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Research Paper

Solution of convex Programing problem by Primal-dual interior-point method

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ABSTRACT : *Modern convex optimization, has been one of the most exciting and active research areas in optimization starting from 1990s. It has attracted researchers with very diverse backgrounds, including experts in convex programming, linear algebra, numerical optimization, combinatorial optimization, control theory, and statistics. This tremendous research activity was spurred by the discovery of important applications in combinatorial optimization and control theory, the development of efficient interior-point algorithms for solving convex programing problem(CPP) and the depth and elegance of the underlying optimization theory. This article includes mathematical formation and application of the primal-dual interior-point method to solve CPP.*
KEYWORDS: *Convex Programing Problem(CPP), KKT, interior-point method , primal-dual interior-point method.*

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I. INTRODUCTION

The first revolution of optimization was during and immediate after the World War- 2 through the linear programming by Dantzig. Then the histories of optimization become more and more popular along with its military purpose the industrial revolution. Later in 1951, Karush-Kuhn-Tucker turns the revolution from linear to nonlinear optimization. Now it reaches to its comfort position, the convex programing.

Convex optimization problems are far more general than linear programming problems, but they share the desirable properties of LP problems: They can be solved quickly and reliably up to very large size i.e., hundreds of thousands of variables and constraints. A convex optimization problem is a problem where all of the constraints are convex functions, and the objective is a convex function if minimizing, or a concave function if maximizing. Linear functions are convex**,** so linear programming problems are convex problems. Conic optimization problems are the natural extension of linear programming problems which are all convex optimization problems. In a convex optimization problem, the feasible region is the intersection of convex constraint functions and is a convex region. With a convex objective and a convex feasible region, there can be only one optimal solution, which is globally optimal**.**

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Day by day, modern techniques come forward and many branches like cone programming (CP), second-order cone programming (SOCP) and semidefinite programming (SDP) comes out as reliable part of convex programming.

Interior-point methods (also referred to as barrier methods) are a certain class of algorithms that solves linear and nonlinear convex optimization problems.

John von Neumann suggested an interior-point method of linear programming which was neither a polynomial time method nor an efficient method in practice. In fact, it turned out to be slower in practice compared to simplex method which is not a polynomial time method. In 1984, Narendra Karmarkar developed a method for linear programming called Karmarkar's algorithm which runs in provably polynomial time and is also very efficient in practice. It enabled solutions of linear programming problems which were beyond the capabilities of the simplex method. Contrary to the simplex method, it reaches a best solution by traversing the interior of the feasible region. The method can be generalized to convex programming based on a self-concordant barrier function used to encode the convex set.

Any convex optimization problem can be transformed into minimizing (or maximizing) a linear function over a convex set by converting to the epigraph form[1]. The idea of encoding the feasible set using a barrier and designing barrier methods was studied by Anthony V. Fiacco, Garth P. McCormick, and others in the early 1960s. These ideas were mainly developed for general nonlinear programming, but they were later abandoned due to the presence of more competitive methods for this class of problems.

Yurii Nesterov and Arkadi Nemirovski came up with a special class of such barriers that can be used to encode any convex set. They guarantee that the number of iterations of the algorithm is bounded by a polynomial in the dimension and accuracy of the solution.

Karmarkar's breakthrough revitalized the study of interior-point methods and barrier problems, showing that it was possible to create an algorithm for linear programming characterized by polynomial complexity and, moreover, that was competitive with the simplex method.

This article focuses on the applications of primal-dual interior-point methods to solve convex programming problem.

II. Method descriptions: Primal-dual interior-point method for CPP

Let the primal problem is

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to,} \quad g_i(x) \leq 0 \\ &\qquad Ax = b \end{aligned}$$

Barrier Primal of CPP (BPCPP)

$$\begin{aligned} &\text{minimize} \quad \psi = tf(x) - \sum_{i=1}^{m} \ln(-g_i(x)) \\ &\text{subject to,} \quad Ax = b \end{aligned}$$

Lagrange multiplier (KKT) optimality conditions for (BPCPP) with Lagrange multiplier vectors λ and w are

$$\begin{cases} \nabla\psi + w\,\nabla(Ax - b) = 0 \\ Ax = b \end{cases}.$$

$$\Rightarrow \begin{cases} t\nabla f(x) - \sum_{i=1}^{m} \frac{1}{-g_i(x)} \nabla g_i(x) + A^T v = 0 \\ Ax = b \end{cases}.$$

$$\Rightarrow \begin{cases} \nabla f(x) - \sum_{i=1}^{m} \frac{1}{-tg_i(x)} \nabla g_i(x) + \frac{1}{t} A^T v = 0 \\ Ax = b \end{cases}.$$

Consider $\lambda_i = \frac{-1}{t\,g_i(x)}$ and $w = \frac{v}{t}$, so that

$$\begin{cases} \nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) + A^T w = 0 \\ -\lambda_i g_i(x) = \frac{1}{t} \quad (i = 1,2, \dots, m) \\ Ax = b \end{cases}$$

.

Or in vector form

$$\begin{cases} \nabla f(x) + Dg(x)^T \lambda + A^T w = 0 \\ -\mathrm{diag}(\lambda) g(x) = \frac{1}{t} e \\ Ax = b \end{cases}$$

Which is the Barrier optimality condition for CPP, where

$$g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ . \\ . \\ . \\ g_m(x) \end{bmatrix}, Dg(x) = \begin{bmatrix} \nabla g_1(x)^T \\ \nabla g_2(x)^T \\ . \\ . \\ . \\ \nabla g_m(x)^T \end{bmatrix}, \mathrm{diag}(\lambda) = \begin{bmatrix} \lambda_1 & 0 & . & . & 0 \\ 0 & \lambda_2 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & \lambda_m \end{bmatrix}$$

Putting $x = x + \Delta x, \lambda = \lambda + \Delta\lambda$, and $w = w + \Delta w$ in (BPDO), we have

$$\begin{cases} f(x + \Delta x) + Dg(x + \Delta x)^T (\lambda + \Delta\lambda) + A^T (w + \Delta w) = 0 \\ -\mathrm{diag}(\lambda + \Delta\lambda) g(x + \Delta x) = \frac{1}{t} e \\ A(x + \Delta x) = b \end{cases}$$

$$\begin{cases} \nabla^2 f(x)\Delta x + \sum_{i=1}^{m} \lambda_i \nabla^2 g_i(x)\Delta x + Dg(x)^t \Delta\lambda + A^t \Delta w = -(\Delta f(x) + \lambda Dg(x) + A^t w) \\ -\mathrm{diag}(\lambda) Dg(x)\Delta x - \mathrm{dig}\big(g(x)\big)\Delta\lambda = -(\mathrm{dig}(\lambda) g(x)^t - \frac{1}{t} e) \\ A\Delta x = -(Ax - b) \end{cases}$$

And in matrix form

$$\begin{bmatrix} \nabla^2 f(x) + \sum_{j=1}^{m} \lambda_j \nabla^2 g_j(x) & Dg(x)^T & A^T \\ -\mathrm{diag}(\lambda) Dg(x) & -\mathrm{diag}(g(x)) & 0 \\ A & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{pd} \\ \Delta\lambda_{pd} \\ \Delta v_{pd} \end{bmatrix} = - \begin{bmatrix} \nabla f(x) + Dg(x)^T \lambda + A^T v \\ \mathrm{diag}(\lambda)\, g(x) + \frac{1}{t} e \\ Ax - b \end{bmatrix}$$

is a linear system of $2n + m$ equations in $2n + m$ variables; thus may be solved for some initial (x, λ, μ) and t.

By the solution of the system, update the primal-dual variables as

$$x = x + r\, \Delta x_{pd}$$
$$\lambda = \lambda + r\, \lambda_{pd}$$
$$\mu = \mu + r\, \mu_{pd}$$

Or,

$$l = l + r\, \Delta l_{pd}$$

For $l = [x,\ \lambda, \mu]$ and for some necessary line search r for each l.

We repeat the same for larger and larger values of t and updated (x, λ, μ) until the optimality conditions.

III. Algorithm: Primal-dual interior-point method for CPP

given: strictly feasible $x, \lambda > 0, \mu > 1$, tolerance $\varepsilon_{feas} > 0$ and $\varepsilon > 0$;

Repeat:

1. *Determine t.* Set $= \mu m/\tilde{\eta}$; [$m \equiv$ no. of inequality constraints, $\tilde{\eta} = -g(x)^T\lambda$]
2. Compute primal-dual search direction Δl_{pd}
3. *Line search and update*.

Determine step length r and update $l := l + r\Delta l_{pd}$

Until. $\|\nabla f(x) + Dg(x)^T\lambda + A^T v\|_2 < \varepsilon_{feas}$, $\left\|-\mathrm{diag}(\lambda)g(x) - \frac{1}{t}e\right\|_2 < \varepsilon_{feas}$, and $\tilde{\eta} < \varepsilon$

IV. Application: Primal-dual interior-point method for CPP

Consider a CPP

$$\begin{array}{ll} \text{minimize} & (x_1-1)^2 + (x_2-1)^2 \\ \text{subject to} & x_1 + x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{array}$$

So the condition for Primal Dual Interior-point method is

$$\begin{bmatrix} \nabla^2 f(x) + \sum_{i=1}^{m} \lambda_i \nabla^2 g_j(x) & Dg(x)^T \\ -\mathrm{diag}(\lambda) Dg(x) & -\mathrm{diag}(g(x)) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\lambda \end{bmatrix} = -\begin{bmatrix} \nabla f(x) + Dg(x)^T\lambda \\ -\mathrm{diag}(\lambda)g(x) - \frac{1}{t}e \end{bmatrix}$$

Here,

$$f(x) = (x_1-1)^2 + (x_2-1)^2$$
$$g_1(x) = x_1 + x_2 - 1$$
$$g_2(x) = -x_1$$
$$g_3(x) = -x_2$$

Therefore, $\nabla f = \begin{pmatrix} 2x_1 - 2 \\ 2x_2 - 2 \end{pmatrix}$, $\nabla^2 f = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\nabla g_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\nabla g_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$,

$\nabla g_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $\nabla^2 g_i = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - 1 \\ -x_1 \\ -x_2 \end{pmatrix},$$

$Dg(x) = \begin{pmatrix} D_1 g_1 & D_2 g_1 \\ D_1 g_2 & D_2 g_2 \\ D_1 g_3 & D_2 g_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $Dg(x)^T = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$

$$\mathrm{diag}(\lambda) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

$$\mathrm{diag}(g(x)) = \begin{pmatrix} g_1(x) & 0 & 0 \\ 0 & g_2(x) & 0 \\ 0 & 0 & g_3(x) \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - 1 & 0 & 0 \\ 0 & -x_1 & 0 \\ 0 & 0 & -x_2 \end{pmatrix}$$

Therefore

$$\left[\nabla^2 f(x) + \sum_{i=1}^{3} \lambda_i \nabla^2 g_i(x)\right] \triangle x = \left[\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \sum_{i=1}^{3} \lambda_i \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right] \triangle x$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \triangle x_1 \\ \triangle x_2 \end{pmatrix} = \begin{pmatrix} 2 \triangle x_1 \\ 2 \triangle x_2 \end{pmatrix}$$

And

$$[-\mathrm{diag}(\lambda) Dg(x)] \triangle x = \left[-\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \right] \triangle x$$

$$= -\begin{pmatrix} \lambda_1 & \lambda_1 \\ -\lambda_2 & 0 \\ 0 & -\lambda_3 \end{pmatrix} \begin{pmatrix} \triangle x_1 \\ \triangle x_2 \end{pmatrix} = \begin{pmatrix} -\lambda_1 \triangle x_1 - \lambda_1 \triangle x_2 \\ \lambda_2 \triangle x_1 \\ \lambda_3 \triangle x_2 \end{pmatrix}$$

Now $[Dg(x)^T] \triangle \lambda = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \triangle \lambda_1 \\ \triangle \lambda_2 \\ \triangle \lambda_3 \end{pmatrix} = \begin{pmatrix} \triangle \lambda_1 - \triangle \lambda_2 \\ \triangle \lambda_1 - \triangle \lambda_3 \end{pmatrix}$

And $-\mathrm{diag}\big(g(x)\big) \triangle \lambda = -\begin{pmatrix} x_1 + x_2 - 1 & 0 & 0 \\ 0 & -x_1 & 0 \\ 0 & 0 & -x_2 \end{pmatrix} \begin{pmatrix} \triangle \lambda_1 \\ \triangle \lambda_2 \\ \triangle \lambda_3 \end{pmatrix}$

$$= \begin{pmatrix} -(x_1 + x_2 - 1) \triangle \lambda_1 \\ x_1 \triangle \lambda_2 \\ x_2 \triangle \lambda_3 \end{pmatrix}$$

Furthermore,

$$-[\nabla f(x) + Dg(x)^T \lambda] = -\begin{pmatrix} 2x_1 - 2 \\ 2x_2 - 2 \end{pmatrix} - \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$$

$$= -\begin{pmatrix} 2x_1 - 2 \\ 2x_2 - 2 \end{pmatrix} - \begin{pmatrix} \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_3 \end{pmatrix} = \begin{pmatrix} -2x_1 + 2 - \lambda_1 + \lambda_2 \\ -2x_2 + 2 - \lambda_1 + \lambda_3 \end{pmatrix}$$

And $\left[\mathrm{diag}(\lambda) g(x) + \frac{1}{t} e\right] = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} x_1 + x_2 - 1 \\ -x_1 \\ -x_2 \end{pmatrix} + \frac{1}{t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} (x_1 + x_2 - 1)\lambda_1 + \dfrac{1}{t} \\ -x_1 \lambda_2 + \dfrac{1}{t} \\ -x_2 \lambda_3 + \dfrac{1}{t} \end{pmatrix}$$

Hence the conditions become

$$\begin{cases} \left[\nabla^2 f(x) + \sum_{i=1}^{m} \lambda_i \nabla^2 g_j(x)\right] \triangle x + [Dg(x)^T] \triangle \lambda = -[\nabla f(x) + Dg(x)^T \lambda] \\ [-\mathrm{diag}(\lambda) Dg(x)] \triangle x - \mathrm{diag}\big(g(x)\big) \triangle \lambda = [\mathrm{diag}(\lambda) g(x) + \frac{1}{t} e] \end{cases}.$$

Or,

$$\begin{cases} \begin{pmatrix} 2 \triangle x_1 \\ 2 \triangle x_2 \end{pmatrix} + \begin{pmatrix} \triangle \lambda_1 - \triangle \lambda_2 \\ \triangle \lambda_1 - \triangle \lambda_3 \end{pmatrix} = \begin{pmatrix} -2x_1 + 2 - \lambda_1 + \lambda_2 \\ -2x_2 + 2 - \lambda_1 + \lambda_3 \end{pmatrix} \\ \begin{pmatrix} -\lambda_1 \triangle x_1 - \lambda_1 \triangle x_2 \\ \lambda_2 \triangle x_1 \\ \lambda_3 \triangle x_2 \end{pmatrix} + \begin{pmatrix} -(x_1 + x_2 - 1) \triangle \lambda_1 \\ x_1 \triangle \lambda_2 \\ x_2 \triangle \lambda_3 \end{pmatrix} = \begin{pmatrix} (x_1 + x_2 - 1)\lambda_1 + \dfrac{1}{t} \\ -x_1 \lambda_2 + \dfrac{1}{t} \\ -x_2 \lambda_3 + \dfrac{1}{t} \end{pmatrix} \end{cases}.$$

Or,
$$\begin{cases} \begin{pmatrix} 2\triangle x_1 \\ 2\triangle x_2 \end{pmatrix} + \begin{pmatrix} \triangle\lambda_1 - \triangle\lambda_1 \\ \triangle\lambda_1 - \triangle\lambda_3 \end{pmatrix} = \begin{pmatrix} -2x_1+2-\lambda_1+\lambda_2 \\ -2x_2+2-\lambda_1+\lambda_3 \end{pmatrix} \\ \begin{pmatrix} -\lambda_1\triangle x_1 - \lambda_1\triangle x_2 \\ \lambda_2\triangle x_1 \\ \lambda_3\triangle x_2 \end{pmatrix} + \begin{pmatrix} -(x_1+x_2-1)\triangle\lambda_1 \\ x_1\triangle\lambda_2 \\ x_2\triangle\lambda_3 \end{pmatrix} = \begin{pmatrix} (x_1+x_2-1)\lambda_1+\frac{1}{t} \\ -x_1\lambda_2+\frac{1}{t} \\ -x_2\lambda_3+\frac{1}{t} \end{pmatrix} \end{cases}$$

Or,
$$\begin{cases} 2\triangle x_1 + \triangle\lambda_1 - \triangle\lambda_2 = -2x_1+2-\lambda_1+\lambda_2 \\ 2\triangle x_2 + \triangle\lambda_1 - \triangle\lambda_3 = -2x_2+2-\lambda_1+\lambda_3 \\ -\lambda_1\triangle x_1 - \lambda_1\triangle x_2 - (x_1+x_2-1)\triangle\lambda_1 = (x_1+x_2-1)\lambda_1+\frac{1}{t} \\ \lambda_2\triangle x_1 + x_1\triangle\lambda_2 = -x_1\lambda_2+\frac{1}{t} \\ \lambda_3\triangle x_2 + x_2\triangle\lambda_3 = -x_2\lambda_3+\frac{1}{t} \end{cases}$$

1st centering step:

Chose a strictly feasible point $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{3}$, $\lambda_1 = 3$, $\lambda_2 = 3$, $,\lambda_3 = 3$, $m = 3, \mu = 10$ and $\varepsilon_{feas} = 0.001, \epsilon = 0.001$. Then $\eta = -g(x)^T\lambda = 3$ and $t = \frac{m\mu}{\eta} = 10$

So,
$$l^{(0)} = \left(\frac{1}{3}, \frac{1}{3}, 3, 3, 3\right)$$

Then
$$\begin{cases} 2\triangle x_1 + \triangle\lambda_1 - \triangle\lambda_2 = 1.333333333 \\ 2\triangle x_2 + \triangle\lambda_1 - \triangle\lambda_3 = 1.333333333 \\ -3\triangle x_1 - 3\triangle x_2 + .333333333\triangle\lambda_1 = -0.9 \\ 3\triangle x_1 + .333333333\triangle\lambda_2 = -0.9 \\ 3\triangle x_2 + .333333333\triangle\lambda_3 = -0.9 \end{cases}$$

By solving we get,
$\triangle x_1 = 0.046, \triangle x_2 = 0.046\,, \triangle\lambda_1 = -1.8724, \triangle\lambda_2 = -3.1138, \triangle\lambda_3 = -3.1138$
So
$$\Delta l^{(0)} = (0.046, 0.046, -1.8724, -3.1138, -3.1138)$$

Backtracking line search 1: Choose $\alpha = 0.2$, $\beta = 0.8$

So, $s^{max} = \min\left\{1, \min\left\{-\frac{3}{-1.8724}, -\frac{3}{-3.1138} - \frac{3}{-3.1138}\right\}\right\} = \min\{1, 0.963453\} = 0.963453.$

set $s = 0.99 \times s^{max} = 0.953818$

And Compute $l^{(0)} + s\Delta l^{(0)} = (0.377209, 0.377209, 1.214070, 0.03, 0.03)$. Then $g_1 = -0.2456 < 0$, $g_2 = -0.3772 < 0$, $g_3 = -0.3772$. and $r_t(x^+, \lambda^+) = 0.250133$ On the other hand $|(1-\alpha s)r_t(x,\lambda)| = 2.3999$

Since $r_t(x^+,\lambda^+) < |(1-\alpha s)r_t(x,\lambda)|$; so we have the next centering point is
$$\begin{aligned} l^{(1)} &= l^{(0)} + s\Delta l^{(0)} \\ &= (0.377209, 0.377209, 1.214070, 0.03, 0.03) \end{aligned}$$

Stopping condition:

We see, $\eta = 3 \nless \epsilon$ and $\|\nabla f(x) + Dg(x)^T\lambda\|_2 = 0.0869 \nless \varepsilon_{feas}$ and

$$\left\|-\text{diag}(\lambda)g(x)-\frac{1}{t}e\right\|_2=0.2354 \nless \varepsilon_{feas}$$

Consequently, we continue more.

2nd centering step:

$l^{(1)}=(0.377209, 0.377209, 1.214070, 0.03, 0.03)$

$$\eta=-g(x)^T\lambda=0.1664 \text{ and } t=\frac{m\mu}{\eta}=182.4808342$$

Putting these values, we get the set of 5 equations

$$\begin{cases} 2\triangle x_1+\triangle\lambda_1-\triangle\lambda_2=0.061512 \\ 2\triangle x_2+\triangle\lambda_1-\triangle\lambda_3=0.061512 \\ -1.214070\triangle x_1-1.214070\triangle x_2+0.245582\triangle\lambda_2=-0.287461 \\ 0.03\triangle x_1+0.377209\triangle\lambda_2=-0.000623 \\ 0.03\triangle x_2+0.377209\triangle\lambda_3=-0.000623 \end{cases}$$

By solving we get,

$\triangle x_1=0.1028, \triangle x_2=0.1028, \triangle\lambda_1=-0.154, \triangle\lambda_2=-0.0098, \triangle\lambda_3=-0.0098$

$\Delta l^{(1)}=(0.1028, 0.1028, -0.154, -0.0098, -0.0098)$

Backtracking line search 2: Choose $\alpha=0.2,\ \beta=0.8$

so, $s^{max}=\min\left\{1,\min\left\{-\frac{1.214070}{-0.154},-\frac{0.03}{-0.0098}-\frac{0.03}{-0.0098}\right\}\right\}=\min\{1,3.061224\}=1$. set $s=0.99\times s^{max}=0.99$

And Compute $l^{(1)}+s\Delta l^{(1)}=(0.478981, 0.478981, 1.061610, 0.020298, 0.020298)$. Then $g_1=-0.0420<0$, $g_2=-0.4798<0$, $g_3=-0.4798$. and $r_t(x^+,\lambda^+)=0.0339$ On the other hand $|(1-\alpha s)r_t(x,\lambda)|=0.2943$

Since $r_t(x^+,\lambda^+)<|(1-\alpha s)r_t(x,\lambda)|$; so we have the next centering point is

$$l^{(2)}=l^{(1)}+s\Delta l^{(1)}$$

$=(0.478981, 0.478981, 1.061610, 0.020298, 0.020298)$

Stopping condition:

We see, $\eta=0.1664 \nless \epsilon$ and $\|\nabla f(x)+Dg(x)^T\lambda\|_2=0.001 \nless \varepsilon_{feas}$ and

$$\left\|-\text{diag}(\lambda)g(x)-\frac{1}{t}e\right\|_2=0.0438 \nless \varepsilon_{feas}$$

Consequently, we continue more.

3rd centering step:

$l^{(2)}=(0.478981, 0.478981, 1.061610, 0.020298, 0.020298)$

$$\eta=-g(x)^T\lambda=0.064073 \text{ and } t=\frac{m\mu}{\eta}=468.2179$$

Putting these values, we get the set of 5 equations

$$\begin{cases} 2\triangle x_1+\triangle\lambda_1-\triangle\lambda_2=0.000726 \\ 2\triangle x_2+\triangle\lambda_1-\triangle\lambda_3=0.000726 \\ -1.061610\triangle x_1-1.061610\triangle x_2+0.042038\triangle\lambda_2=-0.042492 \\ 0.020298\triangle x_1+0.478981\triangle\lambda_2=-0.007587 \\ 0.020298\triangle x_2+0.478981\triangle\lambda_3=-0.007587 \end{cases}$$

By solving we get,

$\triangle x_1=0.0189, \triangle x_2=0.0189, \triangle\lambda_1=-0.0538, \triangle\lambda_2=-0.0166, \triangle\lambda_3=-0.0166$

$\Delta l^{(2)}=(0.0189, 0.0189, -0.0538, -0.0166, -0.0166)$

Backtracking line search 3: Choose $\alpha=0.2,\ \beta=0.8$

so, $s^{max}=\min\left\{1,\min\left\{-\frac{1.061610}{-0.0538},-\frac{0.020298}{-0.0166}-\frac{0.020298}{-0.0166}\right\}\right\}=\min\{1,1.2228\}=1.$ Set $s=0.99\times s^{max}=0.99$

And Compute $l^{(2)} + s\Delta l^{(2)} = (0.497992, 0.497992, 1.008348, 0.003864, 0.003864)$. Then $g_1 = -0.0046 < 0$, $g_2 = -0.4977 < 0$, $g_3 = -0.4977$. and $r_t(x^+, \lambda^+) = 0.002544$ On the other hand $|(1 - \alpha s) r_t(x, \lambda)| = 0.0429690$

Since $r_t(x^+, \lambda^+) < |(1 - \alpha s) r_t(x, \lambda)|$; so we have the next centering point is

$$l^{(3)} = l^{(3)} + s\Delta l^{(3)}$$

$$= (0.497992, 0.497992, 1.008348, 0.003864, 0.003864)$$

Stopping condition:

We see, $\eta = 0.064073 \nless \epsilon$ and $\|\nabla f(x) + Dg(x)^T \lambda\|_2 = 0.0001 < \varepsilon_{feas}$ and

$$\left\| -\mathrm{diag}(\lambda) g(x) - \frac{1}{t} e \right\|_2 = 0.0049 \nless \varepsilon_{feas}$$

Consequently, we continue more.

<u>4th centering step:</u>

$l^{(3)} = (0.497992, 0.497992, 1.008348, 0.003864, 0.003864)$

$$\eta = -g(x)^T \lambda = 0.0085 \text{ and } t = \frac{m\mu}{\eta} = 3529.1078$$

Putting these values, we get the set of 5 equations

$$\begin{cases} 2 \triangle x_1 + \triangle \lambda_1 - \triangle \lambda_2 = 0.000132 \\ 2 \triangle x_2 + \triangle \lambda_1 - \triangle \lambda_3 = 0.000132 \\ -1.008348 \triangle x_1 - 1.008348 \triangle x_2 + 0.004616 \triangle \lambda_2 = -0.004371 \\ 0.003864 \triangle x_1 + 0.497992 \triangle \lambda_2 = -0.001639 \\ 0.003864 \triangle x_2 + 0.497992 \triangle \lambda_3 = -0.001639 \end{cases}$$

By solving we get,

$\triangle x_1 = 0.0022, \triangle x_2 = 0.0022, \triangle \lambda_1 = -0.0075, \triangle \lambda_2 = -0.0033, \triangle \lambda_3 = -0.0033$

$\Delta l^{(3)} = (0.0022, 0.0022, -0.0075, -0.0033, -0.0033)$

Backtracking line search 4

Choose $\alpha = 0.2$, $\beta = 0.8$

so, $s^{max} = \min\left\{1, \min\left\{-\frac{1.008348}{-0.0075}, -\frac{0.003864}{-00.0033} - \frac{0.003864}{-0.0033}\right\}\right\} = \min\{1, 1.1709\} = 1.$ set $s = 0.99 \times s^{max} = 0.99$

And Compute $l^{(3)} + s\Delta l^{(3)} = (0.499869, 0.499869, 1.00932, 0.000579, 0.000579)$. Then $g_1 = -0.00026 < 0$, $g_2 = -0.499869 < 0$, $g_3 = -0.499869$. and $r_t(x^+, \lambda^+) = 0.000099$ On the other hand $|(1 - \alpha s) r_t(x, \lambda)| = 0.004854$

Since $r_t(x^+, \lambda^+) < |(1 - \alpha s) r_t(x, \lambda)|$; so, we have the next centering point is

$$l^{(4)} = l^{(3)} + s\Delta l^{(3)}$$

$$= (0.499869, 0.499869, 1.00932, 0.000579, 0.000579)$$

Stopping condition:

We see, $\eta = 0.0085 \nless \epsilon$ and $\|\nabla f(x) + Dg(x)^T \lambda\|_2 = 0.00009 < \varepsilon_{feas}$ and

$$\left\| -\mathrm{diag}(\lambda) g(x) - \frac{1}{t} e \right\|_2 = 0.0004 < \varepsilon_{feas}$$

Consequently, we continue more.

<u>5th centering step:</u>

$l^{(4)} = (0.499869, 0.499869, 1.00932, 0.000579, 0.000579)$

$$\eta = -g(x)^T \lambda = 0.000857 \text{ and } t = \frac{m\mu}{\eta} = 350001.03$$

Putting these values, we get the set of 5 equations

$$\begin{cases} 2\triangle x_1 + \triangle \lambda_1 - \triangle \lambda_2 = -0.000066 \\ 2\triangle x_2 + \triangle \lambda_1 - \triangle \lambda_3 = -0.000066 \\ -1.00932 \triangle x_1 - 1.00932 \triangle x_2 + 0.00026 \triangle \lambda_2 = -0.000232 \\ 0.000579 \triangle x_1 + 0.499869 \triangle \lambda_2 = -0.000269 \\ 0.000579 \triangle x_2 + 0.499869 \triangle \lambda_3 = -0.000269 \end{cases}$$

By solving we get,

$\triangle x_1 = 0.000116, \triangle x_2 = 0.000116, \triangle \lambda_1 = -0.000838, \triangle \lambda_2 = -0.0166,$
$\triangle \lambda_3 = -0.0166$

$\Delta l^{(4)} = (0.000116, 0.000116, -0.000838, -0.00054, -0.00054)$

Backtracking line search 5: Choose $\alpha = 0.2,\ \beta = 0.8$

so, $s^{max} = \min\left\{1, \min\left\{-\frac{1.00932}{-0.000838}, -\frac{0.000579}{-0.00054} - \frac{0.000579}{-0.00054}\right\}\right\} = \min\{1,1.10556\} = 1.$ set $s = 0.99 \times s^{max} = 0.99$

And Compute $l^{(4)} + s\Delta l^{(4)} = (0.499984, 0.499984, 1.000094, 0.000062, 0.000062)$. Then $g_1 = -0.00003114 < 0,$ $g_2 = -0.499984 < 0,$ $g_3 = -0.499984.$ and $r_t(x^+, \lambda^+) = 0.000099$ On the other hand $|(1 - \alpha s) r_t(x, \lambda)| = 0.004854$

Since $r_t(x^+, \lambda^+) < |(1 - \alpha s) r_t(x, \lambda)|$; so, we have the next centering point is

$$l^{(4)} = l^{(4)} + s\Delta l^{(4)}$$
$$= (0.499984, 0.499984, 1.000094, 0.000062, 0.000062)$$

Stopping condition:

We see, $\eta = 0.000857 < \epsilon$ and $\|\nabla f(x) + Dg(x)^T \lambda\|_2 = 0.0000008 < \varepsilon_{feas}$

and $\left\| -\mathrm{diag}(\lambda) g(x) - \frac{1}{t} e \right\|_2 = 0.000004 < \varepsilon_{feas}$

Consequently, we will stop here.

Result:

$$(x_1, x_2) = (0.499984, 0.499984)$$

The optimum solution (with tolerance $\varepsilon = 0.001$) of the problem.

Table of Results:

Iteration no.	$x =$	$y =$	$f =$	Comment (stopping criteria)
0	0.3333333	0.3333333	0.888889	Continue
1	0.377209	0.377209	0.775537	Continue
2	0.478981	0.478981	0.542922	Continue
3	0.497992	0.497992	0.504024	Continue
4	0.499869	0.499869	0.500262	Continue
5	**0.499984**	**0.499984**	**0.500032**	**Optimum**

Graphical Views: (Using demos graphing calculator: [https://www.desmos.com/calculator])

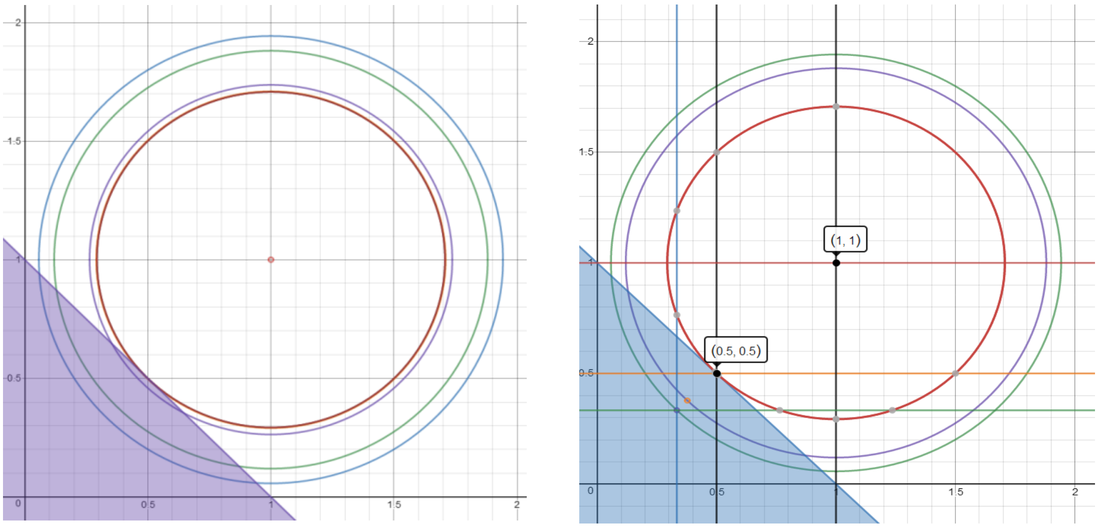

Fig-1(a &b): (a) Objective contour at the iterations (b) (initial: green, 1st: violet, …, optimum: red)

V. Conclusion

Convex programming Problem (CPP) can be considered an extension of linear programming that includes a wide variety of interesting nonlinear convex optimization problems. CPP has applications in such diverse fields as traditional convex constrained optimization, control theory, and combinatorial optimization. In this article work, we have reviewed the theory and algorithm of interior-point methods specially the primal-dual interior point method for convex programming (CPP) and give their satisfactory solutions by our own.

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