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Research Paper

The Area Enclosed by a Plane Fractional Closed Curve

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ABSTRACT: *Based on Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, this paper studies the calculation of the area surrounded by a plane fractional closed curve. The major method we used is the change of variables for fractional calculus. On the other hand, several examples are provided to illustrate how to solve this problem. In fact, these results obtained in this paper are generalizations of the results in classical calculus.*
KEYWORDS: *Jumarie's Modified R-L Fractional Calculus, New Multiplication, Fractional Analytic Functions, Plane Fractional Closed Curve, Change of Variables for Fractional Calculus*

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I. INTRODUCTION

In 1695, the fractional derivative first appeared in a famous letter between L'Hospital and Leibniz. Fractional calculus includes the derivative and integral of any real or complex order. Fractional calculus has attracted many physicists, engineers, scientists, and mathematicians to do this research. In recent decades, fractional calculus has been widely used in many fields such as physics, mechanics, electricity, economics, control theory, and so on [1-8]. For the introduction and application of fractional calculus, we can refer to [9-13].

In this paper, based on Jumarie modification of R-L fractional calculus and a new multiplication of fractional analytic functions, the problem of finding the area surrounded by a plane fractional closed curve is studied. The change of variables for fractional calculus plays an important role in this article. Moreover, we give some examples to illustrate the method to solve this problem. In addition, these results we obtained are natural generalizations of the results in traditional calculus.

II. PRELIMINARIES

In the following, we introduce the fractional calculus used in this paper.

Definition 2.1 ([14])**:** Assume that $0<\alpha\leq 1$, and x_0 is a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt. \tag{1}$$

And the Jumarie type of R-L α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt, \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

Proposition 2.2 ([15])**:** *Suppose that* α,β,x_0,C *are real numbers and* $\beta\geq\alpha>0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C]=0. \tag{4}$$

Next, the definition of fractional analytic function is introduced.

Definition 2.3 ([16])**:** Suppose that x,x_0, and a_k are real numbers for all k, $x_0\in(a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as $f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$, an α-fractional power series on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every

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point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([17])**:** If $0<\alpha\le 1$, and x_0 is a real number. Let $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ be two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes k}, \tag{5}$$

$$g_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes k}. \tag{6}$$

Then we define

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\otimes\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\\&=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(x-x_0)^{k\alpha}.\end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes k}\otimes\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes k}\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes k}.\end{aligned} \tag{8}$$

Definition 2.5 ([18])**:** Let $0<\alpha\le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes k}, \tag{9}$$

$$g_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes k}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha\left(g_\alpha(x^\alpha)\right)=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(g_\alpha(x^\alpha)\right)^{\otimes k}, \tag{11}$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha\left(f_\alpha(x^\alpha)\right)=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(f_\alpha(x^\alpha)\right)^{\otimes k}. \tag{12}$$

Definition 2.6 ([18])**:** Let $0<\alpha\le 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha\circ g_\alpha)(x^\alpha)=(g_\alpha\circ f_\alpha)(x^\alpha)=\frac{1}{\Gamma(\alpha+1)}x^\alpha. \tag{13}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

The following are some fractional analytic functions.

Definition 2.7([18])**:** If $0<\alpha\le 1$, and x, x_0 are real numbers. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(k\alpha+1)}=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{14}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. Moreover, the α-fractional the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2k}, \tag{15}$$

and

$$sin_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(2k+1)}. \tag{16}$$

Definition 2.8: The smallest positive real number T_α such that $E_\alpha(iT_\alpha)=1$, is called the period of $E_\alpha(ix^\alpha)$.

Definition 2.9 [18]**:** Let $0<\alpha\le 1$, and r be a real number. The r-th power of the α-fractional analytic function $f_\alpha(x^\alpha)$ is defined by $[f_\alpha(x^\alpha)]^{\otimes r}=E_\alpha\left(rLn_\alpha\left(f_\alpha(x^\alpha)\right)\right)$.

III. METHODS AND EXAMPLES

In this section, we introduce the main methods used in this paper and give some examples to illustrate how to evaluate the area enclosed the plane fractional closed curve.

Theorem 3.1 (change of variables for fractional calculus)[ ]**:** *Let $0<\alpha\le 1$, $u_\alpha(x^\alpha)$ is an α-fractional analytic function defined on an interval I, and $f_\alpha\left(u_\alpha(x^\alpha)\right)$ is an α-fractional analytic function such that the range of u_α contained in the domain of f_α, then*

$$\left({}_{c}I_{d}^{\alpha}\right)\left[f_{\alpha}\left(u_{\alpha}(x^{\alpha})\right)\otimes\left({}_{c}D_{x}^{\alpha}\right)[u_{\alpha}(x^{\alpha})]\right]=\left({}_{u_{\alpha}(c^{\alpha})}I_{u_{\alpha}(d^{\alpha})}^{\alpha}\right)[f_{\alpha}(u_{\alpha})], \tag{17}$$

for $c, d \in I$.

Definition 3.2: If $0<\alpha\leq 1$, $f_{\alpha}(x^{\alpha})$ is an α-fractional analytic function defined on an interval I, and $f_{\alpha}(x^{\alpha})\geq 0$, then the area under $f_{\alpha}(x^{\alpha})$ from $x=c$ to $x=d$ is defined by

$$\left({}_{c}I_{d}^{\alpha}\right)[f_{\alpha}(x^{\alpha})]. \tag{18}$$

Definition 3.3: Assume that $0<\alpha\leq 1$, $\rho_{\alpha}(\theta^{\alpha})$ is the polar coordinate equation of a plane α-fractional analytic closed curve, then the area under $\rho_{\alpha}(\theta^{\alpha})$ from $\theta=\lambda$ to $\theta=\mu$ is defined by

$$\left({}_{\lambda}I_{\mu}^{\alpha}\right)\left[\frac{1}{2}\left(\rho_{\alpha}(\theta^{\alpha})\right)^{\otimes 2}\right]. \tag{19}$$

Some examples are provided below to illustrate the method of calculating the area surrounded by a plane fractional closed curve.

Example 3.4: Suppose that $0<\alpha\leq 1$, $p, q>0$. The parametric equation of the α-fractional elliptic curve is

$$\begin{cases} x_{\alpha}(t^{\alpha})=p\cdot sin_{\alpha}(t^{\alpha}) \\ y_{\alpha}(t^{\alpha})=q\cdot cos_{\alpha}(t^{\alpha}) \end{cases},\ 0\leq t^{\alpha}\leq T_{\alpha}. \tag{20}$$

Find the the area enclosed by this plane α-fractional closed curve.

Solution Since this α-fractional elliptic curve satisfies

$$\frac{[x_{\alpha}(t^{\alpha})]^{\otimes 2}}{p^2}+\frac{[y_{\alpha}(t^{\alpha})]^{\otimes 2}}{q^2}=1. \tag{21}$$

It follows that

$$y_{\alpha}\left(x_{\alpha}(t^{\alpha})\right)=\left(q^2-\left(\frac{q}{p}\right)^2[x_{\alpha}(t^{\alpha})]^{\otimes 2}\right)^{\otimes\frac{1}{2}}\ \text{ for } 0\leq t^{\alpha}\leq\frac{T_{\alpha}}{4}. \tag{22}$$

Then by change of variable for fractional calculus, the area enclosed by this plane α-fractional closed curve is

$$\begin{aligned}
&4\cdot\left({}_{x=0}I_{x=p}^{\alpha}\right)\left[y_{\alpha}(x_{\alpha})\right]\\
&=4\cdot\left({}_{x=0}I_{x=p}^{\alpha}\right)\left[\left(q^2-\left(\frac{q}{p}\right)^2[x_{\alpha}]^{\otimes 2}\right)^{\otimes\frac{1}{2}}\right]\\
&=4\cdot\left({}_{t=0}I_{t=\left(\frac{T_{\alpha}}{4}\right)^{\frac{1}{\alpha}}}^{\alpha}\right)\left[\left(q^2-\left(\frac{q}{p}\right)^2[p\cdot sin_{\alpha}(t^{\alpha})]^{\otimes 2}\right)^{\otimes\frac{1}{2}}\otimes\left({}_{0}D_{t}^{\alpha}\right)[p\cdot sin_{\alpha}(t^{\alpha})]\right]\\
&=4\cdot\left({}_{0}I_{\left(\frac{T_{\alpha}}{4}\right)^{\frac{1}{\alpha}}}^{\alpha}\right)\left[q\cdot cos_{\alpha}(t^{\alpha})\otimes\left(p\cdot cos_{\alpha}(t^{\alpha})\right)\right]\\
&=4pq\cdot\left({}_{0}I_{\left(\frac{T_{\alpha}}{4}\right)^{\frac{1}{\alpha}}}^{\alpha}\right)\left[\left(cos_{\alpha}(t^{\alpha})\right)^{\otimes 2}\right]\\
&=4pq\cdot\left({}_{0}I_{\left(\frac{T_{\alpha}}{4}\right)^{\frac{1}{\alpha}}}^{\alpha}\right)\left[\frac{1}{2}+\frac{1}{2}cos_{\alpha}(2t^{\alpha})\right]\\
&=4pq\cdot\left(\frac{1}{2}\cdot\frac{1}{\Gamma(\alpha+1)}\cdot\frac{T_{\alpha}}{4}+\frac{1}{4}sin_{\alpha}\left(\frac{T_{\alpha}}{2}\right)\right).\\
&=\frac{1}{\Gamma(\alpha+1)}\cdot\frac{T_{\alpha}}{2}pq.
\end{aligned} \tag{23}$$

Remark 3.5: If $\alpha=1$, then the fractional elliptic curve becomes the classical elliptic curve, and its area is πpq.

Example 3.6: Let $0<\alpha\leq 1, p>0$. The polar coordinate equation of the α-fractional cardioid is

$$\rho_{\alpha}(\theta^{\alpha})=p\left(1+cos_{\alpha}(\theta^{\alpha})\right)\ \text{ for } 0\leq\theta^{\alpha}\leq T_{\alpha}. \tag{24}$$

Find the area enclosed by this plane α-fractional curve.

Solution By Definition 3.4, the area is

$$2\cdot\left({}_{0}I_{\left(\frac{T_{\alpha}}{2}\right)^{\frac{1}{\alpha}}}^{\alpha}\right)\left[\frac{1}{2}\left(p\left(1+cos_{\alpha}(\theta^{\alpha})\right)\right)^{\otimes 2}\right]$$

$$= 2 \cdot \left({}_{0}I^{\alpha}_{\left(\frac{T_\alpha}{2}\right)^{\frac{1}{\alpha}}} \right) \left[\frac{p^2}{2} \left(\left(\cos_\alpha(\theta^\alpha) \right)^{\otimes 2} + 2 \cdot \cos_\alpha(\theta^\alpha) + 1 \right) \right]$$
$$= p^2 \cdot \left({}_{0}I^{\alpha}_{\left(\frac{T_\alpha}{2}\right)^{\frac{1}{\alpha}}} \right) \left[\left(\frac{1}{2} + \frac{1}{2} \cos_\alpha(2\theta^\alpha) \right) + 2 \cdot \cos_\alpha(\theta^\alpha) + 1 \right]$$
$$= p^2 \cdot \left({}_{0}I^{\alpha}_{\left(\frac{T_\alpha}{2}\right)^{\frac{1}{\alpha}}} \right) \left[\frac{1}{2} \cos_\alpha(2\theta^\alpha) + 2 \cdot \cos_\alpha(\theta^\alpha) + \frac{3}{2} \right]$$
$$= p^2 \cdot \left[\frac{1}{4} \cdot \sin_\alpha(T_\alpha) + 2 \cdot \sin_\alpha\left(\frac{T_\alpha}{2} \right) + \frac{3}{2} \cdot \frac{1}{\Gamma(\alpha+1)} \cdot \frac{T_\alpha}{2} \right]$$
$$= \frac{3}{4} p^2 \cdot \frac{1}{\Gamma(\alpha+1)} \cdot T_\alpha \ . \tag{25}$$

Remark 3.7: If $\alpha = 1$, then the fractional cardioid is the traditional cardioid, and the area enclosed is $\frac{3}{2}\pi p^2$.

Example 3.8: Suppose that $0 < \alpha \leq 1, q$ is a real number. The polar coordinate equation of the α-fractional lemniscate is

$$[\rho_\alpha(\theta^\alpha)]^{\otimes 2} = q^2 \cdot \cos_\alpha(2\theta^\alpha). \tag{26}$$

Find the area enclosed by this plane α-fractional curve.

Solution Using Definition 3.4, the area is

$$4 \cdot \left({}_{0}I^{\alpha}_{\left(\frac{T_\alpha}{8}\right)^{\frac{1}{\alpha}}} \right) \left[\frac{1}{2} q^2 \cdot \cos_\alpha(2\theta^\alpha) \right]$$
$$= 2q^2 \cdot \left[\frac{1}{2} \cdot \sin_\alpha\left(\frac{T_\alpha}{4} \right) \right]$$
$$= q^2 \cdot \sin_\alpha\left(\frac{T_\alpha}{4} \right). \tag{27}$$

Remark 3.9: If $\alpha = 1$, then the fractional lemniscate is the classical lemniscate, and the area enclosed is q^2.

IV. CONCLUSION

The main purpose of this paper is to evaluate the area enclosed by a plane fractional closed curve based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions. The change of variables for fractional calculus plays an important role in this article. In fact, these results we obtained are generalizations of those in traditional calculus. In the future, we will continue to use these methods to study problems in other fields such as fractional calculus and engineering mathematics.

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