin{cases} H_1 = -I^\alpha(B_0)_x = -I^\alpha(H_0.U_0)_x = 0 \\ U_1 = -I^\alpha(\frac{1}{2}.A_0 + H_0 + Fd)_x = I^\alpha(\frac{1}{4}sech(x).tanh(x)) \end{cases} \tag{48}$$

$$(FSV)_1 = \begin{cases} H_1 = 0 \\ U_1 = \frac{1}{4}sech(x).tanh(x)\frac{t^\alpha}{\Gamma(\alpha+1)} \end{cases} \tag{49}$$

$$(FSV)_2 = \begin{cases} H_2 = -I^\alpha(B_1)_x = -I^\alpha(H_0.U_1)_x \\ U_2 = -I^\alpha\frac{2xe^{-x^2}}{(1+e^{-x^2})^2} = \frac{-2xe^{-x^2}}{(1+e^{-x^2})^2}\frac{t^\alpha}{\Gamma(\alpha+1)} \end{cases} \tag{50}$$

with :

$$H_2 = ((\frac{1}{80(1+e^{-x^2})^2}sech(x)\{(15sech(x)e^{-2x^2} + 30sech(x)e^{-x^2} + 44e^{-2x^2} + 15sech(x) + 48e^{-x^2} + 4)tanh^2(x) \tag{51}$$

$$+ 40xtanh(x)e^{-x^2} - 5sech(x)e^{-2x^2} - 10sech(x)e^{-x^2} - 22e^{-2x^2} - 5sech(x) - 24e^{-x^2} - 2)\}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{52}$$

(53)

$$(FSV)_3 = \begin{cases} H_3 = -I^\alpha(B_2)_x = -I^\alpha(H_0.U_2)_x \\ U_3 = -I^\alpha(\frac{1}{2}A_2 + H_2 + Fd)_x \end{cases} \tag{54}$$

with :

$$H_3 = \frac{e^{-x^2}}{10(1+e^{-x^2})^4}\{-(5xe^{-2x^2} + 10xe^{-x^2} + 5)sech(x)tanh(x) \tag{55}$$

$$+ (10x^2e^{-2x^2} - 10x^2 + 5e^{-2x^2} + 10e^{-x^2} + 5)sech(x) \tag{56}$$

$$+ 44x^2e^{-2x^2} - 80x^2e^{-x^2} - 4x^2 + 22e^{-2x^2} + 24e^{-x^2} + 2\}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{57}$$

and :

$$U_3 = (\frac{1}{16}sech^2(x)tanh(x)(2tanh^2(x) - 1)\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)\Gamma^2(\alpha+1)} \tag{58}$$

$$+\frac{sech(x)}{40(1+e^{-x^2})^3}\{(30tanh^3(x) - 20tanh^2(x))(e^{-3x^2} + 3e^{-2x^2} + 3e^{-x^2} + 1)sech(x) \tag{59}$$

$$+ (66e^{-3x^2} + 138e^{-2x^2} + 78e^{-x^2} + 6)tanh^3(x) + 80xe^{-x^2}(1+e^{-x^2})tanh^2(x)+ \tag{60}$$

$$- (40x^2e^{-2x^2} - 40x^2e^{-x^2} + 55e^{-3x^2} + 135e^{-2x^2} + 85e^{-x^2} + 5)tanh(x) \tag{61}$$

$$- 40xe^{-x^2}(e^{-x^2} + 1)\}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + (\frac{-2xe^{-x^2}}{(1+e^{-x^2})^2})\frac{t^{\alpha}}{\Gamma(\alpha+1)} \tag{62}$$

These series terms permit to get approximate solution for an arbitrary fractional parameter α.

Particular case $\alpha = 1$:

$$(FSV)_1 = \begin{cases} H_1 = 0 \\ U_1 = \frac{1}{4}sech(x).tanh(x)\frac{t^1}{\Gamma(1+1)} \end{cases} \tag{63}$$

$$(FSV)_2 = \begin{cases} H_2 = -I(B_1)_x = -I^{\alpha}(H_0.U_0)_x \\ U_2 = -I(\frac{2xe^{-x^2}}{1+e^{-x^2}}) = (\frac{-2xe^{-x^2}}{1+e^{-x^2}})\frac{t}{\Gamma(2)} \end{cases} \tag{64}$$

with :

$$H_2 = ((\frac{1}{80(1+e^{-x^2})^2}sech(x)\{(15sech(x)e^{-2x^2} + 30sech(x)e^{-x^2} + 44e^{-2x^2} + 15sech(x) + 48e^{-x^2} + 4)tanh^2(x) \tag{65}$$

$$+ 40xtanh(x)e^{-x^2} - 5sech(x)e^{-2x^2} - 10sech(x)e^{-x^2} - 22e^{-2x^2} - 5sech(x) - 24e^{-x^2} - 2)\}\frac{t^2}{\Gamma(3)} \tag{66}$$

(67)

$$(FSV)_3 = \begin{cases} H_3 = -I(B_2)_x = -I(H_0.U_2)_x \\ U_3 = -I(\frac{2xe^{-x^2}}{1+e^{-x^2}}) = (\frac{-2xe^{-x^2}}{1+e^{-x^2}})\frac{t^1}{\Gamma(2)} \end{cases} \tag{68}$$

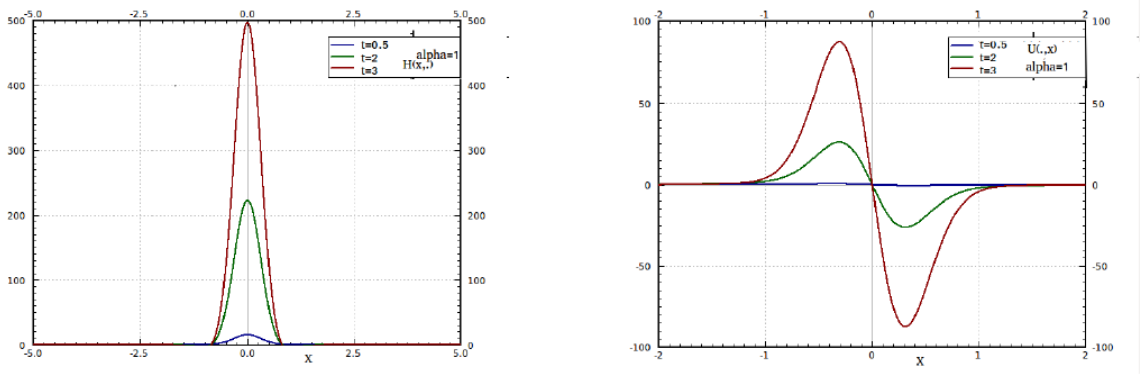

Figure 4: Variation of velocity U(x,.) and H(x,.) for $t = 0.5, 2, 3$ and $\alpha = 1$

with :

$$H_3 = \frac{e^{-x^2}}{10(1+e^{-x^2})^4}\{-(5xe^{-2x^2} + 10xe^{-x^2} + 5)sech(x)tanh(x) \tag{69}$$

$$+ (10x^2e^{-2x^2} - 10x^2 + 5e^{-2x^2} + 10e^{-x^2} + 5)sech(x) \tag{70}$$

$$+ 44x^2e^{-2x^2} - 80x^2e^{-x^2} - 4x^2 + 22e^{-2x^2} + 24e^{-x^2} + 2\}\frac{t^2}{\Gamma(3)} \tag{71}$$

$$U_3 = (\frac{1}{16}sech^2(x)tanh(x)(2tanh^2(x) - 1)\frac{t^3}{3} \tag{72}$$

$$+ \frac{sech(x)}{40(1+e^{-x^2})^3}\{(30tanh^3(x) - 20tanh^2(x))(e^{-3x^2} + 3e^{-2x^2} + 3e^{-x^2} + 1) \tag{73}$$

$$+ (66e^{-3x^2} + 138e^{-2x^2} + 78e^{-x^2} + 6)tanh^3(x) + 80xe^{-x^2}(1+e^{-x^2})tanh^2(x) + \tag{74}$$

$$- (40x^2e^{-2x^2} - 40x^2e^{-x^2} + 55e^{-3x^2} + 135e^{-2x^2} + 85e^{-x^2} + 5)tanh(x) \tag{75}$$

$$- 40xe^{-x^2}(e^{-x^2} + 1)\}\frac{t^3}{6} + (\frac{-2xe^{-x^2}}{(1+e^{-x^2})^2})t \tag{76}$$

The obtained approximate solution is thus

$$(FSV)_{sw} = \begin{cases} U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + U_3(x,t) \\ H(x,t) = H_0(x,t) + H_1(x,t) + H_2(x,t) + H_3(x,t) \end{cases} \tag{77}$$

These solutions are presented in figures 4 to 8 for various fixed parameters. The time and space evolution of U are clearly presented. At fixed time, the variations of U and H with respect to x are presented in figure 4. The effect of the fractional derivative order α on the velocity U and height H is presented in figures 5-6 and 7 for $\alpha = 1$,$\alpha = 0.9$ and $\alpha = 0.5$. The variation of U(x=0,t) for various values of α is presented in figure 8. These presentations show that the fractional derivative order may have a strong effect on the dynamic effect of Saint-Venant system and some physical phenomena may be captured by the presented model.

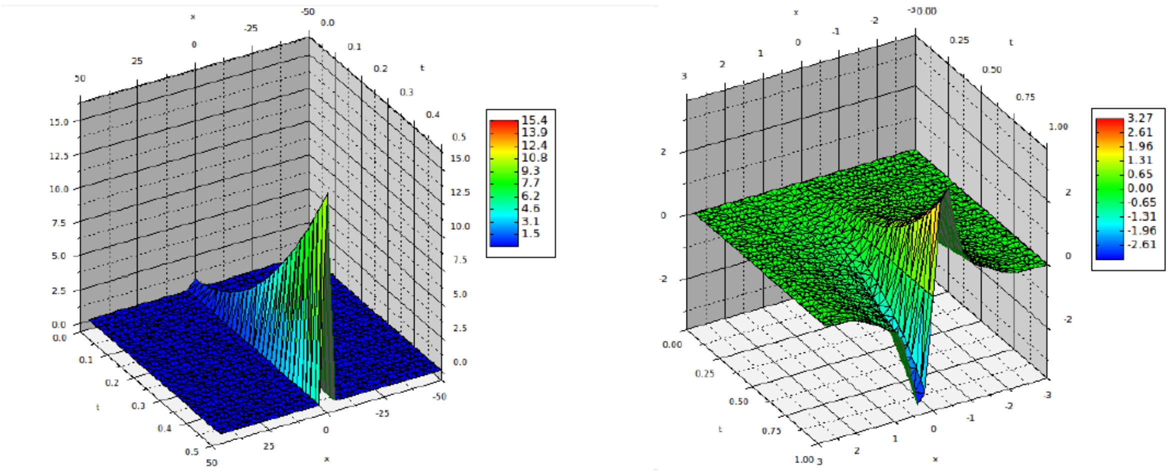

Figure 5: Velocity U(x,t) and height H(x,t) for $\alpha = 1$

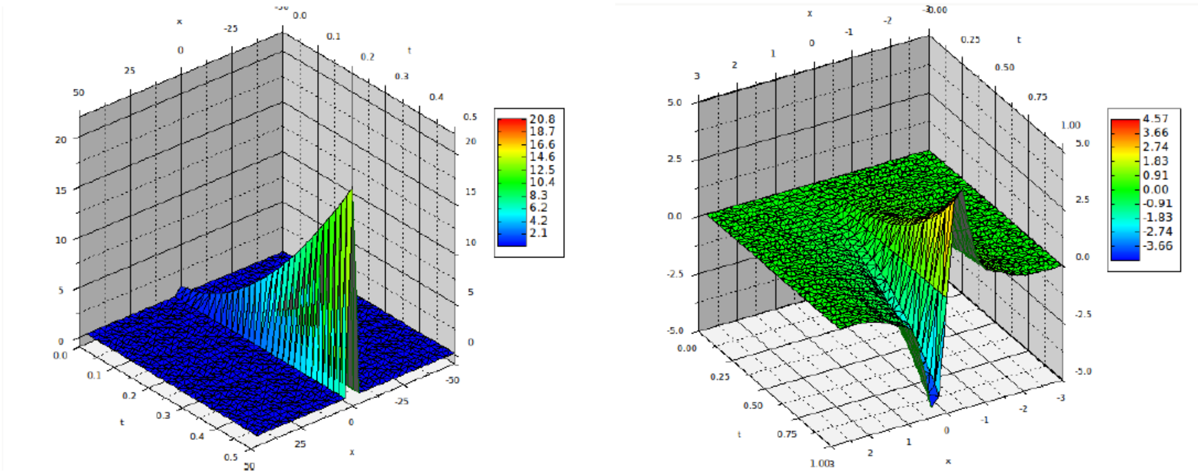

Figure 6: Velocity U(x,t) and height H(x,t) for $\alpha = 0.9$

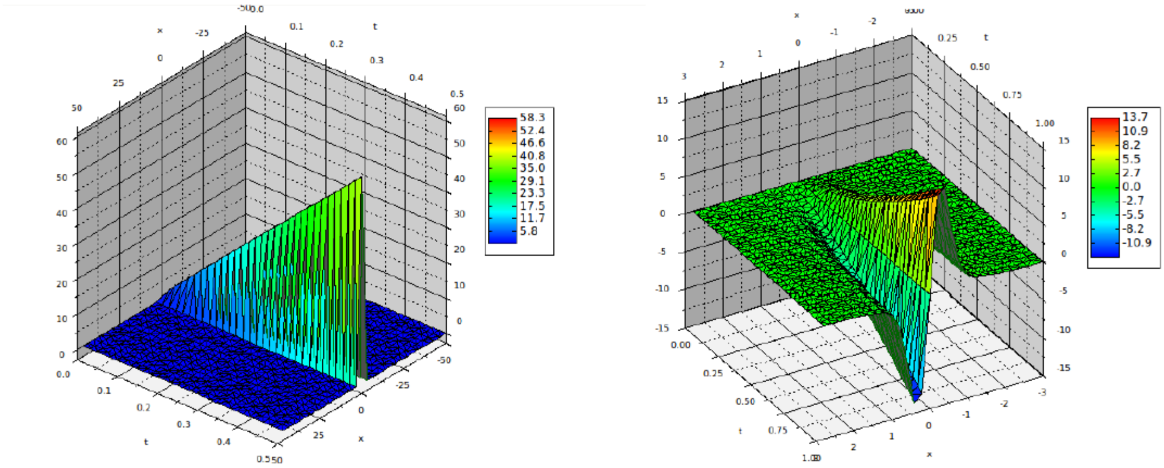

Figure 7: Velocity U(x,t) and height H(x,t) for $\alpha = 0.5$

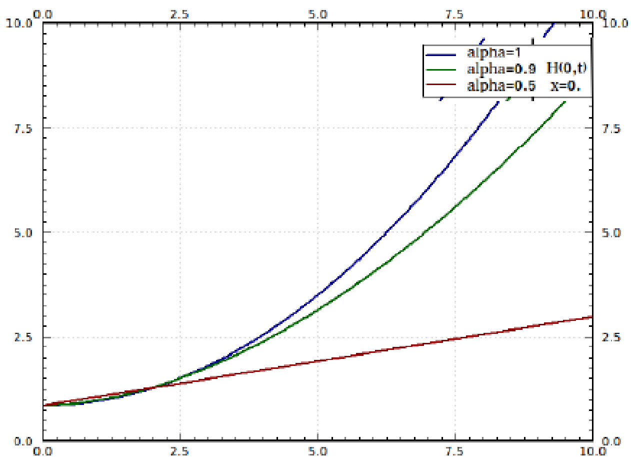

Figure 8: Variation of velocity U(0,t) for x=0 and $\alpha = 1$, $\alpha = 0.9$, $\alpha = 0.5$

4.4. Application 2

For more validation tests, the obtained formulations are also tested in the following fractional Saint-Venant equations.

$$(FSV)_{sw} = \begin{cases} D_t^{\alpha} H(x,t) + (H.U)_x(x,t) = 0 \\ D_t^{\alpha} U + (\frac{1}{2}U^2 + H)_x + Fd_x = 0 \\ U(x,0) = 0,\ H(x,0) = x^p. \end{cases} \quad (78)$$

with the bed topography plate. The previous formulations lead to :

$$(FSV)_0 = \begin{cases} H_0(x,t) = x^p \\ U_0(x,t) = 0 \end{cases} \quad (79)$$

$$(FSV)_1 = \begin{cases} H_1 = -I^{\alpha}(H_0.U_0)_x = 0. \\ U_1 = -I^{\alpha}(\frac{1}{2}.A_0 + H_0)_x = -px^{p-1}\frac{t^{\alpha}}{\Gamma(\alpha+1)} \end{cases} \quad (80)$$

$$(FSV)_2 = \begin{cases} H_2 = -I^{\alpha}(B_1)_x = -I^{\alpha}(H_0.U_1 + H_1.U_0)_x = p(2p-1)x^{2p-2}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ U_2 = -I^{\alpha}(\frac{1}{2}.A_1 + H_1)_x = 0. \end{cases} \quad (81)$$

$$(FSV)_3 = \begin{cases} H_3 = -I^{\alpha}(B_2)_x = 0 \\ U_3 = -I^{\alpha}(\frac{1}{2}.A_2 + H_2)_x \end{cases} \quad (82)$$

$$U_3 = -\left(\frac{p^2(p-1)}{\Gamma^2(\alpha+1)} + \frac{p(2p-1)(2p-2)}{\Gamma(2\alpha+1)}\right) x^{2p-3}\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)} \quad (83)$$

The resulting approximate solution is

$$\begin{cases} H(x,t) = x^p + p(2p-1)x^{2p-2}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ U(x,t) = -px^{p-1}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + -\left(\frac{p^2(p-1)}{\Gamma^2(\alpha+1)} + \frac{p(2p-1)(2p-2)}{\Gamma(2\alpha+1)}\right) x^{2p-3}\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)} \end{cases} \quad (84)$$

Particular case :$\alpha = 1$

$$(FSV)_0 = \begin{cases} H_0(x,t) = x^p \\ U_0(x,t) = 0 \end{cases} \quad (85)$$

$$(FSV)_1 = \begin{cases} H_1 = 0. \\ U_1 = -px^{p-1}t \end{cases} \quad (86)$$

$$(FSV)_2 = \begin{cases} H_2 = -I^{\alpha}(B_1)_x = p(2p-1)x^{2p-2}\frac{t^2}{2} \\ U_2 = 0. \end{cases} \quad (87)$$

$$(FSV)_3 = \begin{cases} H_3 = -I^{\alpha}(B_2)_x = 0 \\ U_3 = -\left(p^2(p-1) + p(2p-1)(p-1)\right) x^{2p-3}\frac{t^3}{3} \end{cases} \quad (88)$$

And the resulting approximate solution is

$$\begin{cases} H(x,t) = x^p + p(2p-1)x^{2p-2}\frac{t^2}{2} \\ U(x,t) = -px^{p-1}t + -\left(p^2(p-1) + p(2p-1)(p-1)\right) x^{2p-3}\frac{t^3}{3} \end{cases} \quad (89)$$

Obviously, these presented solutions can be easly used for parametric analysis of the solution.

5. Coupled Boussinesq Saint-Venant system

For a more general purpose, two types of coupled Boussinesq Saint-Venant systems are considered. Approximate solution of the considered coupled fractional nonlinear PDS are elaborated based on the ADM.

5.1. Coupled Boussinesq Saint-Venant system 1

Let us consider the following coupled Boussinesq Saint-Venant system.

$$(FBSV)_{sw} = \begin{cases} D_t^\alpha(D_t^\alpha U(x,t)) - aU_{xx}(x,t) - b(U^2(x,t))_{xx} - \varepsilon U_{xxxx}(x,t) = f(x,t) \\ D_t^\alpha H(x,t) + ((HU)(x,t))_x = 0 \\ D_t^\alpha((HU)(x,t)) + (HU^2(x,t) + \dfrac{gH^2(x,t)}{2})_x + gH(x,t)(Fd(x))_x = 0 \\ U_t(x,0) = g_1(x),\ U(x,0) = g_2(x),\ H(x,0) = g_3(x) \end{cases} \tag{90}$$

By substitution one gets the following nonlinear fractional partial differential equation.

$$(FBSV)_{sw} = \begin{cases} D_t^\alpha(D_t^\alpha U(x,t)) - aU_{xx}(x,t) - b(U^2(x,t))_{xx} - \varepsilon U_{xxxx}(x,t) = f(x,t) \\ D_t^\alpha D_t^\alpha H(x,t) - (HU^2(x,t) + \dfrac{gH^2(x,t)}{2})_{xx} - g(H(x,t)(Fd(x))_x)_x = 0 \\ U_t(x,0) = g_1(x),\ U(x,0) = g_2(x),\ H(x,0) = g_3(x) \end{cases} \tag{91}$$

where $U = U(x,t)$, $H = H(x,t)$, are the unknowns, that are to be considered the velocity and the height of the wave respectively. f is a source term and g_1, g_2 and g_3 are initial conditions and g is the acceleration due to the gravity. $Fd(x)$ is the shape of the bottom, a, b are model parameters.

It should be noted that an integer derivative Boussinesq Saint-Venant shallow water system is defined in [1, 34] :

$$(BSV)_{sw} = \begin{cases} U_{tt}(x,t) - aU_{xx}(x,t) - b(U^2(x,t))_{xx} - \varepsilon U_{xxxx}(x,t) = f(x,t) \\ H_t(x,t) + (HU)_x(x,t) = 0 \\ (HU)_t(x,t) + ((HU^2)(x,t) + \dfrac{gH^2(x,t)}{2})_x + gH(x,t)(Fd(x))_x = 0 \\ U_t(x,0) = g_1(x),\ U(x,0) = g_2(x),\ H(x,0) = g_3(x) \end{cases} \tag{92}$$

The fractional nonlinear PDE (90) is a general formulation that contains the Boussinesq equation and the effet of the Boussinesq solution on the Saint-Venant height. This nonlinear fractional PDE will be investigated here and the associated analytical solution will be elaborated based on the decomposition method.

5.1.1. FBSV decomposition

This subsection aims to develop the decomposition method for the coupled Boussinesq Saint-Venant nonlinear fractional differential system (91). Similarly, the inverse operator I^α in both parts of the equations (91) is applied :

$$(FBSV)_{sw} = \begin{cases} D_t^\alpha U(x,t) = \Sigma_{k=0}^{m-1} U^{(k)}(x,0^+)\dfrac{t^k}{k!} + I^\alpha(aU_{xx}) + bI^\alpha\left((U^2)_{xx}\right) + I^\alpha(\varepsilon U_{xxxx}) + I^\alpha(f) \\ D_t^\alpha H(x,t) = \Sigma_{k=0}^{m-1} H^{(k)}(x,0^+)\dfrac{t^k}{k!} + I^\alpha(HU^2(x,t) + \dfrac{gH^2(x,t)}{2})_{xx} + gI^\alpha(H(x,t)(Fd(x))_x)_x \end{cases} \tag{93}$$

By reapplying the inverse operator I^α in both parts of the first equation and using integral's propriety one gets:

$$(FBSV)_{sw} = \begin{cases} U(x,t) = \Sigma_{k=0}^{m-1} U^{(k)}(x,0^+)\dfrac{t^k}{k!} + I^{2\alpha}(aU_{xx}) + bI^{2\alpha}\left((U^2)_{xx}\right) + I^{2\alpha}(\varepsilon U_{xxxx}) + I^{2\alpha}(f) \\ H(x,t) = \Sigma_{k=0}^{m-1} H^{(k)}(x,0^+)\dfrac{t^k}{k!} + I^{2\alpha}(HU^2(x,t) + \dfrac{gH^2(x,t)}{2})_{xx} + gI^{2\alpha}(H(x,t)(Fd(x))_x)_x \end{cases} \quad (94)$$

The following decompositions are used:

$$U_t(x,t) = \Sigma_{n=0}^{\infty} U_{t,n}(x,t) \quad (95)$$

$$U(x,t) = \Sigma_{n=0}^{\infty} U_n(x,t) \quad (96)$$

$$H(x,t) = \Sigma_{n=0}^{\infty} H_n(x,t), \quad (97)$$

For nonlinear terms we use

$$U^2 = \Sigma_{n=0}^{\infty} A_n(x,t), \quad (98)$$

$$H^2 = \Sigma_{n=0}^{\infty} B_n(x,t), \quad (99)$$

$$H.U^2 = \Sigma_{n=0}^{\infty} D_n(x,t), \quad (100)$$

The insertion of these series in (94) and using some mathematical developments and identifications, the following sequences are obtained

- $A_1 = 2U_0U_1$
- $A_2 = 2U_0U_2 + U_1^2$
- $A_3 = 2U_0U_3 + 2U_1U_2$
- ...
- $A_n = \Sigma_{i=0}^{n} U_iU_{n-i}$

Sequence of B_n

- $B_0 = H_0^2$
- $B_1 = 2H_0H_1$
- $B_2 = 2H_0H_2 + H_1^2$
- $B_3 = 2H_0H_3 + 2H_1H_2$

- $B_n = \Sigma_{i=0}^{n} H_i H_{n-i}$

Sequence of D_n

- $D_0 = H_0.A_0$
- $D_1 = H_0 A_1 + H_1 A_0$
- $D_2 = H_0 A_2 + H_1 A_1 + H_2 A_0$
- $D_3 = H_0 A_3 + H_1 A_2 + H_2 A_1 + H_3 A_0$
- ...
- $D_n = \Sigma_{i=0}^{n} H_i A_{n-i} = \Sigma_{i=0}^{n} H_i (\Sigma_{k=0}^{n-i} U_k U_{n-i-k})$

We assume that g is constant (gravity) and that Fd(.) does not depend on time.
Using the previous decompositions and inversed equation (94), one gets :

$$(FBSV)_{sw} = \begin{cases} \Sigma_{n=0}^{\infty} U_n = \Sigma_{k=0}^{m-1} U_t^{(k)}(x, 0^+)\frac{t^k}{k!} + I^{2\alpha}(\Sigma_{n=0}^{\infty}(a(U_n)_{xx} + \varepsilon(U_n)_{xxxx})) + bI^{2\alpha}(\Sigma_{n=0}^{\infty}(A_n)_{xx}) + I^{2\alpha}(f) \\ \Sigma_{n=0}^{\infty} H_n = \Sigma_{k=0}^{m-1} H^{(k)}(x, 0^+)\frac{t^k}{k!} + I^{2\alpha}(\Sigma_{n=0}^{\infty}(D_n)_x) + \frac{g}{2} I^{2\alpha}(\Sigma_{n=0}^{\infty}(B_n)_x) + g(Fd(x))_x I^{2\alpha}(\Sigma_{n=0}^{\infty} H_n) \end{cases} \tag{101}$$

The identification of these decompositions leads to:

$$(FBSV)_0 = \begin{cases} U_0 = \Sigma_{k=0}^{m-1} U_t^{(k)}(x, 0^+)\frac{t^k}{k!} \\ H_0 = \Sigma_{k=0}^{m-1} H^{(k)}(x, 0^+)\frac{t^k}{k!} \end{cases} \tag{102}$$

$$(FBSV)_1 = \begin{cases} U_1 = I^{2\alpha}(a(U_0)_{xx} + \varepsilon(U_0)_{xxxx}) + b((A_0)_{xx} + f) \\ H_1 = I^{2\alpha}((D_0)_{xx}) + \frac{g}{2} I^{2\alpha}((B_0)_{xx}) + gI^{2\alpha}(Fd(x)_x H_0)_x \end{cases} \tag{103}$$

The recursivity leads to:

$$(FBSV)_{n+1} = \begin{cases} U_{n+1} = I^{2\alpha}(a(U_n)_{xx} + \varepsilon(U_n)_{xxxx}) + b(A_n)_{xx} + f) \\ H_{n+1} = I^{2\alpha}((D_n)_{xx}) + \frac{g}{2} I^{2\alpha}((B_n)_{xx}) + gI^{2\alpha}(Fd(x)_x H_n)_x \end{cases} \tag{104}$$

This recurrent relation can be rewritten as

$$(FBSV)_{n+1} = \begin{cases} U_{n+1} = I^{2\alpha}(a(U_n)_{xx} + \varepsilon(U_n)_{xxxx}) + b(\Sigma_{i=0}^{n} U_i U_{n-i})_{xx} + f) \\ H_{n+1} = -I^{\alpha}(\Sigma_{i=0}^{n+1} U_i H_{n+1-i})_x \end{cases} \tag{105}$$

The presented recurrent relationships allow one to get analytical solution of the considered coupled nonlinear system. Again, the higher orders can be computed in the presented straithforward maner by specifying the coefficients the source and initial conditions using a symbolic software.

5.1.2. Application

The presented formulations are tested in the following coupled Boussinesq Saint-Venant equations with specified terms and initial conditions.

$$\begin{cases} D_t^\alpha D_t^\alpha(U) - \frac{1}{2}.(U^2)_{xx} - U_{xxxx} = 2x^3 - 15x^4t^4 \\ D_t^\alpha H + (HU)_x = 0 \\ D_t^\alpha(HU) + (H.U^2 + \frac{gH^2}{2})_x + gH(Fd(x))_x = 0 \\ U_t(x,0) = 0,\ U(x,0) = 0,\ H(x,0) = x^p \end{cases} \tag{106}$$

At order 0, the presented formulations lead to :

$$\begin{cases} U_0 = U(x,0^+) + U_t(x,0^+).t + I^{2\alpha}(f) \\ H_0 = x^p \end{cases} \tag{107}$$

Thus,

$$\begin{cases} U_0 = \frac{2}{\Gamma(2\alpha+1)}x^3t^{2\alpha} - \frac{15.4!}{\Gamma(2\alpha+5)}x^4t^{2\alpha+4} \\ H_0 = x^p \end{cases} \tag{108}$$

Compactly, one uses

$$U_0 = a_3x^3t^{2\alpha} + a_0x^4t^{2\alpha+4} \tag{109}$$

with

$$a_0 = \frac{-360}{\Gamma(2\alpha+5)}, a_3 = \frac{2}{\Gamma(2\alpha+1)}$$

U_1 and H_1 are given by:

$$\begin{cases} U_1 = I^{2\alpha}((U_0)_{xxxx}) + \frac{1}{2}.I^{2\alpha}((A_0)_{xx}) \\ H_1 = I^{2\alpha}((H_0U_0^2)_{xx}) + \frac{g}{2}I^{2\alpha}((H_0^2)_{xx}) \end{cases} \tag{110}$$

Thus,

$$\begin{cases} U_1 = \frac{60.\Gamma(4\alpha+1)}{\Gamma(6\alpha+1)\Gamma^2(2\alpha+1)}x^4.t^{6\alpha} + 28.(\frac{15.4!}{\Gamma(2\alpha+5)})^2.\frac{\Gamma(4\alpha+9)}{\Gamma(6\alpha+9)}x^6.t^{6\alpha+8} + \\ \quad - \frac{30240.\Gamma(4\alpha+5)}{\Gamma(2\alpha+1).\Gamma(2\alpha+5)\Gamma(6\alpha+5)}x^5.t^{6\alpha+4} - \frac{8640}{\Gamma(4\alpha+5)}.t^{4\alpha+4} \\ H_1 = a_3^2(6+p)(5+p)\frac{\Gamma(4\alpha+1)}{\Gamma(6\alpha+1)}x^{p+4}.t^{6\alpha} + a_0^2.p(p-1)\frac{\Gamma(5\alpha+9)}{\Gamma(6\alpha+9)}x^{p-2}.t^{6\alpha+8} + \\ \quad 2a_3a_0(p+3)\frac{\Gamma(4\alpha+5)}{\Gamma(6\alpha+4)}x^{p+2}.t^{6\alpha+4} + g.p(2p-1)\frac{1}{\Gamma(2\alpha+1)}x^{2p-2}.t^{2\alpha} \end{cases} \tag{111}$$

Equivalently,

$$U_1 = b_4.x^4t^{6\alpha} + b_6x^6t^{6\alpha+8} + b_5x^5.t^{6\alpha+4} + b_0t^{4\alpha+4} \tag{112}$$

$$H_1 = c_{p+4}x^{p+4}.t^{6\alpha} + c_{p-2}x^{p-2}.t^{6\alpha+8} + c_{p+2}x^{p+2}.t^{6\alpha+4} + c_{2p-2}x^{2p-2}.t^{2\alpha} \tag{113}$$

with

$$\begin{cases} b_0 = \dfrac{-8640}{\Gamma(4\alpha+5)}, \\ b_4 = \dfrac{60\Gamma(4\alpha+1)}{\Gamma(6\alpha+1)\Gamma^2(2\alpha+1)}, \\ b_5 = \dfrac{-30240.\Gamma(4\alpha+5)}{\Gamma(2\alpha+1).\Gamma(2\alpha+5)\Gamma(6\alpha+5)}, \\ b_6 = \dfrac{3628800.\Gamma(4\alpha+9)}{\Gamma^2(2\alpha+5).\Gamma(6\alpha+9)} \end{cases} \tag{114}$$

$$\begin{cases} c_{p+4} = a_3^2(6+p)(5+p)\dfrac{\Gamma(4\alpha+1)}{\Gamma(6\alpha+1)}, \\ c_{p-2} = a_0^2(p)(p-1)\dfrac{\Gamma(5\alpha+9)}{\Gamma(6\alpha+9)}, \\ c_{p+2} = 2a_3a_0(p+3)\dfrac{\Gamma(4\alpha+5)}{\Gamma(6\alpha+4)}, \\ c_{2p-2} = g.(p)(2p-1)\dfrac{1}{\Gamma(2\alpha+1)} \end{cases} \tag{115}$$

Similarly, U_2 and H_2 are given by :

$$\begin{cases} U_2 = I^{2\alpha}((U_1)_{xxxx} + (U_0U_1)_{xx}) \\ H_2 = I^{2\alpha}((2H_0U_0U_1 + H_1U_0^2 + gH_0H_1)_{xx}) \end{cases} \tag{116}$$

$$\begin{aligned} I^{2\alpha}((U_1)_{xxxx}) &= 24.b_4.\frac{\Gamma(6\alpha+1)}{\Gamma(8\alpha+1)}t^{8\alpha} + 360.b_6.\frac{\Gamma(6\alpha+9)}{\Gamma(8\alpha+9)}x^2.t^{8\alpha+8} + 120.b_5\frac{\Gamma(6\alpha+5)}{\Gamma(8\alpha+5)}x.t^{8\alpha+4} \\ I^{2\alpha}((U_0U_1)_{xx}) &= 42.b_4a_3\frac{\Gamma(8\alpha+1)}{\Gamma(10\alpha+1)}x^5t^{10\alpha} + 72.b_6a_3\frac{\Gamma(8\alpha+9)}{\Gamma(10\alpha+9)}x^7.t^{10\alpha+8} + \\ &56.b_5a_3\frac{\Gamma(8\alpha+5)}{\Gamma(10\alpha+5)}x^6.t^{10\alpha+4} + 6.b_0a_3\frac{\Gamma(6\alpha+5)}{\Gamma(8\alpha+5)}x.t^{8\alpha+4} \\ &12.b_4a_0\frac{\Gamma(8\alpha+5)}{\Gamma(10\alpha+5)}x^2.t^{10\alpha+4} + 30.b_6a_0\frac{\Gamma(8\alpha+13)}{\Gamma(10\alpha+13)}x^4.t^{10\alpha+12} + 20.b_5a_0\frac{\Gamma(8\alpha+9)}{\Gamma(10\alpha+9)}x^3.t^{10\alpha+8} \end{aligned} \tag{117}$$

Mathematical developments lead to:

$$\begin{cases} U_2 = 24.b_4.\dfrac{\Gamma(6\alpha+1)}{\Gamma(8\alpha+1)}t^{8\alpha} + 360.b_6.\dfrac{\Gamma(6\alpha+9)}{\Gamma(8\alpha+9)}x^2.t^{8\alpha+8} + 120.b_5\dfrac{\Gamma(6\alpha+5)}{\Gamma(8\alpha+5)}x.t^{8\alpha+4} \\ 42.b_4a_3\dfrac{\Gamma(8\alpha+1)}{\Gamma(10\alpha+1)}x^5t^{10\alpha} + 72.b_6a_3\dfrac{\Gamma(8\alpha+9)}{\Gamma(10\alpha+9)}x^7.t^{10\alpha+8} + \\ 56.b_5a_3\dfrac{\Gamma(8\alpha+5)}{\Gamma(10\alpha+5)}x^6.t^{10\alpha+4} + 6.b_0a_3\dfrac{\Gamma(6\alpha+5)}{\Gamma(8\alpha+5)}x.t^{8\alpha+4} \\ 12.b_4a_0\dfrac{\Gamma(8\alpha+5)}{\Gamma(10\alpha+5)}x^2.t^{10\alpha+4} + 30.b_6a_0\dfrac{\Gamma(8\alpha+13)}{\Gamma(10\alpha+13)}x^4.t^{10\alpha+12} + 20.b_5a_0\dfrac{\Gamma(8\alpha+9)}{\Gamma(10\alpha+9)}x^3.t^{10\alpha+8} \end{cases} \tag{118}$$

And

$$\begin{cases} H_2 = a_3b_4(p+7)(p+6)x^{p+5}\frac{\Gamma(8\alpha+1)}{\Gamma(10\alpha+1)}t^{10\alpha} + a_3b_6(p+9)(p+8)x^{p+7}\frac{\Gamma(8\alpha+9)}{\Gamma(10\alpha+9)}t^{10\alpha+8} \\ \quad + a_3b_5(p+8)(p+7)x^{p+6}\frac{\Gamma(8\alpha+5)}{\Gamma(10\alpha+5)}t^{10\alpha+4} + a_3b_0(p+3)(p+2)\frac{\Gamma(6\alpha+5)}{\Gamma(8\alpha+5)}t^{8\alpha+4} \\ \quad + a_0b_4(p+5)(p+4)x^{p+2}\frac{\Gamma(8\alpha+5)}{\Gamma(10\alpha+5)}t^{10\alpha+4} + a_0b_6(p+6)(p+5)x^{p+4}\frac{\Gamma(8\alpha+13)}{\Gamma(10\alpha+13)}t^{10\alpha+12} \\ \quad + a_0b_5(p+5)(p+4)x^{p+3}\frac{\Gamma(8\alpha+9)}{\Gamma(10\alpha+9)}t^{10\alpha+8} + a_0b_0(p)(p-1)x^{p-2}\frac{\Gamma(6\alpha+9)}{\Gamma(8\alpha+9)}t^{8\alpha+8} \\ \quad a_3^2c_{p+4}(p+10)(p+9)x^{p+8}\frac{\Gamma(10\alpha+1)}{\Gamma(12\alpha+1)}t^{12\alpha} + a_3^2c_{p-2}(p+4)(p+3)x^{p+2}\frac{\Gamma(10\alpha+9)}{\Gamma(12\alpha+9)}t^{12\alpha+8} \\ \quad + a_3^2c_{p+2}(p+8)(p+7)x^{p+6}\frac{\Gamma(10\alpha+5)}{\Gamma(12\alpha+5)}t^{12\alpha+4} + a_3^2c_{2p-2}(2p+4)(2p+3)x^{2p+2}\frac{\Gamma(6\alpha+1)}{\Gamma(8\alpha+1)}t^{8\alpha} \\ \quad + a_0^2c_{p+4}(p+4)(p+3)x^{p+2}\frac{\Gamma(10\alpha+9)}{\Gamma(12\alpha+9)}t^{12\alpha+8} + a_0^2c_{p-2}(p-2)(p-3)x^{p-4}\frac{\Gamma(10\alpha+17)}{\Gamma(12\alpha+17)}t^{12\alpha+16} \\ \quad + a_0^2c_{p+2}(p+2)(p+1)x^{p}\frac{\Gamma(10\alpha+13)}{\Gamma(12\alpha+13)}t^{12\alpha+12} + a_0^2c_{2p-2}(2p-2)(2p-3)x^{2p-4}\frac{\Gamma(6\alpha+9)}{\Gamma(8\alpha+9)}t^{8\alpha+8} \\ \quad + 2a_0a_3c_{p+4}(p+7)(p+6)x^{p+5}\frac{\Gamma(10\alpha+5)}{\Gamma(12\alpha+5)}t^{12\alpha+4} + 2a_0a_3c_{p-4}(p+1)(p)x^{p-1}\frac{\Gamma(10\alpha+13)}{\Gamma(12\alpha+13)}t^{12\alpha+12} \\ \quad + 2a_0a_3c_{p+2}(p+5)(p+4)x^{p+3}\frac{\Gamma(10\alpha+9)}{\Gamma(12\alpha+9)}t^{12\alpha+8} + 2a_0a_3c_{2p-2}(2p+1)(2p)x^{2p-1}\frac{\Gamma(6\alpha+5)}{\Gamma(8\alpha+5)}t^{8\alpha+4} \\ \quad g.c_{p+4}(2p+4)(2p+3)x^{2p+2}\frac{\Gamma(6\alpha+1)}{\Gamma(8\alpha+1)}t^{8\alpha} + g.c_{p-2}(2p-2)(2p-3)x^{2p-4}\frac{\Gamma(6\alpha+9)}{\Gamma(8\alpha+9)}t^{8\alpha+8} \\ \quad + g.c_{p+2}(2p+2)(2p+1)x^{2p}\frac{\Gamma(6\alpha+5)}{\Gamma(8\alpha+5)}t^{8\alpha+4} + g.c_{2p-2}(3p-2)(3p-3)x^{3p-4}\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}t^{4\alpha}) \end{cases} \tag{119}$$

For the particular case, $\alpha = 1$, these formulations lead to the following explicite formula of $U_0,\ U_1$, and U_2.

$$\begin{cases} U_0(x,t) = x^3t^2 - \frac{1}{2}x^4t^6 \\ U_1(x,t) = \frac{-3}{14}t^8 + \frac{1}{2}x^4t^6 - \frac{7}{30}x^5t^{10} + \frac{1}{26}x^6t^{14} \\ U_2(x,t) = \frac{3}{14}t^8 + \frac{3}{52}x^2.t^{16} - \frac{7}{33}x.t^{12} + \frac{7}{30}x^5t^{10} + \frac{2}{221}x^7.t^{18} + \\ \qquad - \frac{14}{195}x^6.t^{14} - \frac{3}{308}x.t^{12} - \frac{3}{182}x^2.t^{14} - \frac{15}{572}x^4.t^{22} + \frac{7}{918}x^3.t^{18} \end{cases} \tag{120}$$

It is stated that similar results than those presented in [35]for the considered uncoupled Boussinesq equation are obtained.

On the otherhand, the following functions H_0,H_1,H_2 are also obtained for $\alpha = 1$.

$$\begin{cases} H_0(x,t) = x^p \\ H_1(x,t) = c_{p+4}x^{p+4}.t^6 + c_{p-2}x^{p-2}.t^{14} + c_{p+2}x^{p+2}.t^{10} + c_{2p-2}x^{2p-2}.t^2 \\ H_2(x,t) = a_3b_4(p+7)(p+6)x^{p+5}\frac{1}{90}t^{10} + a_3b_6(p+9)(p+8)x^{p+7}\frac{1}{18.17}t^{18} \\ \qquad + a_3b_5(p+8)(p+7)x^{p+6}\frac{1}{14.13}t^{14} + a_3b_0(p+3)(p+2)\frac{1}{12.11}t^{12} \\ \qquad + a_0b_4(p+5)(p+4)x^{p+2}\frac{1}{14.13}t^{14} + a_0b_6(p+6)(p+5)x^{p+4}\frac{1}{22.21}t^{22} \\ \qquad + a_0b_5(p+5)(p+4)x^{p+3}\frac{1}{18.17}t^{18} + a_0b_0(p)(p-1)x^{p-2}\frac{1}{16.15}t^{16} \\ \qquad a_3^2c_{p+4}(p+10)(p+9)x^{p+8}\frac{1}{12.11}t^{12} + a_3^2c_{p-2}(p+4)(p+3)x^{p+2}\frac{1}{20.19}t^{20} \\ \qquad + a_3^2c_{p+2}(p+8)(p+7)x^{p+6}\frac{1}{16.15}t^{16} + a_3^2c_{2p-2}(2p+4)(2p+3)x^{2p+2}\frac{1}{56}t^{8} \\ \qquad + a_0^2c_{p+4}(p+4)(p+3)x^{p+2}\frac{1}{20.19}t^{20} + a_0^2c_{p-2}(p-2)(p-3)x^{p-4}\frac{1}{28.27}t^{28} \\ \qquad + a_0^2c_{p+2}(p+2)(p+1)x^{p}\frac{1}{24.23}t^{24} + a_0^2c_{2p-2}(2p-2)(2p-3)x^{2p-4}\frac{1}{16.15}t^{16} \\ \qquad + 2a_0a_3c_{p+4}(p+7)(p+6)x^{p+5}\frac{1}{16.15}t^{16} + 2a_0a_3c_{p-4}(p+1)(p)x^{p-1}\frac{1}{24.23}t^{24} \\ \qquad + 2a_0a_3c_{p+2}(p+5)(p+4)x^{p+3}\frac{1}{20.19}t^{20} + 2a_0a_3c_{2p-2}(2p+1)(2p)x^{2p-1}\frac{1}{12.11}t^{12} \\ \qquad g.c_{p+4}(2p+4)(2p+3)x^{2p+2}\frac{1}{56}t^{8} + g.c_{p-2}(2p-2)(2p-3)x^{2p-4}\frac{1}{16.15}t^{16} \\ \qquad + g.c_{p+2}(2p+2)(2p+1)x^{2p}\frac{1}{12.11}t^{12} + g.c_{2p-2}(3p-2)(3p-3)x^{3p-4}\frac{1}{12}t^{4} \end{cases} \tag{121}$$

Based on these analytical solutions the effect of the fractional parameter α as well as the power coefficient p can be analysed.

5.2. Coupled Boussinesq Saint-Venant system 2

The following coupled Boussinesq Saint-Venant system is considered.

$$(FBSV)_{sw}=\begin{cases} D_t^\alpha(D_t^\alpha U(x,t))-aU_{xx}(x,t)-b(U^2(x,t))_{xx}-\varepsilon U_{xxxx}(x,t)=f(x,t)\\ D_t^\alpha H(x,t)+((HU)(x,t))_x=0\\ D_t^\alpha U+(\frac{1}{2}U^2+gH)_x+gFd_x=0\\ U(x,0)=g_1(x),U_t(x,0)=g_2(x)\ ,U_x(x,0)=g_3(x),\ H(x,0)=g_4(x)\end{cases} \quad (122)$$

Eliminating the nonlinear term $(U^2)_{xx}$ from the first equation and using substitution one gets the following fractional partial differential system.

$$(FBSV)_{sw}=\begin{cases} D_t^\alpha H(x,t)+((HU)(x,t))_x=0\\ D_t^\alpha(D_t^\alpha U(x,t))+2bD_t^\alpha U_x(x,t))+2bg(H+Fd)_{xx}=a.U_{xx}(x,t)+\varepsilon U_{xxxx}(x,t)+f(x,t)\\ U(x,0)=g_1(x),U_t(x,0)=g_2(x)\ ,U_x(x,0)=g_3(x),\ H(x,0)=g_4(x)\end{cases} \quad (123)$$

where U=U(x,t), H= H(x,t), are the unknowns, f is a source term and g_1, g_2 and g_3, g_4 are initial conditions and g is the acceleration due to the gravity. Fd(x) is the shape of the bottom, a, b are model parameters. For $\alpha=1$, this equation is reduced to :

$$(BSV)_{sw}=\begin{cases} H_t(x,t)+(H.U)_x(x,t)=0\\ U_{tt}(x,t))+2bU_{tx}(x,t))+2bg(H+Fd)_{xx}=a.U_{xx}(x,t)+\varepsilon U_{xxxx}(x,t)+f(x,t)\\ U_t(x,0)=g_1(x),U_t(x,0)=g_2(x)\ ,U_x(x,0)=g_3(x),\ H(x,0)=g_4(x)\end{cases} \quad (124)$$

5.2.1. Adomian decomposition

Similarly, the inverse operator I^α in both parts of the equations (123) is applied :

$$(FBSV)_{sw}=\begin{cases} H=\Sigma_{k=0}^{m-1}H^{(k)}(x,0^+).\frac{t^k}{k!}-I^\alpha(HU)_x\\ D_t^\alpha U=U(x,0)+2bU_x(x,0)-2bU_x+I^\alpha(a.U_{xx}+\varepsilon U_{xxxx}-2bgH_{xx}-2bgFd_{xx}+f\end{cases} \quad (125)$$

By reapplying the inverse operator I^α in both parts of the seconde equation and using integral's propriety one gets:

$$(FBSV)_{sw}=\begin{cases} H=\Sigma_{k=0}^{m-1}H^{(k)}(x,0^+).\frac{t^k}{k!}-I^\alpha(HU)_x\\ U=2bU_x(x,0)t+\Sigma_{k=0}^{m-1}U^{(k)}(x,0^+).\frac{t^k}{k!}-2bI^\alpha U_x+I^{2\alpha}\left(a.U_{xx}+\varepsilon U_{xxxx}-2bgH_{xx}-2bgFd_{xx}+f\right)\end{cases} \quad (126)$$

The following decompositions are used :

$$U=\Sigma_{n=0}^{\infty}U_n(x,t), \quad (127)$$

$$H=\Sigma_{n=0}^{\infty}H_n(x,t), \quad (128)$$

$$H.U=\Sigma_{n=0}^{\infty}B_n(x,t), \quad (129)$$

The insertion of these series decomposition in (123) and using some mathematical developments and identifications, the following sequences are resulted

Sequence of B_n

- $B_0 = H_0.U_0$
- $B_1 = H_0U_1 + H_1U_0$
- $B_2 = H_0U_2 + H_1U_1 + H_2U_0$
- $B_3 = H_0U_3 + H_1U_2 + H_2U_1 + H_3U_0$
- ...
- $B_n = \Sigma_{i=0}^{n} H_iU_{n-i}$

Assume that g is constant (gravity) and that Fd(.) does not depend on time, the previous decompositions and inversed equation, lead to :

$$(FBSV)_{sw} = \begin{cases} H_0 = \Sigma_{k=0}^{m-1} H^{(k)}(x, 0^+).\dfrac{t^k}{k!} \\ U_0 = 2bU_x(x, 0) + \Sigma_{k=0}^{m-1} U^{(k)}(x, 0^+).\dfrac{t^k}{k!} \end{cases} \tag{130}$$

$$(FBSV)_{sw} = \begin{cases} H_{n+1} = -I^{\alpha}(B_n)_x \\ U_{n+1} = -2bI^{\alpha}(U_n)_x + I^{2\alpha}\left(a.(U_n)_{xx} + \varepsilon(U_n)_{xxxx} - 2bg(H_n)_{xx} - 2bgFd_{xx} + f\right) \end{cases} \tag{131}$$

5.2.2. Application

$$(CFBSV)_{sw} = \begin{cases} D_t^{\alpha}H + (H.U)_x = 0 \\ D_t^{\alpha}(D_t^{\alpha}U(x,t)) + 2bD_t^{\alpha}U_x(x,t)) + 2b(H + Fd)_{xx} = a.U_{xx}(x,t) + \varepsilon U_{xxxx}(x,t) \\ U(x,0) = 0,\ U_t(x,0) = U_x(x,0) = 0,\ H(x,0) = x^p \end{cases} \tag{132}$$

with the bed topography plat, f=0, g=1.

$$(CFBSV)_0 = \begin{cases} H_0 = x^p \\ U_0 = 0 \end{cases} \tag{133}$$

$$(CFBSV)_1 = \begin{cases} H_1 = -I^{\alpha}(H_0.U_0)_x = 0. \\ U_1 = -2I^{\alpha}(U_0)_x + I^{2\alpha}((U_0)_{xx} + (U_0)_{4x} - 2(H_0)_{xx}) = -2p(p-1)x^{p-2}\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \end{cases} \tag{134}$$

$$(CFBSV)_2 = \begin{cases} H_2 = -I^{\alpha}(B_1)_x = 2p(p-1)(2p-2)x^{2p-3}\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ U_2 = -2bI^{\alpha}(U_1)_x + I^{2\alpha}((aU_1)_{xx} + \varepsilon(U_1)_{4x} - 2b(H_1)_{xx}) \end{cases} \tag{135}$$

with :

$$U_2 = 4bp(p-1)(p-2)x^{p-3}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \tag{136}$$

$$-2ap(p-1)(p-2)(p-3)x^{p-4}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - \quad 2\varepsilon p(p-1)(p-2)(p-3)(p-4)(p-5)x^{p-6}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} \tag{137}$$

(138)

$$(CFBSV)_3 = \begin{cases} H_3 = -I^{\alpha}(B_2)_x \\ U_3 = -2bI^{\alpha}(U_2)_x + I^{2\alpha}(a(U_2)_{xx} + \varepsilon(U_2)_{4x} - 2b(H_2)_{xx}) \end{cases} \tag{139}$$

with :

$$H_3 = -4bp(p-1)(p-2)(2p-3)x^{2p-4})\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + 2ap(p-1)(p-2)(p-3)(2p-4)x^{2p-5})\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} \tag{140}$$

$$+ 2\varepsilon bp(p-1)(p-2)(p-3)(p-4)(p-5)(2p-6)x^{2p-7})\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} \tag{141}$$

(142)

$$U_3 = -8bp(p-1)(p-2)(p-3)x^{p-4})\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} \tag{143}$$

$$+ \{4ap(p-1)(p-2)(p-3)(p-4)x^{p-5}) + \varepsilon 4p(p-1)(p-2)(p-3)(p-4)(p-5)x^{p-7}) \tag{144}$$

$$- 4bp(2p-1)(2p-2)(2p-3)x^{2p-4}))\}\frac{\Gamma(4\alpha+1)t^{5\alpha}}{\Gamma(3\alpha+1)\Gamma(5\alpha+1)} \tag{145}$$

Remark : This model generalizes the model of Saint-Venant by taking into account the factors of Saint-Venant and Boussinesq's one in particular, the shape of the bottom and the previous wave which is not true for Boussinesq when the wave has no initial speed and remains. By taking $a = \varepsilon = f = 0$ and b=1 under the condition that the constant of integration is null we find the second equation of Saint-Venant as follow with $\alpha = 1$

$$U(x,t)_{tt} + 2bU_{tx}(x,t)) + 2bg(H + Fd)_{xx} = a.U_{xx}(x,t) + \varepsilon U_{xxxx}(x,t) \tag{146}$$

$$U(x,t)_{tt} + 2U_{tx}(x,t)) + 2g(H + Fd)_{xx} = 0 \tag{147}$$

$$U_{tt} = 0.U_{xx}(x,t) + 1.(U^2(x,t))_{xx} + 0U_{xxxx}(x,t) = 1.(U^2(x,t))_{xx} \tag{148}$$

By substitution and integration :

$$(U^2(x,t))_{xx} + 2U_{tx}(x,t)) + 2g(H + Fd)_{xx} = 0 \tag{149}$$

$$(U^2(x,t))_x + 2U_t(x,t)) + 2g(H + Fd)_x = 0 \tag{150}$$

For this particular case, the following terms are obtained.

$$(CFBSV)_0 = \begin{cases} H_0 = x^p \\ U_0 = 0 \end{cases} \tag{151}$$

$$(CFBSV)_1 = \begin{cases} H_1 = 0. \\ U_1 = -p(p-1)x^{p-2}t^2 \end{cases} \tag{152}$$

$$(CFBSV)_2 = \begin{cases} H_2 = p(p-1)(2p-2)x^{2p-3}\dfrac{t^3}{3} \\ U_2 = 2p(p-1)(p-2)x^{p-3}\dfrac{t^3}{3} \end{cases} \tag{153}$$

$$(CFBSV)_3 = \begin{cases} H_3 = -p(p-1)(p-2)(2p-3)x^{2p-4}\dfrac{t^4}{6} \\ U_3 = -8p(p-1)(p-2)(p-3)x^{p-4})\dfrac{t^4}{4!} \\ \quad - 4p(2p-1)(2p-2)(2p-3)x^{2p-4})\dfrac{4!.t^5}{3!5!} \end{cases} \tag{154}$$

6. Conclusion

Analytical solutions are investigated for fractional Boussinesq, Saint-Venant and coupled Boussinesq Saint-Venant partial differential equations based on the Adomian decomposition. Existence of the solution and the convergence of the decomosition procedure are also demonstrated. Explicit formulations are developed for each case in general frameworks as well as for several particular exemples.

Based on the presented recurrent relationships, higher orders can be computed by a symbolic software when parameters, excitation and initial conditions are specified. The obtained solutions are explicite and can be used as references for numerical methods. The applicability and effectivens of the elaborated methodological approach are demonstrated and comparaison are made with available results.

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annexes A. Exacte solution of Boussinesq equation

In order to investigate the exact solutions for the Boussinesq equation a general Kudryashov algorithm [36, 37]is used.

The methodological approach is elaborated here and mathematical steps are :

Step 1 :

This traveling wave transformation is defined by : $U(x,t) = U(\xi)$, and $\xi = kx - ct$, with :

$$U(\xi) = \frac{\Sigma_{j=0}^{N} a_j Q(\xi)^j}{\Sigma_{i=0}^{M} b_i Q(\xi)^i}$$

where the coefficients a_j, b_i are constants to be determined and also the positive integer N and M by considering the homogeneous balance between the highest order derivatives and nonlinear term in (153). Using traveling wave Eqs. transform is applied to following partial differential equation.

- $U_{tt} = aU_{xx} + b(U^2)_{xx} + \varepsilon U_{4x}$

- Using the derivative transformation lead to the following nonlinear differential equation :

$$c^2U'' = ak^2U" + bk^2(U^2)'' + \varepsilon k^4U'''' \tag{A.1}$$

- By integration and taking constant of integration null we get :

$$c^2U = ak^2U + bk^2(U^2) + \varepsilon k^4U'' \tag{A.2}$$

$$\varepsilon k^4U'' + bk^2U^2 + (ak^2 - c^2)U = 0. \tag{A.3}$$

Step 2 :

Using the Kudryashov algorithm we put :

$$D(U^p.(\frac{d^qU}{d\xi^q})^s) = (N - M)p + s(N - M + q) \tag{A.4}$$

where N, M are integer to be determined.

Now, balancing the highest order derivative U'' and nonlinear term U^2, we get :

$$2(N - M) = N - M + 2 \Leftrightarrow N = M + 2$$

Setting M = 0, we obtain N = 2. Therefore

$$U(\xi) = \alpha + \beta\mathbf{Q} + \gamma\mathbf{Q}^2 \tag{A.5}$$

where α, β and γ are unknown coefficients and Q is a function of ξ

Step 3 :

Substituting Eq. (155) into Eq. (153), we get a polynomial of **Q**,(k = 0, 1, 2, . . .). Equating the coefficients of this polynomial of the same powers of Q to zero. The following nonlinear algebraic system is resulted.

$$(P) = \begin{cases} 6\gamma\varepsilon k^4 + b\gamma^2k^2 = 0 \\ \varepsilon k^4(2\beta - 10\gamma) + 2b\beta\gamma k^2 = 0 \\ \varepsilon k^4(-3\beta + 4\gamma) + bk^2(2\alpha\gamma + \beta^2) + (ak^2 - c^2) = 0 \\ \varepsilon k^4\beta + 2\beta\alpha + \beta(ak^2 - c^2) = 0 \\ bk^2\alpha^2 + (ak^2 - c^2)\alpha = 0 \end{cases}$$

By simplification we get :

$$(P) = \begin{cases} 6\varepsilon k^2 = -b\gamma \\ \beta = -\gamma \\ \alpha = \dfrac{\gamma}{6} \\ c^2 = ak^2 + \dfrac{bk^2\gamma}{6} \end{cases}$$

by substitution :

$$U(\xi) = \alpha + \beta \mathbf{Q} + \gamma \mathbf{Q}^2 \tag{A.6}$$

$$U(\xi) = \frac{\gamma}{6} - \gamma \mathbf{Q} + \gamma \mathbf{Q}^2 \tag{A.7}$$

and knowing that $\mathbf{Q}(\xi) = \frac{1}{1 \pm A.e^{kx-ct}}$ leads to :

$$U(\xi) = \gamma(\frac{1}{6} - \frac{1}{1 \pm A.e^{kx-ct}} + \frac{1}{(1 \pm A.e^{kx-ct})^2}) \tag{A.8}$$

using

$$sech^2(\frac{\xi}{2}) = \frac{1}{(\frac{e^{\xi}+1}{2e^{\xi/2}})^2}$$

it gets :

$$U(x,t) = \varepsilon \left(c_0 - \frac{2K}{3b} + \frac{K}{b} \sec h^2 (\sqrt{\frac{K}{6}(x-c)t} - \frac{\ln(A)}{2})) \right) \tag{A.9}$$

with :

$$c = \sqrt{2(\frac{a}{2} + \frac{\varepsilon k}{3})}$$

and c_0 is an integration constant.