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Research Paper

Wiener Indices of Generalized HYPERCUBES

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Abstract: Given $m \geq 2$, the *generalized hypercube*, Q_n^m, is defined recursively by $Q_1^m = K_m$ and $Q_n^m = Q_{n-1}^m \times K_m$, where K_m is the complete graph of order m. The *Wiener Index* of a graph, G, denoted $W(G)$, is the sum of the distances between all pairs of vertices in G. We calculate the Wiener Index of the generalized hypercube. Wiener Indices play an important role in chemical graph theory.

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I. INTRODUCTION

We use the notation of [1]. V and E denote the vertex set and edge set, respectively, of graph, G. The *Wiener Index* of graph G, $W(G)$, is the sum of the distances between all pairs of vertices in G. The *average distance* between two vertices in G, $AD(G)$, is given by

$$AD(G) = \frac{W(G)}{\binom{n}{2}} = \frac{2W(G)}{n(n-1)}$$

where n is the order of G.

Examples: For the path, P_n, on n vertices, $W(P_n) = \binom{n+1}{3}$ and $AD(P_n) = \frac{n+1}{3}$. For the star, S_n, on n vertices, $W(S_n) = (n-1)^2$ and $AD(S_n) = \frac{2(n-1)}{n}$.

II. Main Result

The *generalized hypercube*, Q_n^m, is defined recursively by:

1) $Q_1^m = K_m$

2) $Q_n^m = Q_{n-1}^m \times K_m$

where K_m is the complete graph of order m. Note that Q_n^m is $n(m-1)$-regular. When m = 2, we have the traditional hypercube, an important structure in the study of massively parallel computing. It is the subject of much research. See [2-7].

Lemma: Given the generalized hypercube, Q_n^m, the following hold:

(1) $|V| = m^n$

(2) $|E| = \dfrac{n(m-1)|V|}{2}$

Proof: (1) When constructing Q_n^m, with each recursion the number of vertices increases by a factor of m. (2) Q_n^m is $n(m-1)$-regular, so $n(m-1)|V|$ gives the degree sum of the graph.

Theorem: Given the generalized hypercube Q_n^m, we have

1) $$W(Q_n^m) = \frac{|V||E|}{m}$$

2) $$AD(Q_n^m) = \frac{2|E|}{m(|V|-1)}$$

Proof: The standard vertex addressing system for Q_n^m uses vectors, the entries of which are numbers between 1 and m. The distance between two vertices equals the number of corresponding entries that differ. For example, the distance between vertices with addresses 0102 and 1100 in Q_n^m is 2. Since each slot in a vertex address can have m different digits, a slot in a vertex address can be changed to $m-1$ different digits. Hence, if we change k slots, there are $(m-1)^k$ ways in which the new address can differ from the original address.

Since there are $\binom{n}{k}$ different ways of choosing k slots, we have $\binom{n}{k}(m-1)^k$ new addresses that can be constructed. Then $f(n) = \sum_{k=1}^{n}\binom{n}{k}k(m-1)^k$ is the sum of all distances from an arbitrarily selected vertex v to all the other vertices in Q_n^m. Differentiating both sides of $(1+x)^n = 1+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{k}x^k+\ldots+\binom{n}{n}x^n$ yields

$$n(1+x)^{n-1} = \binom{n}{1}+\binom{n}{2}2x+\ldots+\binom{n}{k}kx^{k-1}+\ldots+\binom{n}{n}nx^{n-1}$$

Putting $x = m-1$ and multiplying both sides by $m-1$, we have:

$$(m-1)nm^{n-1} = \binom{n}{1}(m-1)+\binom{n}{2}2(m-1)^2+\ldots+\binom{n}{k}k(m-1)^k+\ldots+\binom{n}{n}n(m-1)^n = f(n)$$

Combining the above with $|V| = m^n$ and $|E| = \dfrac{n(m-1)|V|}{2}$ yields:

$$f(n) = (m-1)nm^{n-1} = \frac{n(m-1)m^n}{m} = \frac{n(m-1)|V|}{m} = \frac{2|E|}{m}$$

Since $W(Q_n^m) = \dfrac{f(n)|V|}{2}$, we have $W(Q_n^m) = \dfrac{|V||E|}{m}$. Since $AD(Q_n^m) = \dfrac{W(Q_n^m)}{\binom{|V|}{2}}$, we have

$$AD(Q_n^m) = \frac{2|E|}{m(|V|-1)}.$$

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