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Research Paper

Minimum and Maximum Unit Squares inside the Circumscribed Circle

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ABSTRACT: *Through this paper, in a limited number of observations, we will try to find the minimum and a maximum number of unit squares inside the circumscribed (smallest) circle of the given square. In addition to verifying the images for n = 2, 5, 7, 9 presented by Erich Friedman in 1997, we will also try to prove that "the maximum n number of unit squares can be packed inside the smallest circle with a radius of r for n = 3, 4". Also, we will highlight similar problems for solving by mathematicians. By applying concepts from Euclidean geometry, algebra, and the Cartesian coordinate system, we will try to inspire the general reader of mathematics to engage in solving complex packing problems in geometry. Hopefully, new readers, especially juniors and those concerned with solving packing problems, will find a rhythm here that can find new ways to develop the content.*

KEYWORDS: *Square, Circle, Packing, Friedman, Inscribed, Circumscribed.*
Mathematics Subject Classification (2020): *Primary: 52C15, 52C17; Secondary: 05B40, 51E23*

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I. INTRODUCTION

Solving packing problems of various shapes is one of the critical endeavors in geometry. For the best packing of n unit squares in the known smallest circle of radius r has seen a lot of pictures in [1]. There mentioned that out of these pictures, only the cases $n = 1$ and 2 have been proved optimal. An article has found that the given task has been verified through computer-based programming for $n = 3$, see [2]. Many scholarly articles have been found in journals, and web portals regarding the solution to various packing problems, see [3, 4] and watch [5]. With few exceptions, we have no general formula for determining the maximum or a minimum number of small geometric areas of different shapes that can be packed inside another geometric area. Despite having various journal articles on geometric shape packing, ordinary math readers like me find it difficult to understand or lose sight of these elevated texts. As a result, the study of this subject ebbs among them, and the inquisitive mentality disappears. We will try to discuss the subject matter of geometric packing using some tools of general geometry, thinking that it will be easy for a general reader like me, and try to prove the statement described in the abstract.

II. GENERAL RESULTS

2.1 Inscribed square and circumscribed circle: Let us consider a square with the side of $\sqrt{2}r$ and a circle passes through all the vertices of the square. From the definition of the inscribed square of a circle, we know that this inscribed square is the largest in that circle. Also, this circle is called the circumscribed circle of this square, and this is the smallest circle containing the largest square shown in the following figure:

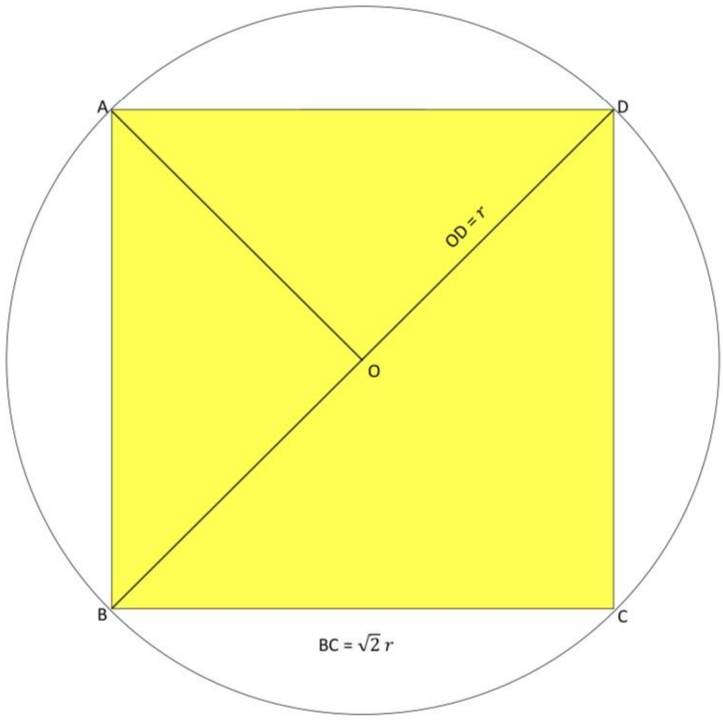


Fig: 1 (Inscribed square and circumscribed circle)

In the above figure, the side length of the square is $\mathrm{AB} = \mathrm{BC} = \mathrm{CD} = \mathrm{AD} = \sqrt{2}r$. Therefore, the radius of the circle is $\mathrm{OA} = \mathrm{OB} = \mathrm{OD} = r$, for inscribed and circumscribed, please see [6].

The circle area is πr^2, and the square area is $\sqrt{2}r \times \sqrt{2}r = 2\,r^2$.

The area of the triangle $\mathrm{AOD} = \frac{1}{4} \times \text{Square Area} = \frac{r^2}{2}$………………………………………… (1)

The area of sector $\mathrm{OAD} = \frac{1}{4} \times \text{Circle Area} = \frac{\pi r^2}{4}$ ……………………………………………… (2)

So, the area between the perimeter of the circle and the line AD

= Sector $\mathrm{OAD} - \Delta\mathrm{AOD} = \frac{\pi r^2}{4} - \frac{r^2}{2}$ [by equation (1) and (2)] …………………………………... (3),

and the remaining area of the circle except square is = Circle – Square

= $4 \times \left(\frac{\pi r^2}{4} - \frac{r^2}{2} \right) = (\,\pi - 2\,)r^2$……….. (4)

The maximum height from the line AD to the perimeter of the circle is

= Radius of the circle – Height of the triangle AOD

$= r - \frac{1}{2} \times \mathrm{AD} = r - \frac{r}{\sqrt{2}} = \left(1 - \frac{1}{\sqrt{2}}\right) r$ [because, $\Delta\mathrm{AOD}$ is an isosceles triangle]………… (5)

2.2 Minimum and maximum unit squares in the circumscribed circle of the square with the side length of an integer:

If the side length of the inscribed square is an integer $t = \sqrt{2}r$, then the radius of the circumscribed (smallest) circle is r. So, intuitively the maximum number of unit squares inside the inscribed square $= m =$ the minimum number of unit squares which can pack inside the circle and $m = \frac{t^2}{1} = t^2$, where $t = 1, 2, 3, \dots \dots$ ……………………………………………………………………………… (6), see [3].

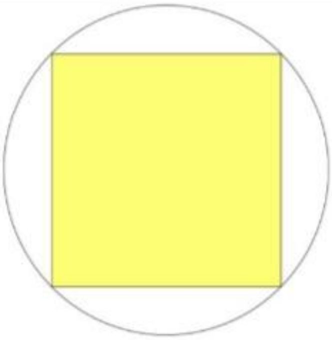
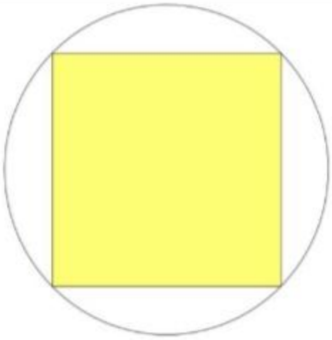
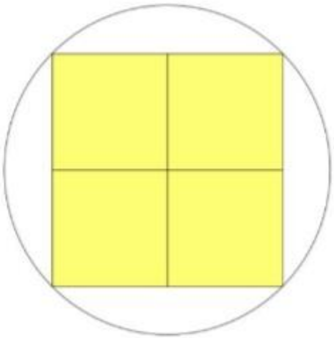
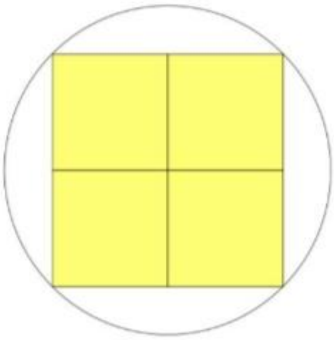
For this case, the maximum number of unit squares in the circumscribed circle is n, where apparently,

$t^2 \leq n \leq [\frac{\pi t^2}{2}]$ for $t = 1, 2, 3, \dots \dots$ .... (7)

It is hard to fix the value of n when t increases. However, only one unit square can be packed inside the circumscribed or smallest circle with a radius of $\frac{1}{\sqrt{2}}$, see also in [1].

Through our article, we can easily find the value of $m \ \forall t \in N$, but to find the value of n, only we can prove it optimally for $t = 1$, and it is $n = 1$. In this article, we will try to find the value of n for $t = 2$ in later.

Table 1 (Minimum and Maximum number of unit squares in smallest circle)

Side length (t)	Minimum unit squares (m)	Maximum unit square (n)
1		
2		

The above table indicates the minimum (m) and maximum (n) number of unit squares in the circumscribed circle of squares with the sides of an integer (t) by trial and error method.

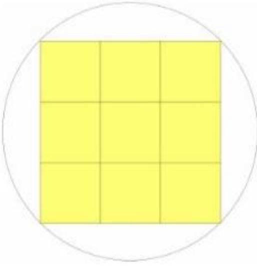
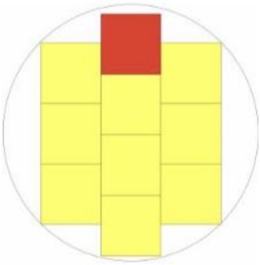
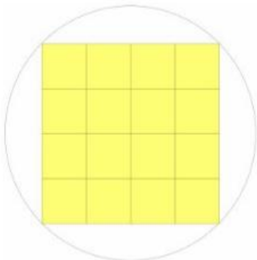
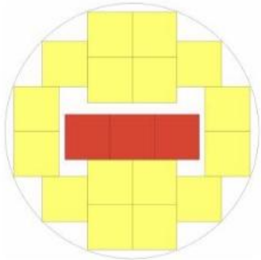
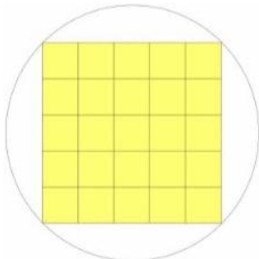
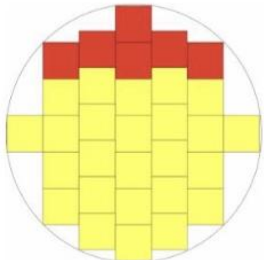
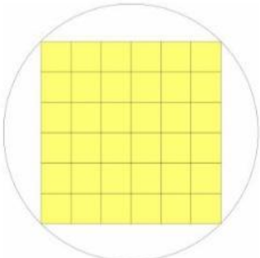
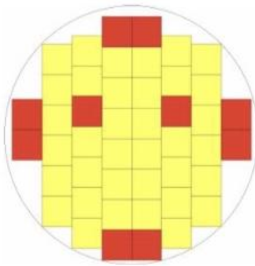
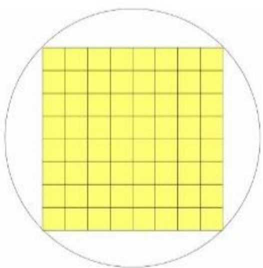
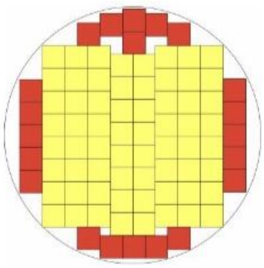
Here one thing is remarkable that the minimum and the maximum number of unit squares inside the circumscribed circle of the respective square of side length = unity (1) is the same, i.e., $m = n = 1$.

For $t = 1$, there is no scope for rearranging this one unit square to find blank space to insert more unit squares into the circumscribed circle. So, it is trivial to prove that only one unit square can be packed inside the circle.

For $t = 2$, this may be a matter of rearranging the four unit squares.

We are waiting for what will be happened if we increase the side length of the square, which will be searched in the tables below:

Table 2 (Minimum and Maximum number of unit squares in smallest circle when side length of inscribed square is increased in integer)

Side length (t)	Minimum unit squares (m)	Maximum unit squares (n)
3		
4		
5		
6		
8		

From tables 1 and 2 we have got an important observation. We see there, the number of maximum unit squares (n) inside the respective circumscribed circles are as follows:

For $t = 1, n = 1^2 = 1$,

For $t = 2, n = 2^2 = 4$,

For $t = 3, n = 3^2 + 1 = 10$,

For $t = 4, n = 4^2 + 1 + 2 = 19$,

For $t = 5, n = 5^2 + 1 + 2 + 3 = 31$,

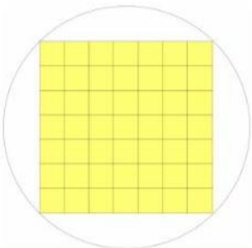
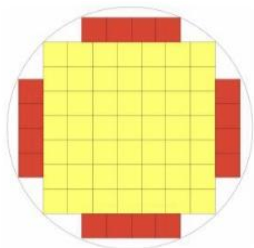
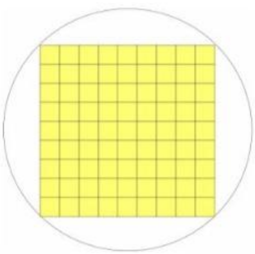
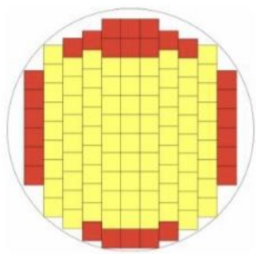
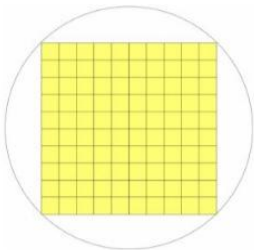
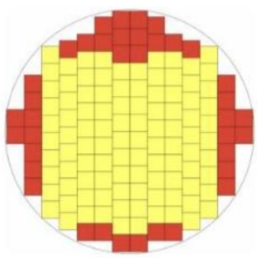
For $t = 6, n = 6^2 + 1 + 2 + 3 + 4 = 46$,

For $t = 8, n = 8^2 + 1 + 2 + 3 + 4 + 5 + 6 = 85$

Therefore, there is a general formula for the side length of the inscribed square $t = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 8$ that the number of maximum non-overlapping unit squares inside the circumscribed circle is $n = t^2 + \frac{(t-2)(t-1)}{2}$ [By inequality (7)], where $t^2 = m =$ the minimum number of unit squares that can be packed inside the respective circles [By equation (6)].

Now we follow the table below outside of the above formula:

Table 3 (Unit squares inside the circle without following the formula in table 2)

Side length (t)	**Minimum unit squares (m)**	**Maximum unit squares (n)**
7		
9		
10		

In table 3, we don't find the characteristics of table 2. We get $n = t^2 + 16 = 65,\ t^2 + 16 + 11 = 108,$ $t^2 + 16 + 11 + 11 = 138$ for $t = 7,\ 9,\ 10$. The trial and error method is completed for the values of t in this case, from 1 to 11.

Here, our paused question to the respected readers of this article is, **"How can be found the number n through a general formula, or is it impossible?"**

2.2.1 Nature and trends of increasing the number of unit squares:

Tables 1, 2, and 3 established that "If no unit square of the inscribed square can move towards the edge of the circle, there is no scope for increasing the number of unit squares inside the circle." From tables 2 and 3, we can say more non-overlapping unit squares can be packed inside the circumscribed circle other than the inserted unit squares into the inscribed square while the side length of the inscribed square is increasing.

2.3 If we take the side of the inscribed square t = 11 units, then the radius of the circumscribed circle is $r = \frac{11}{\sqrt{2}} = 7.78$, and the minimum number of unit squares inside the circle will be $11^2 = 121$, as shown in the figure below:

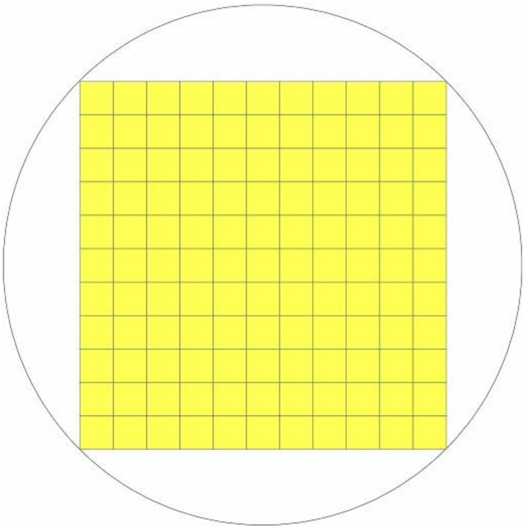


Fig: 2 (Number of unit squares when the side length of the inscribed square is 11 units)

Another problem for solving by the mathematicians has been left in the figure below (**How can we prove optimally that the maximum number of unit squares inside the circumscribed circle with a radius of $\frac{11\sqrt{2}}{2}$ is 169?**):

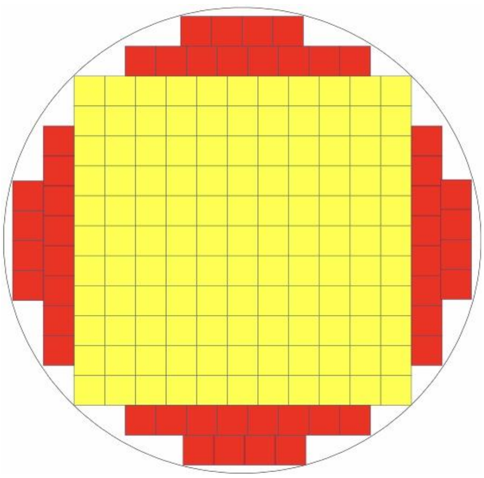


Fig: 3 (Number of unit squares when the radius of the circumscribed circle is $\frac{11\sqrt{2}}{2}$ units)

2.4 Minimum and maximum unit squares in the circumscribed circle with the radius of an integer:

If the radius of the circumscribed (smallest) circle is an integer t, then since the length of the sides of the inscribed square is $\sqrt{2} \times t$, the maximum number of unit squares inside the inscribed square $= m =$ the minimum number of unit squares which can pack inside the circle, where $\frac{([\sqrt{2}\,t])^2}{1} \leq m < \frac{2\,t^2}{1}$ for $t = 1, 2, 3, \dots \dots \dots$… (8) because, the greatest integer which doesn't exceed $\sqrt{2}\,t$ should take as the sum of the length of the side of the total inner unit squares in the blue-colored inscribed square and the suitable integer which is less than $2\,t^2$ should take as the sum of the area of the total inner unit squares in the blue-colored inscribed square, see [4].

In this case, the maximum number of unit squares is n, where apparently, $\frac{2\,t^2}{1} \leq n \leq [\pi\, t^2]$ except for $t = 1$ ………………………………… (9). It is hard to fix the value of both m and n. By trial and error method, we get the following picture (inside the circumscribed circle) with a unit radius containing a minimum of $m = ([\sqrt{2} \times 1\,])^2 = 1$ and a maximum of $n = m = 1$ unit square [By inequality (8) and (9)]:

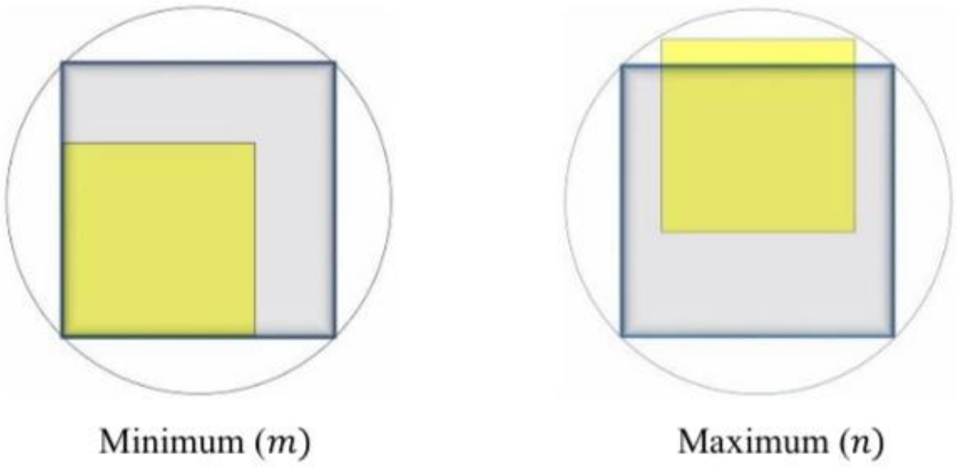


Fig: 4 (Number of unit squares inside a unit circle)

The following picture (inside the circumscribed circle) with a radius of 2 contains a minimum of a suitable integer $m > ([\sqrt{2} \times 2])^2 = 5 < 2\,t^2 = 8$ and a maximum of $n = 2\,t^2 = 8$ unit squares, see [7]:

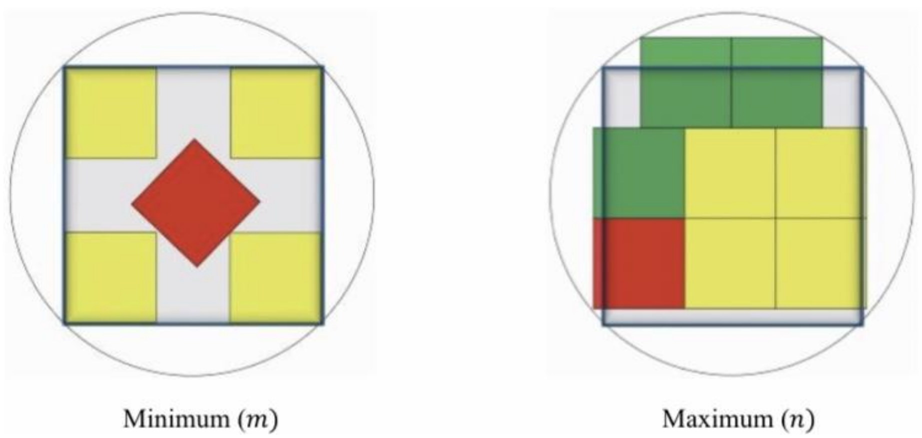
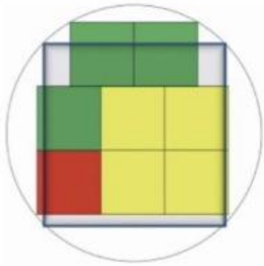
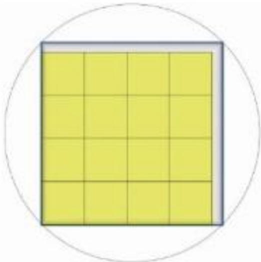
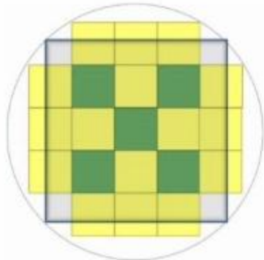
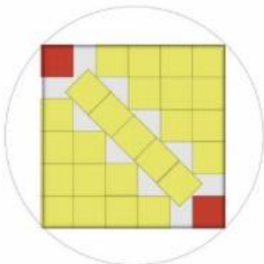
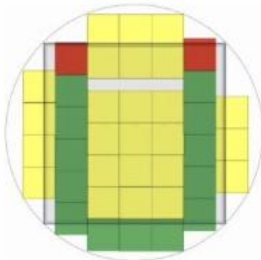
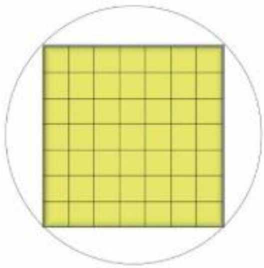
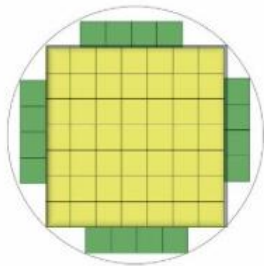


Fig: 5 (Number of unit squares inside the circle of radius 2 units)

Table 4 (Minimum and Maximum number of unit squares in smallest circle when radius of circumscribed circle is increased in integer)

Radius (t)	Minimum unit squares (m)	Maximum unit squares (n)
2		
3		
4		
5		

From figure 4 and table 4 we see there, the number of maximum unit squares (n) inside the respective circumscribed circles are as follows:

For $t = 1, n = 1^2 = 1$,

For $t = 2, n = 2^2 + 4 = 8$,

For $t = 3, n = 3^2 + 4 + 8 = 21$,

For $t = 4, n = 4^2 + 4 + 8 + 12 = 40$,

For $t = 5, n = 5^2 + 4 + 8 + 12 + 16 = 65$

Therefore, there is a general formula for the radius of the circumscribed circle $t = 1,\ 2,\ 3,\ 4,\ 5$ that the number of maximum non-overlapping unit squares inside the circumscribed circle is $n = t^2 + \frac{(t-1)}{2} \times \{2a + (t-2)\ d\} = t^2 + \frac{(t-1)}{2}\{2 \times 4 + (t-2)\ 4\} = 3t^2 - 2t$, see [8].

Also, the number of minimum unit squares (m) inside the circumscribed circles for $t = 2,\ 3,\ 4$ are as follows:

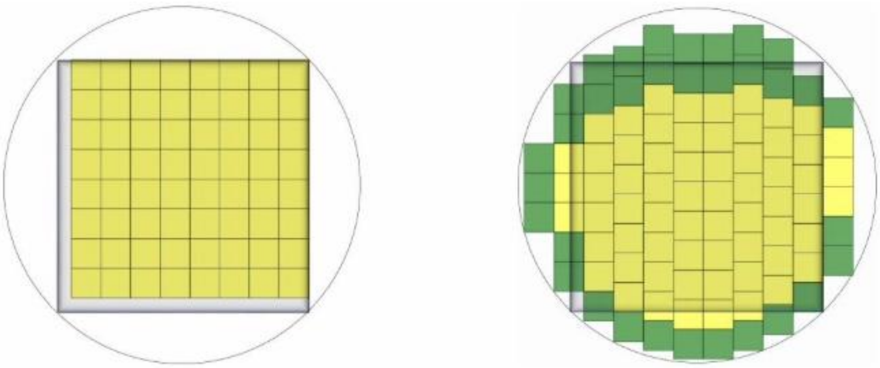
For $t = 2$, $m = 5$,

For $t = 3$, $m = 5 + 11 = 16$,

For $t = 4$, $m = 5 + 11 + 11 = 27$. Therefore, there is a general formula for the radius of the circumscribed circle $t = 2,\ 3,\ 4$ that the number of minimum non-overlapping unit squares inside the circumscribed circle is $m = 5 + (t-2) \times 11$. The trial and error method is completed for the value of t in this case, from 1 to 6.

The question to the readers is, **"Can we establish a general formula for finding the minimum number of unit squares inside the circumscribed circle with the radius of $t = 1,\ 2,\ 3,\ 4,\ 5$?"**

Another problem for solving by the mathematicians has been left in the figure below (**How can we prove optimally that the minimum and the maximum number of unit squares inside the circumscribed circle with a radius of 6 are 64 and 97?**):



Minimum ($m = 64$) Maximum ($n = 97$)

Fig: 6 (Number of Min. and Max. unit squares inside the circle of radius 6 units)

2.5 Minimum and maximum unit squares with a radius of $\sqrt{3}$ multiple:

If the radius of the circumscribed circle (smallest) is t-integer multiple of $\sqrt{3}$, then since the length of the sides of the inscribed square is $\sqrt{2} \times t\sqrt{3} = t\sqrt{6}$, the maximum number of unit squares inside the inscribed square $= m =$ the minimum number of unit squares which can pack inside the circle, where $([\sqrt{6}\,t])^2 \leq \mathrm{m} < 6\,t^2$ for $t = 1, 2, 3, \ldots$ .... because, the greatest integer which doesn't exceed $\sqrt{6}\,t$ should take as the sum of the length of the side of the total inner unit squares in the blue-colored inscribed square and the suitable integer which is less than $6\,t^2$ should take as the sum of the area of the total inner unit squares in the blue-colored inscribed square, see [4]. In this case, the maximum number of unit squares is n, where $6\,t^2 \leq \mathrm{n} \leq [\pi(t\sqrt{3})^2]$ for $t = 1, 2, 3, \ldots$ ....

The following picture (inside the circumscribed circle) with a radius of $\sqrt{3}$ contains a minimum of $m = ([\sqrt{6}\,])^2 = 4$ and a maximum of a suitable integer $n = 6\,t^2 = 6$ unit squares, see [9]:

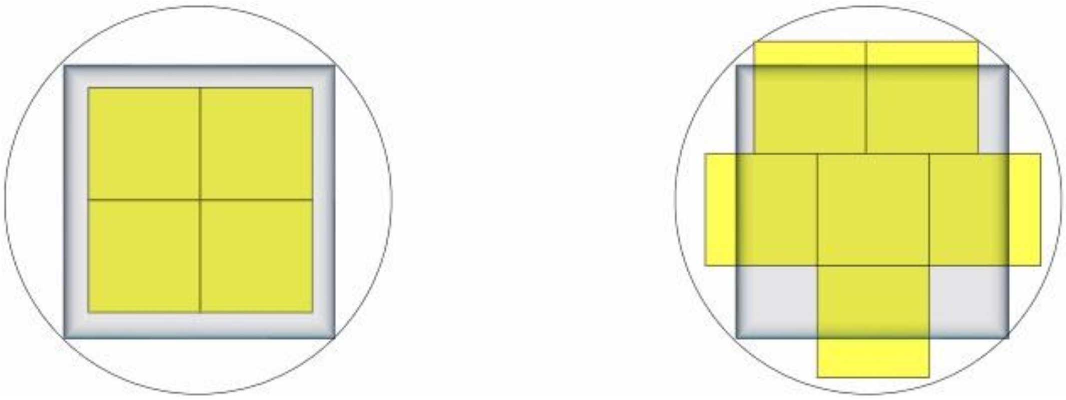


Fig: 7 (Number of Min. and Max. unit squares inside the circle of radius $\sqrt{3}$ units)

2.6 Inequality for the minimum and maximum unit squares when radius $r \in R$:

From the experience of the above subsection 2.1 to 2.5, we may take the following pretty decision:
If the radius of the circumscribed (smallest) circle is $r \in R$, then the minimum number m and the maximum number n of unit squares inside the circumscribed circle follow the inequality $([\sqrt{2}\,r])^2 \leq m \leq [2\,r^2] \leq n \leq [\pi\,r^2]$, where $r \in R$ except for $r = 1$ in the case of n …………......... (10)
Example: Suppose the radius of the circumscribed circle of a square is $r = 1.5$, so for the minimum and the maximum number of unit squares m and n packed inside the circle, the inequality (10) $([\sqrt{2}\,r])^2 \leq m \leq [2\,r^2] \leq n \leq [\pi\,r^2]$ has become $4 \leq m \leq 4 \leq n \leq 7$. Now a minimum of $m = 4$ unit squares and a maximum of $n = 4$ unit squares can pack inside the circumscribed circle, which has shown in the figure below:

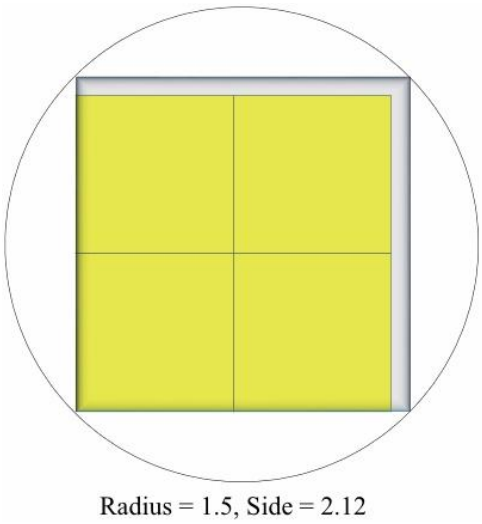


Fig: 8 (Number of Min. and Max. unit squares are equal when radius = 1.5)

III. VERIFICATION OF ERICH FRIEDMAN PICTURES FOR $n = 2,\ 5,\ 7,\ 9$

3.1 For $n = 2$

Let a circumscribed circle of a square with radius r contains two unit squares from [1]. Now we are going to find the radius of the circumscribed circle using a two-dimensional Cartesian coordinate system:

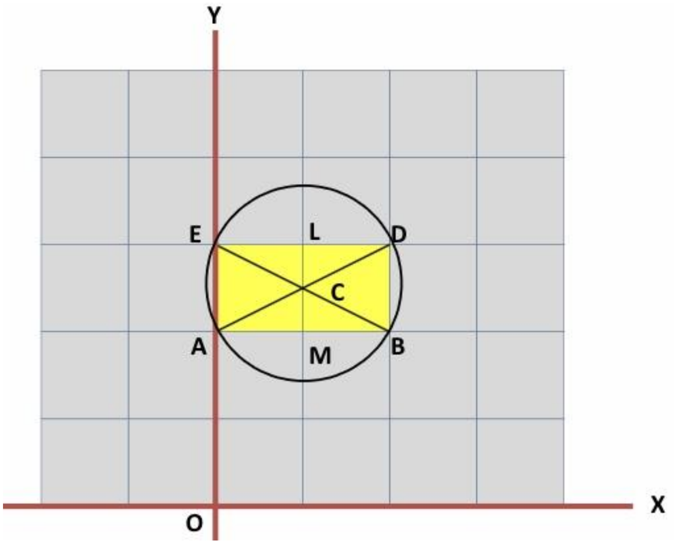


Fig: 9 (Verification of Friedman picture for $n = 2$)

Suppose two yellow-colored unit squares are AMLE and MBDL, and the midpoint of LM is C. Now $\Delta\mathrm{ACE}$ and $\Delta\mathrm{DCB}$ are isosceles congruent triangles, where $\mathrm{CA} = \mathrm{CB} = \mathrm{CD} = \mathrm{CE}$. So, if we draw a circle with radius $\mathrm{CA} = \mathrm{r}$ and center at $\mathrm{C}(1,\ 2.5)$, it will pass through all the four points $\mathrm{A}(0,\ 2)$, $\mathrm{B}(2,\ 2)$, $\mathrm{D}(2,\ 3)$, and $\mathrm{E}(0,\ 3)$. This is the smallest circle where no more unit squares can insert into this circle other than two yellow-colored unit squares.

Therefore, the radius of the smallest circle that contains a maximum of two unit squares is $r = \mathrm{CA} = \sqrt{(1-0)^2 + (2.5-2)^2} = 1.118 +$

3.2 For $n = 5$

Consider the picture from [1] and analyze it with the help of coordinate geometry as given below:

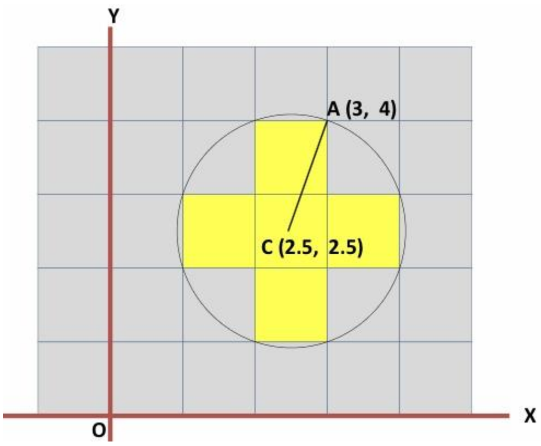


Fig: 10 (Verification of Friedman picture for $n = 5$)

Since the smallest circle passes through eight vertices of yellow-colored unit squares, the center of the circle is at the intersecting point of the diagonals of the central unit square. Let the center $C(2.5,\ 2.5)$ and the coordinate of A be $(3,\ 4)$.

Therefore, the radius of the smallest circle that contains a maximum of five unit squares is $r = CA = \sqrt{(3-2.5)^2 + (4-2.5)^2} = 1.581+$

3.3 For $n = 7$

Now collect the picture from [1] and proceed by Euclidean geometry.

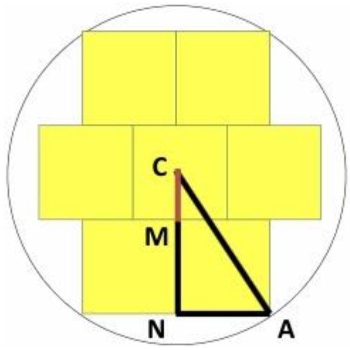


Fig: 11 (Verification of Friedman picture for $n = 7$)

Let the center of the smallest circle be $C \therefore MN = 1$, $CM = 0.5$, radius $CA = r$. From the right-angled triangle CNA, we get $r^2 = (1+0.5)^2 + 1^2 \therefore r = 1.802+$

Therefore, the radius of the smallest circle that contains a maximum of seven unit squares is $r = CA = 1.802+$

3.4 For $n = 9$

Finally, we take the picture from [1] and proceed by Euclidean geometry.

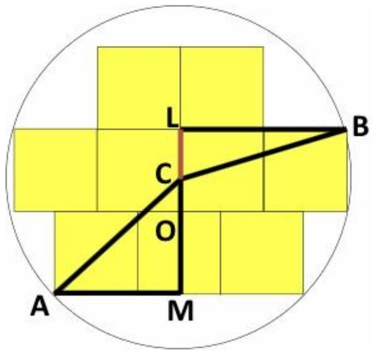


Fig: 12 (Verification of Friedman picture for $n = 9$)

Let the center of the smallest circle be C, $CO = x \ \therefore MC = 1 + x$, $AM = 1.5$, $CL = 1 - x$, and radius $= CA = CB = r$.

From the right-angled triangle AMC and CLB, we get

$r^2 = (1.5)^2 + (1+x)^2$ and $r^2 = 2^2 + (1-x)^2$

$\therefore 2.25 + (1+x)^2 = 4 + (1-x)^2$ Or $4x = 1.75 \ \therefore x = 0.4375$. So, $r = 2.077+$

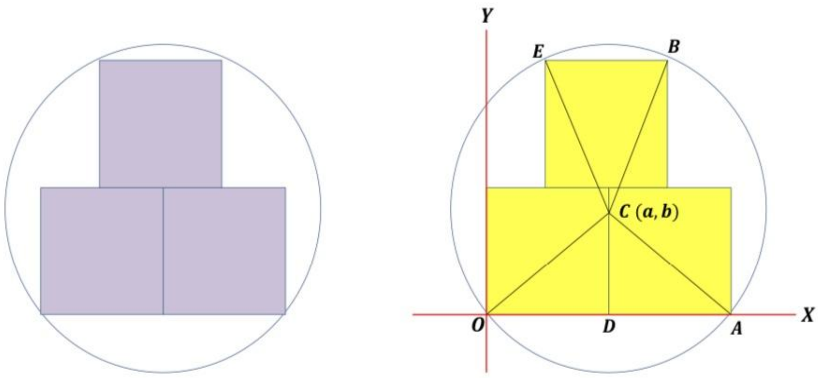
Therefore, the radius of the smallest circle that contains a maximum of nine unit squares is $r = \text{CA} = \text{CB} = 2.077 +$

All subsections 3.1 to 3.4 match the radii in [1].

IV. SPECIAL RESULTS

4.1 The Maximum of three unit squares can be packed inside the smallest circle of radius $r = \frac{5\sqrt{17}}{16} = 1.288+$

To prove the above statement or rely on this statement, we collect the Friedman's picture and try to proceed with two-dimensional Cartesian coordinate geometry in the following way:



Found by Erich Friedman in 1997

Fig: 13 (Proof of Friedman picture for $n = 3$)

In the above figure, we synchronize the left picture with the two-dimensional Cartesian coordinate system in the right. Let the center of the smallest circle of Erich Friedman, where the maximum of 3 unit squares can be packed inside this circle, is at the point $\text{C}(a,\ b)$, and the radius of this circle be $\text{CO} = \text{CA} = \text{CB} = \text{CE} = r$. The coordinate of the points O, D, A, B, and E are $(0,\ 0), (1,\ 0), (2,\ 0)$, $(\frac{3}{2},\ 2)$, and $(\frac{1}{2},\ 2)$, respectively. Here, OA is a chord of the circle. Since the perpendicular drawn from the center of a circle divides the chord into two equal parts, the circle's center lies on the straight line $x = 1$. So, the abscissa of the center C is $a = \text{OD} = 1$.

Again, $\text{CE} = \text{CO}$

Or $\sqrt{(a - \frac{1}{2})^2 + (b-2)^2} = \sqrt{(a-0)^2 + (b-0)^2}$

Or $\sqrt{(1 - \frac{1}{2})^2 + (b-2)^2} = \sqrt{(1-0)^2 + (b-0)^2}$

$\therefore$ the ordinate of the center C is $b = \frac{13}{16}$.

Therefore, the radius of the smallest circle is

$r = \text{CO} = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{(1-0)^2 + (\frac{13}{16} - 0)^2} = \frac{5\sqrt{17}}{16}$

Or $r = \text{CA} = \sqrt{(a-2)^2 + (b-0)^2} = \sqrt{(1-2)^2 + (\frac{13}{16} - 0)^2} = \frac{5\sqrt{17}}{16}$

Or $r = \text{CB} = \sqrt{(a - \frac{3}{2})^2 + (b-2)^2} = \sqrt{(1 - \frac{3}{2})^2 + (\frac{13}{16} - 2)^2} = \frac{5\sqrt{17}}{16}$

Or $r = \text{CE} = \sqrt{(a-\frac{1}{2})^2 + (b-2)^2} = \sqrt{(1-\frac{1}{2})^2 + (\frac{13}{16}-2)^2} = \frac{5\sqrt{17}}{16}$, which all prove the radius of Erich Friedman.

Conversely, we analyze the figure below:

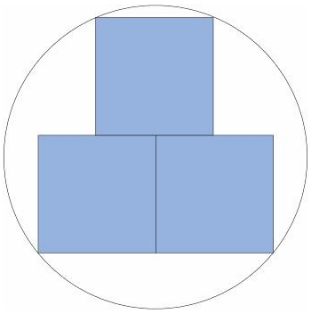


Fig: 14

Here, in the picture of Erich Friedman, the radius of the smallest circle is $r = \frac{5\sqrt{17}}{16} = 1.288$. Now we want to find the maximum number n of the unit squares that can pack inside the smallest circle. For the minimum and the maximum number of the unit squares m and n packed inside the circle, we have the inequality (10), as $([\sqrt{2}\,r])^2 \leq m \leq [2\,r^2] \leq n \leq [\pi\,r^2]$. For $r = 1.288$, we get $1 \leq m \leq 3 \leq n \leq 5$. For the value of n, inequality becomes $3 \leq n \leq 5$.
If possible, suppose 4 unit squares can fit inside the circle. In that case, the minimum radius of the circle will be $\sqrt{2} = 1.414$, which contradicts the existing radius of $r = 1.288$, see optimal packing [10]. So, $n \neq 4$, and accordingly $n \neq 5$. Therefore, a maximum of $n = 3$ unit squares can be packed inside the circle.

4.2 The Maximum of four unit squares can be packed inside the smallest circle of radius $r = \sqrt{2} = 1.414+$

Now we turn our attention toward the proof of the picture of Friedman in the case of $n = 4$.
For proving the above statement or relying on this statement, we collect the Friedman's picture and try to proceed with Euclidean geometry in the following way:
Since the achieved formulae (1) – (5) in subsection 2.1 are general, we can use these for any arbitrary value of r. Let the radius of the smallest circle containing the largest square is $r = \sqrt{2}$. So, the length of the sides of this largest square contained in the smallest circle is $\sqrt{2}r = 2$, which is an integer, see [3].
Therefore, the area of the largest square (orange-colored) inside the circle is $2^2 = 4$, and the minimum number of unit squares inside the circle is $\frac{4}{1} = 4$ [By equation (6)]. The figure, in that case, is given below:

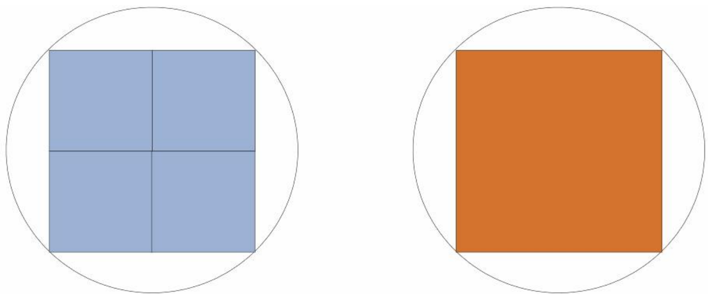


Fig: 15

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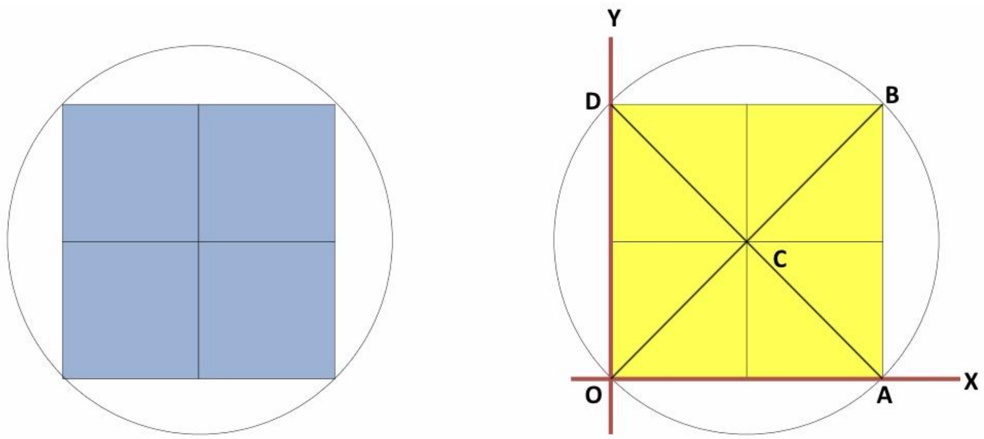
In figure 15, we have found four equally shaped unpacked regions at the edges of the circle. Now we will have to check whether these regions can be made capable of containing more unit squares? Since the inscribed square filled with four unit squares is the greatest, we can't create any blank space between the inscribed square and these four unit squares by rearranging these unit squares, see sub-sub section 2.2.1. The area of every four unpacked regions at the edges of the circle is $\frac{\pi r^2}{4} - \frac{r^2}{2} = 0.571$ sq. unit [From the formula (3)] which does not cover a single unit square. Also, the maximum height from the side of the square towards the edge of the circle is $\left(1 - \frac{1}{\sqrt{2}}\right) r = 0.414$ unit [From the formula (5)] which is less than 1, and it indicates that no unit square can fit in each unpacked region. Again, these four unpacked or discrete regions can't connect altogether, so more unit squares can't insert there. Therefore, although there are a total of $(\pi - 2)r^2 = 2.283$ sq. units [From the formula (4)] area remains unused for packing nonetheless, can't insert more non-overlapping unit squares in these regions because these regions are scattered. Also, by the formula in subsection 2.2, the maximum number of unit squares that can be packed inside the circumscribed circle is $n = t^2 + \frac{(t-2)(t-1)}{2} = 2^2 + \frac{(2-2)(2-1)}{2} = 4$. At the end of the analysis, we can say that the maximum number of unit squares inside the circle is 4, which satisfies $4 \leq n \leq 6$, with the help of the inequality (10).

Conversely, we proceed with the two-dimensional Cartesian coordinate system in the following figure:



Found by Erich Friedman in 1997

Fig: 16 (Proof of Friedman picture for $n = 4$)

In the above figure, let the center of the smallest circle of Erich Friedman, where the maximum of 4 unit squares can be packed inside this circle, at the point $C(a, b)$, and the radius of this circle be $CO = CA = CB = CD = r$. The coordinate of the points O, A, B, and D are $(0, 0)$, $(2, 0)$, $(2, 2)$, and $(0, 2)$, respectively. Here, since C is the midpoint of both diagonals, the coordinate of the center is $C(1, 1)$.

Therefore, the radius of the smallest circle is $r = CO = CA = CB = CD = \sqrt{2}$, which all prove the radius of Erich Friedman.

Therefore, it is clear that the maximum number of unit squares is 4 that can be packed inside the respective circle.

4.3 Table for the values of m and n:

The minimum and the maximum number of unit squares inside the circumscribed circle of the square for some particular cases are given below in table 5:

Table 5 (Numerical information for Min. and Max. unit squares inside the circle)

Radius of the circumscribed circle (r)	Side length of the inscribed square (a)	Minimum number of unit squares inside the circumscribed circle (m)	Maximum number of unit squares inside the circumscribed circle (n)
0.707	1	1	1
1	1.414	1	1
1.414	2	4	4
1.5	2.121	4	4
1.732	2.449	4	6
2	2.828	5	8
2.121	3	9	10
2.828	4	16	19
3	4.243	16	21
3.536	5	25	31
4	5.657	27	40
4.243	6	36	46
4.950	7	49	65
5	7.071	49	65
5.657	8	64	85
6	8.485	64	97
6.364	9	81	108
7.071	10	100	138
7.778	11	121	169

V. DISCUSSION

Unit circle contains only one unit square, and the amount of wastage area is $\pi - 1 = 2.142$ sq. units, see fig: 4. Again, the smallest circle with the radius of $\frac{1}{\sqrt{2}} = 0.707$ also contains a single unit square, where the amount of wastage area is $\frac{\pi}{2} - 1 = 0.571$ sq. unit, see equation (7). Mathematically and logically, both cases are okay, but for the issue of best packing, the second case is more acceptable. Therefore although the described pictures in this article are true against the radius, some radii may decrease, which is marked, in table 5.

General formulae for finding the minimum and a maximum number of unit squares m and n in the circumscribed circle are applicable for the limited radius.

With some exceptions, the lower bound for packing unit squares in the circle is $m \geq ([\sqrt{2}\, r])^2$, and the upper bound for packing unit squares in the circle is $n < [\pi\, r^2]$, where r is the radius of the circumscribed circle, similar [11].

In a few cases, $m = n$ occurs.

To find the maximum number of unit squares inside the circle associated with the smallest radius through Erich Friedman and other pictures in [1] is more acceptable than our described pictures in this article. The advantage of our demonstrated pictures is that these are related to the following two matters simultaneously: a) It helps to find the maximum number of unit squares in a particular square, and b) It helps to find the maximum number of unit squares that can be packed inside the smallest circle. The values m and n indicate the maximum number of unit squares that can be packed into the largest inscribed square and the smallest circumscribed circle, respectively.

VI. CONCLUSION

Famous mathematicians Erich Friedman, David W. Cantrell, Maurizio Morandi, DC after MM, and David Eppstein found that "n unit squares can pack inside the smallest known circle (of radius r)." All these pictures except for n = 1, 2, and 3 are still mathematically unproven, see [1]. Using simple geometry, algebra, and logic, we tried to prove the given picture for n = 3, 4, and we have mentioned some clues regarding the packing unit squares inside the circle for a newcomer like me to this topic. We don't claim that this is complete, rigorous, or optimal work for finding the value of n, but may a talented math reader using these clues could move forward to find proofs for packing geometrical shapes that are our satisfaction.

ACKNOWLEDGMENTS

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