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Research Paper

Locally Attractivity Result for Second Order Nonlinear Functional Differential Equation

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ABSTRACT: *In this paper, we discuss the existence result for SecondOrder nonlinear functional differential equationin $\mathcal{R}_+$ by usingfixed point theorem due to Krisnoselkii's. Also locally attractivity results are obtained.*
KEYWORDS:*Banach algebras, hybridfixed point theorem, functional differential equation, existence result, locally attractive solution.*

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I. INTRODUCTION

Differential and integral equations are one of the most useful Mathematical tools in both applied and pure Mathematics. Moreover the theory of Differential and Integral equations is rapidly developing using the tools of Topology, Functional Analysis and Fixed point theory. This is particularly true for problems in the related fields of Engineering, Mechanical Vibrations and Mathematical Physics. There are numerous applications of differential and integral equations of integer and fractional orders in Electrochemistry, Viscoelasticity, Control theory, Electromagnetism and Porous media etc. [5-16, 20-24,32]
To study the existence the solution of second order nonlinear functional differential equation,we obtain the result by using fixed point theorem for two operators in Banach space .
We consider the following second order nonlinear functional differential equations:

$$\left.\begin{aligned}\mathcal{D}^2\left[x(t)-f\left(t,x\big(\theta_1(t)\big)\right)\right]=g\big[t,x\big(\theta_2(t)\big)\big], \qquad t\in\mathcal{R}_+\\ x(0)=0\end{aligned}\right\} \qquad (2.1.1)$$

Where, $f(t,x):\mathcal{R}_+\times\mathcal{R}\to\mathcal{R}-\{0\}$, $g(t,x):\mathcal{R}_+\times\mathcal{R}\to\mathcal{R}$ and $\theta_1,\theta_2:\mathcal{R}_+\to\mathcal{R}$
Here the solution of nonlinear differential equations (2.1.1) we mean a function $x\in BC(\mathcal{R}_+,\mathcal{R})$ such that:

(i) The function $t\to\left[\frac{x(t)}{f\left(t,x(\theta_1(t))\right)}\right]$ is bounded and continuous for each $x\in\mathcal{R}$.

(ii) x satisfies (2.1.1)

II. PRELIMINARIES

In this section we collectthe definitions, notation, hypothesis and preliminary tools
Let $X=BC(\mathcal{R}_+,\mathcal{R})$ be the space of bounded real valued continuous function on $\mathcal{R}_+$ and S be a subset of X. Let a mapping $\mathcal{A}:X\to X$ be an operator and consider the following operator equation in X, namely,
$x(t)=(\mathcal{A}x)(t)$, for all $t\in\mathcal{R}_+$(2.2.1)

Definition 2.2.1[31]: Let (X, d) be the metric space and $a\in X$ and for some real number $r>0$ the set $B_r[a]=\{x\in X: d(x,a)\le r\}$ is called closed ball centered at a with radius r.

Definition 2.2.2[22]: We say that solution of the equation (2.2.1) is locally attractive if there exists a closed ball $B_r[0]$ in the space $BC(\mathcal{R}_+,\mathcal{R})$for some $x_0\in BC(\mathcal{R}_+,\mathcal{R})$and for some real number $r>0$ such that for arbitrary solution $x=x(t)$ and $y=y(t)$ of equation (2.2.1) belonging to $B_r[0]\cap S$ we have that, $\lim_{t\to\infty}\big(x(t)-yt=0$(2.2.2)

Definition 2.2.3[22]: Let X be a Banach space. A mapping $\mathcal{A}:X\to X$ is called Lipschitz if there is a constant $\alpha>0$ such that,$\|\mathcal{A}x-\mathcal{A}y\|\le\alpha\|x-y\|$for all $x,y\in X$. If $\alpha<1$, then $\mathcal{A}$ is called a contraction on X with the contraction constant α.

Definition 2.2.4[18]:.An operator $\mathcal{U}$ on a Banach space X into itself is called compact if for any bounded subset $S$of X, $\mathcal{U}(S)$ is relatively compact subset of$X$. If $\mathcal{U}$ is continuous and compact, then it is called completely continuous on X.

Definition 2.2.5[18]: Let X be a Banach space with the norm $\|\cdot\|$ and let $\mathcal{U}: X \to X$ be an operator (in general nonlinear). Then $\mathcal{U}$ is called

i. Compact if $\mathcal{U}(X)$ is relatively compact subset of X.
ii. Totally bounded if $\mathcal{U}(S)$ is totally bounded subset of X for any bounded subset S of X.
iii. Completely continuous if it is continuous and totally bounded operator on X

Definition 2.2.6[21]: Let $f \in \mathcal{L}^1[0,\mathcal{T}]$and $\alpha > 0$. The Riemann – Liouville fractional derivative of order ζ of real function f is defined as

$$\mathcal{D}^{\zeta} f(t) = \frac{1}{\Gamma(1-\zeta)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^{\zeta}} ds \quad , \qquad 0 < \zeta < 1$$

Such that $\mathcal{D}^{-\zeta} f(t) = I^{\zeta} f(t) = \frac{1}{\Gamma(\zeta)} \int_0^t \frac{f(s)}{(t-s)^{1-\zeta}} ds$ respectively.

Definition 2.2.6.1 [21]: The Riemann-Liouville fractional integral of order $\zeta \in (0,1)$ of the function $f \in \mathcal{L}^1[0,\mathcal{T}]$ is defined by the formula:

$$I^{\zeta} f(t) = \frac{1}{\Gamma(\zeta)} \int_0^t \frac{f(s)}{(t-s)^{1-\zeta}} ds\,, \quad t \in [0,\mathrm{T}]$$

Where $\Gamma(\zeta)$ denote the Euler gamma function. The Riemann-Liouville fractional derivative operator of order ζ defined by

$$\mathcal{D}^{\zeta} = \frac{d^{\zeta}}{dt^{\zeta}} = \frac{d}{dt} \circ I^{1-\zeta}$$

Theorem 2.2.1 [6] :(Arzela-Ascoli Theorem) If every uniformly bounded and equicontinuous sequence $\{f_n\}$ of functions in $\mathcal{C}(\mathcal{R}_+,\mathcal{R})$, then it has a convergent subsequence.

Theorem 2.2.2[6]: A metric space *X* is compact iff every sequence in *X* has a convergent subsequence.

Theorem 2.2.3[26,27]: Let X be a Banach Space and D be a non-empty bounded closed convex subset of X. Let$\mathcal{A},\mathcal{B}$ maps D into X s.t. $\mathcal{A}u + \mathcal{B}v \in D$,for every *(u, v)*$\in D$.If $\mathcal{A}$ is a contraction and $\mathcal{B}$ is completely continuous then the equation $\mathcal{A}w + \mathcal{B}w = w$ has a solution w on D. i.e.

a) $\mathcal{A}$ is a contraction
b) $\mathcal{B}$ is completely continuous
c) $\mathcal{A}u + \mathcal{B}v \in D \;\; for\; (u,v) \in D$

III. EXISTENCE THEORY

For the solution of (2.2.1) in the space $BC(\mathcal{R}_+,\mathcal{R})$of bounded and continuous realvalued functions defined on $\mathcal{R}_+$, Define a standard norm $\|\cdot\|$ and a multiplication " $\cdot$ " in $BC(\mathcal{R}_+,\mathcal{R})$by,$\|x\| = sup\{|x(t)|: t \in$ *ℛ+, xyt=xtyt, t∈ℛ+* (2.3.1)

Clearly, $BC(\mathcal{R}_+,\mathcal{R})$ becomes a Banach space with respect to the above norm and the multiplication in it. By $\mathcal{L}^1(\mathcal{R}_+,\mathcal{R})$ we denote the space of Lebesgue-integrable function in$\mathcal{R}_+$with the norm $\|\cdot\|_{\mathcal{L}^1}$ defined by $\|x\|_{\mathcal{L}} = \int_0^{\infty} |x(t)| dt$(2.3.2)

Definition 2.3.1[6]: A mapping $g: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is Caratheodory if:

i) $t \to g(t,x)$ is measurable for each $x \in \mathcal{R}$ and
ii) $x \to g(t,x)$is continuous almost everywhere for $t \in \mathcal{R}_+$.

Furthermore a Caratheodory function g is $\mathcal{L}^1$ −Caratheodory if:

iii) For each real number $r > 0$ there exists a function $h_r \in \mathcal{L}^1(\mathcal{R}_+,\mathcal{R})$ such that$|g(t,x)| \le h_r(t) a.e.\;\; t \in \mathcal{R}_+$ for all $x \in \mathcal{R}$ with $|x| \le r$

Finally a caratheodory function g is $\mathcal{L}_X^1$ −caratheodory if:

iv) There exists a function $h \in \forall\mathcal{L}^1(\mathcal{R}_+,\mathcal{R})$ such that $|g(t,x)| \le h(t), \;\; a.e.\;\; t \in \mathcal{R}_+$ for all $x \in \mathcal{R}$

For convenience, the function h is referred to as a bound function for g.

IV. MAIN RESULT

We need following hypothesis for existence of solution of second order nonlinear functional differential equation (2.1.1)

$(\boldsymbol{H_1})$ The functions $\theta_1, \theta_2: \mathcal{R}_+ \to \mathcal{R}$ are continuous.

$(\boldsymbol{H_2})$The function $f: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is continuous and bounded with bound F= $sup_{(t,x(\theta_1(t)))\epsilon\mathcal{R}_+\times\mathcal{R}} \left|f\left(t, x(\theta_1(t))\right)\right|$there exist a bounded function $l: \mathcal{R}_+ \to \mathcal{R}$ with bound L satisfying

$\left|f\left(t,x(\theta_1(t))\right)-f\left(t,y(\theta_1(t))\right)\right|$

$\leq \frac{l(t)|x-y|}{2(N+|x-y|)}$ $t\in\mathcal{R}_+$, for all $x,y\in\mathcal{R}$ and $0<L\leq$ N
and vanishes as $\lim_{t\to\infty}$

(H_3) The function $g:\mathcal{R}_+\times\mathcal{R}\to\mathcal{R}$ is satisfying caratheodory condition with continuous function $h(t):\mathcal{R}_+\to\mathcal{R}$ such that
$g(t,x)\leq h(t)\ \forall t\in\mathcal{R}_+$ and $x,y\in\mathcal{R}$.
(H_4) The function $v:\mathcal{R}_+\to\mathcal{R}$ defined by the formulas $v(t)=\int_0^t(t-s)h(s)\,ds$ is bounded on $\mathcal{R}_+$ and vanish at infinity, that is $\lim_{t\to\infty}v(t)=0$.
Remark 2.4.1: Note that the **(H_3)**and **(H_4)** hold, then there exists a constant $K_1>0$ such that $K_1=\sup\{v(t):t\epsilon\mathcal{R}_+\}$
Lemma 2.4.1: The function f,g satisfying SNFDE (2.1.1) then x is the solution of the SNFDE (2.1.1) if and only if it is the solution of integral equation

$$x(t)=\left[f\left(t,x(\theta_1(t))\right)\right]+\left[\int_0^t(t-s)g\left(s,x(\theta_2(s))\right)ds\right],\ t\in\mathcal{R}_+ \qquad (2.4.1)$$

Proof: Integrating equation (2.1.1) of second order, we get,

$$I\mathcal{D}^2\left[x(t)-f\left(t,x(\theta_1(t))\right)\right]_0^t=I\left[g\left(s,x(\theta_2(s))\right)\right]$$
$$\mathcal{D}\left[x(t)-f\left(t,x(\theta_1(t))\right)\right]_0^t=I\left[g\left(s,x(\theta_2(s))\right)\right]$$
$$\mathcal{D}\left[x(t)-f\left(t,x(\theta_1(t))\right)\right]=I\left[g\left(s,x(\theta_2(s))\right)\right]$$

Again integrating, we get

$$\left[x(t)-f\left(t,x(\theta_1(t))\right)\right]=I^2\left[g\left(s,x(\theta_2(s))\right)\right]$$
$$x(t)=\left[f\left(t,x(\theta_1(t))\right)\right]+\left[I^2\left[g\left(s,x(\theta_2(s))\right)\right]\right]$$
$$x(t)=\left[f\left(t,x(\theta_1(t))\right)\right]+\frac{1}{(2-1)!}\int_0^t(t-s)g\left(s,x(\theta_2(s))\right)ds$$
$$x(t)=\left[f\left(t,x(\theta_1(t))\right)\right]+\left[\int_0^t(t-s)g\left(s,x(\theta_2(s))\right)ds\right],t\in\mathcal{R}_+$$

Since$\int_0^t f(t)dt^n=\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}f(s)ds$, Where $n=0,1,2,3,\dots\dots\dots$
Conversely differentiate (2.4.1) of order 2 w.r.to t, we get,

$$\mathcal{D}^2\left[x(t)-f\left(t,x(\theta_1(t))\right)\right]=\mathcal{D}^2\left[\int_0^t(t-s)g\left(s,x(\theta_2(s))\right)ds\right]$$
$$\mathcal{D}^2\left[x(t)-f\left(t,x(\theta_1(t))\right)\right]=\mathcal{D}^2\left[\frac{1}{\Gamma(2)}\int_0^t(t-s)^{2-1}g\left(s,x(\theta_2(s))\right)ds\right]$$
$$\mathcal{D}^2\left[x(t)-f\left(t,x(\theta_1(t))\right)\right]=\quad g\left(s,x(\theta_2(t))\right)$$

Theorem 2.4.2: Assume that condition **(H_1)-(H_4)** hold. Then (2.1.1) has a solution in the space $BC(\mathcal{R}_+,\mathcal{R})$, moreover solution of (2.1.1) are locally attractive on$\mathcal{R}_+$.
Proof: By a solution of SNFDE (2.1.1) we mean a continuous function $x:\mathcal{R}_+\to\mathcal{R}$ that satisfies SNFDE (2.1.1) on $\mathcal{R}_+$. Let$X=BC(\mathcal{R}_+,\mathcal{R})$ and define a subset $B_r[0]$ of X as $B_r[0]=\{x\in X:\|x\|\leq r\}$.where r satisfies the inequality, $F+K_1\leq r$.
Let $X=BC(\mathcal{R}_+,\mathcal{R})$ be Banach algebra of all bounded continuous real-valued function on $\mathcal{R}_+$ with the norm
$\|x\|=sup|x(t)|,t\in\mathcal{R}_+$(2.4.2)
Under some suitable conditions involved in (2.1.1) we obtain the solution of SNFDE (2.1.1) Now the SNFDE (2.1.1) is equivalent to the SNFIE

$$x(t)=\left[f\left(t,x(\theta_1(t))\right)\right]+\left[\int_0^t(t-s)g\left(s,x(\theta_2(s))\right)ds\right]$$

Let us define the two mappings $\mathcal{A}:X\to X$
and $\mathcal{B}:B_r[0]\to X$ by
$\mathcal{A}x(t)=f\left(t,x(\theta_1(t))\right),t\in\mathcal{R}_+$(2.4.3)

$\mathcal{B}x(t) = \int_0^t (t-s) g\left(s, x(\theta_2(s))\right) ds, t \in \mathcal{R}_+$(2.4.4)

Thus from the SNDE (2.1.1), we obtain the operator equation as follows:

$x(t) = \mathcal{A}x(t) + \mathcal{B}x(t),\ t \in \mathcal{R}_+$(2.4.5)

If the operator $\mathcal{A}$ and $\mathcal{B}$ satisfy all the hypothesis of theorem (2.2.3), then the operator equation (2.4.5) has a solution on $B_r[0]$.

Step I: Firstly we show that $\mathcal{A}$ is contraction mapping X Let $x, y \in X$; then

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| = \left|f\left(t, x(\theta_1(t))\right) - f\left(t, y(\theta_1(t))\right)\right|$$

$$\le \frac{l(t)\left|x(\theta_1(t)) - y(\theta_1(t))\right|}{2(N + |x-y|)}$$

$$\le \frac{L\left|x(\theta_1(t)) - y(\theta_1(t))\right|}{2(N + |x-y|)} \text{for all} \quad t \in \mathcal{R}_+$$

Taking supremum over t

$$\|\mathcal{A}x - \mathcal{A}y\| \le \frac{L\|x-y\|}{2(N + \|x-y\|)} \quad \text{for all } x, y \in X$$

This shows that $\mathcal{A}$ is contraction mappingwith

$$L_1 = \frac{L}{2(N + \|x-y\|)}$$

Step II: Secondly we show that $\mathcal{B}$ is completely continuous operator on $B_r[0]$using Granas at [18], it can be shown that $\mathcal{B}$ is continuous operator on $B_r[0]$.

Let us fix arbitrary $\epsilon > 0$ and take $x, y \in B_r[0]$such that$\|x-y\| \le \epsilon$.

$$|\mathcal{B}x(t) - \mathcal{B}y(t)| = \left|\int_0^t (t-s) g\left(s, x(\theta_2(s))\right) ds - \int_0^t (t-s) g\left(s, y(\theta_2(s))\right) ds\right|$$

$$\le \left|\int_0^t (t-s) g\left(s, x(\theta_2(s))\right) ds\right| + \left|\int_0^t (t-s) g\left(s, y(\theta_2(s))\right) ds\right|$$

$$\le \int_0^t (t-s) h(s)\, ds + \int_0^t (t-s) h(s)\, ds$$

$$\le 2\int_0^t (t-s) h(s)\, ds \ ,$$

$$\le 2\int_0^t \frac{1}{(t-s)^{1-2}} h(s) ds \le 2v(t) \quad (\text{ by Hypothesis } H_8)$$

As $v(t) \le \frac{\epsilon}{2}$, $|\mathcal{B}x(t) - \mathcal{B}y(t)| \le \epsilon$.

Thus $\mathcal{B}$ is continuous.

Step III: Now we will show that $\mathcal{B}$ is compact on $\mathcal{B}(B_r[0])$

a) First we prove that every sequence $\{\mathcal{B}x_n\}$ in $\mathcal{B}(B_r[0])$has uniformly bounded sequence and $\{\mathcal{B}x_n\}$ is equicontinuous set in $B_r[0]$. Since $g\left(t, x(\theta_2(t))\right)$ is $\mathcal{L}_X^1 -$ caratheodary, we have

$$|\mathcal{B}x_n(t)| = \left|\int_0^t (t-s) g\left(s, x_n(\theta_2(s))\right) ds\right|$$

$$\le \int_0^t (t-s) \left|g\left(s, x_n(\theta_2(s))\right)\right| ds$$

$$\le \int_0^t (t-s) h(s) ds$$

$$\le \int_0^t \frac{1}{(t-s)^{1-2}} h(s) ds \le v(t) \quad (\text{ by Hypothesis } H_4)$$

Taking supremum over t, we obtain, $\|\mathcal{B}x_n\| \le K_1$ for all $x \in B_r[0]$

Where, $K_1 = sup_{t \in \mathcal{R}_+}\{v(t)\}$

This shows that $\{\mathcal{B}x_n\}$ is uniformly bounded sequence in $\mathcal{B}(B_r[0])$

To show that $\{\mathcal{B}x_n\}$ is an equicontinuous sequence, let $t_1, t_2 \in [0, T]$ be arbitrary. Then for any $x \in B_r[0]$ (2.4.5) implies

$$|\mathcal{B}x_n(t_2)-\mathcal{B}x_n(t_1)| = \left| \int_0^{t_2}(t_2-s)g\left(s,x_n(\theta_2(s))\right)ds - \int_0^{t_1}(t_1-s)g\left(s,x_n(\theta_2(s))\right)ds \right|$$

$$= \left|\int_0^{t_2}(t_2-s)h(s)ds - \int_0^{t_1}(t_1-s)h(s)ds\right|$$

$$\le |v(t_2)-v(t_1)|$$

The right hand side of the above inequality doesn't depend on x and tends to zero as $t_1 \to t_2$.Therefore$|\mathcal{B}x_n(t_2)-\mathcal{B}x_n(t_1)| \to 0$ as $t_1 \to t_2$.

If $t_1, t_2 \ge T$ then we have

$$|\mathcal{B}x_n(t_2)-\mathcal{B}x_n(t_1)| = \left| \int_0^{t_2}(t_2-s)g\left(s,x_n(\theta_2(s))\right)ds - \int_0^{t_1}(t_1-s)g\left(s,x_n(\theta_2(s))\right)ds \right|$$

$$= \left|\int_0^{t_2}(t_2-s)h(s)ds - \int_0^{t_1}(t_1-s)h(s)ds\right|$$

$$\le \left|\int_0^{t_2}(t_2-s)h(s)ds\right| + \left|\int_0^{t_1}(t_1-s)h(s)ds\right| \quad \text{(by Hypothesis } H_8 \text{)}$$

$$\le v(t_2)+v(t_1) \le \frac{\epsilon}{2}+\frac{\epsilon}{2} \le \epsilon \text{ as } t_1 \to t_2$$

If $t_1, t_2 \in \mathcal{R}_+$ then we have

$|\mathcal{B}x_n(t_2)-\mathcal{B}x_n(t_1)| \le |\mathcal{B}x_n(t_2)-\mathcal{B}x_n(T)| + |\mathcal{B}x_n(T)-\mathcal{B}x_n(t_1)|$

If $t_1 \to t_2$, then $t_1 \to T \, and \, T \to t_2$

Therefore $|\mathcal{B}x_n(t_2)-\mathcal{B}x_n(T)| \to 0$ $\quad$ $|\mathcal{B}x_n(T)-\mathcal{B}x_n(t_1)| \to 0$

So $|\mathcal{B}x_n(t_2)-\mathcal{B}x_n(t_1)| \to 0$ as $t_1 \to t_2$

Hence, $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in $\mathcal{B}(B_r[0])$ so $\mathcal{B}(B_r[0])$ is relatively compact by Arzela-Ascoli theorem. By definition 2.2.4 $\mathcal{B}$ is compactwhich gives, $\mathcal{B}$ is compact and continuous operator on $B_r[0]$.

Thus $\mathcal{B}$ is completely continuous on $B_r[0]$

Step IV: Next we show that $\mathcal{A}x + \mathcal{B}x \in B_r[0]$

Let $x, y \in B_r[0]$such that $x = \mathcal{A}x + \mathcal{B}x$

$$|\mathcal{A}x(t)+\mathcal{B}x(t)| \le |\mathcal{A}x(t)| + |\mathcal{B}x(t)|$$

$$\le \left|f\left(t,x(\theta_1(t))\right)\right| + \left|\int_0^t (t-s)g\left(s,x(\theta_2(s))\right)ds\right|$$

$$\le \left|f\left(t,x(\theta_1(t))\right)\right| + \int_0^t (t-s)\left|g\left(s,x(\theta_2(s))\right)\right|ds$$

$$\le F + \int_0^t (t-s)h(s)ds \le F + v(t) \text{(by Hypothesis } H_4 \text{)}$$

Taking supremum over ton $\mathcal{R}_+$, we obtain $\|\mathcal{A}x+\mathcal{B}x\| \le F + K_1$, $\forall x \in B_r[0]$

That is we have,$\|x\| = \|\mathcal{A}x+\mathcal{B}x\| \le r$, $\forall x \in B_r[0]$.

which gives $x = \mathcal{A}x + \mathcal{B}y \in B_r[0]$

Hence assumption(c) of theorem (2.2.3) is proved.

Hence all the conditions of theorem (2.2.3) are satisfied and therefore the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution in $B_r[0]$. As a result, (2.1.1) has a solution defined on $\mathcal{R}_+$.

Step VI: Finally we show the locally attractivity of the solutions for (2.1.1). Let x and y be two solutions of (2.1.1) in $B_r[0]$ defined on $\mathcal{R}_+$ .Then we have

$$|x(t)-y(t)| = \left| \left[f\left(t,x(\theta_1(t))\right)\right] + \left[\int_0^t (t-s)g\left(s,x(\theta_2(s))\right)ds\right] - \left[f\left(t,y(\theta_1(t))\right)\right] + \left[\int_0^t (t-s)g\left(s,y(\theta_2(s))\right)ds\right] \right|$$

$$\leq \left| \left[f\left(t, x(\theta_1(t))\right) \right] + \left[\int_0^t (t-s) g\left(s, x(\theta_2(s))\right) ds \right] \right| +$$

$$\left| \left[f\left(t, y(\theta_1(t))\right) \right] + \left[\int_0^t (t-s) g\left(s, y(\theta_2(s))\right) ds \right] \right|$$

$$\leq \left| f\left(t, x(\theta_1(t))\right) \right| + \int_0^t (t-s) \left| g\left(s, x(\theta_2(s))\right) \right| ds +$$

$$\left| f\left(t, y(\theta_1(t))\right) \right| + \int_0^t (t-s) \left| g\left(s, y(\theta_2(s))\right) \right| ds$$

$$\leq F + \left\{ \int_0^t (t-s) h(s)\, ds \right\} + F + \left\{ \int_0^t (t-s) h(s)\, ds \right\}$$

$$\leq 2F + 2 \int_0^t (t-s) h(s)\, ds \ \leq 2F + 2[v(t)] \qquad \text{(by Hypothesis } \mathrm{H}_4 \text{)}$$

For all $t \in \mathcal{R}_+$ as $\lim_{t\to\infty} v(t) = 0$ this gives that $\lim_{t\to\infty} \sup|x(t) - y(t)| = 0.$ for all $t \geq T$. This completes the proof.

V. CONCLUSION

In this paper we have studied the existence and locally attractivity of solutions to the second order nonlinear functional differential equation in Banach Space by fixed point theorem.

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