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Research Paper

Lie Symmetry Solution of Bernoulli Differential Equation of First Order

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ABSTRACT: *Bernoulli equation is of importance in different fields of science and engineering. It has its applications in areas like fluid dynamics and weather forecasting among others. In some cases, higher order non-linear differential equations are reduced to Bernoulli differential equation of first order. There are different methods of finding general solution of this equation which is non-linear in nature. Lie symmetry method is one of those methods which are used to find general solution of differential equations of any order by finding lie symmetries. Lie symmetries are the solutions of Lie invariance equation. In this paper, this method is used to solve Bernoulli differential equation of first order.*

KEYWORDS: *Lie Symmetry Method, Similarity Solutions, Bernoulli ODE, Lie Invariance*

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I. INTRODUCTION

In mathematical modelling, differential equations have played a vital role to describe various phenomenon of different walks of life. Such models contain differential equations of various order. The equations could be linear and non-linear. Mathematicians and Scientists have devised different methods of solving such equations whether ordinary or partial, linear or non-linear. Bernoulli differential equation of first order is a non-linear equation arising in solutions of problems in different fields like fluid dynamics, circuit theory, weather forecasting among others. In 19^{th} Century, a Norwegian Mathematician, Sophus Lie obtained symmetries of differential equations to find their general solutions. Symmetries of differential equations are nothing but the transformations of both the dependent and independent variables. These transformations also include the derivatives of the dependent variable. Lie used these transformations to reduce the order of any differential equation and then obtained its general solution. This method is also used to transform first order differential equation whether linear or non-linear to canonical form. This form is then solved by usual method to find its general solution. In this work, Lie symmetries are obtained for Bernoulli differential equation and is then transformed to canonical form so as to find its general solution.

II. LITERATURE REVIEW

Olver and Rosenau (1987) studied group invariant solutions of differential equations. **Dresner (1988)** presented similarity solutions of second-order partial differential equations in one dependent and two independent variables**. Bluman (1990)** solved algebraic equation to find invariant solution for a differential equation. A new class of symmetries were introduced by **Muriel and Romero (2001)** based on Lie symmetries to reduce the order of ordinary differential equation. **Craddock and Platen (2004)** used Lie symmetry group methods to study partial differential equations. **Moitsheki (2007)** constructed invariant solutions using Lie Point and Potential symmetries. **Zachary Martinot (2014)** explained this method by reduction of order of a differential equation using symmetry transformations which are part of the theory of Lie groups. **Manjit Singh (2015)** reduced the order of ordinary differential equations using Lie group of transformations. He showed that for an ordinary differential equation admitting one parameter Lie group symmetry, order of differential equation, in principle, can always be reduce by one. **Muhsen and Maan (2016)** applied Lie group analysis to second order Neutral Delay Diffrential Equations (NDDE). **Hu and Du (2019)** studied first integrals of a second order ordinary differential equations.

III. LIE SYMMETRY METHOD

In this section, Lie symmetry method to solve first order ordinary differential equation is presented.
Let us consider general form of a first order ODE as –

$$\frac{dx}{dt} = G(t, x), \qquad (1)$$

where *x* dependent and t is independent variable.
To solve (1) , we shall consider Lie group in one parameter as

$$\bar{t} = f(t, x; \varepsilon), \ \bar{x} = g(t, x; \varepsilon). \qquad (2)$$

We assume that (1) is invariant under (2), so that we have –

$$\frac{d\bar{x}}{d\bar{t}} = G(\bar{t}, \bar{x}), \qquad (3)$$

Solving (3), we get

$$\frac{g_t + g_x \mathrm{G}(t, x)}{f_t + f_x \mathrm{G}(t, x)} = G\left(f\left(t, x, \varepsilon\right), g(t, x, \varepsilon)\right). \qquad (4)$$

Now we need to find transformations/infinitesimal. For this we use Lie's invariance condition

$$X_t + (X_x - T_t)G - T_x G^2 = TG_t + XG_x \qquad (5)$$

The infinitesimals *X* and *T* are given by the solutions of equation (5). Now to solve (5) we assume some form of one infinitesimal, say *T*. The other infinitesimal *X* is then obtained by using Lagrange's method for partial differential equation of Ist order.
From *X* and *T*, we transform (1) to canonical coordinates *(r, s)* using -

$$r_t T + r_x X = 0 \qquad (6)$$

$$s_t T + s_x X = 1 \qquad (7)$$

From Lagrange's method of characteristics, we find *r* and *s* from (6) and (7). We then obtain -

$$\frac{ds}{dr} = \frac{s_t + s_x \frac{dx}{dt}}{r_t + r_x \frac{dx}{dt}} \qquad (8)$$

Equation (8) is then to find the general solution of equation (1).

IV. EXAMPLES

In this section, Lie symmetry method for Bernoulli ODE of Ist order is explained with the help of examples.
Example 4.1 Consider first order ODE -

$$\frac{dx}{dt} = 2x + tx^2 \qquad (9)$$

Here *x* is dependent and *t* is independent variable. Equation (9) is a linear ODE in dependent variable *x*.

Comparing equation(9) with $\frac{dx}{dt} = G(t, x)$, we get

$$G(t, x) = 2x + tx^2 \qquad (10)$$

Lie invariance condition is given by

$$X_t + (X_x - T_t)G - T_x G^2 = TG_t + XG_x \qquad (11)$$

where X and T are infinitesimals to be obtained.
From equation (10), we get

$$G_x = 2 + 2tx, G_t = x^2 \qquad (12)$$

Putting these values in Lie's invariance condition (11), we get -

$$X_t + (X_x - T_t)(2x + tx^2) - T_x(4x^2 + t^2x^4 + 4x^3t) = Tx^2 + X(2 + 2tx) \qquad (13)$$

Here we assume:

$$T = 0 \qquad (14)$$

Putting this in equation (13), we get -

$$X_t + X_x(2x + tx^2) = 2X(1 + tx) \tag{15}$$

We shall use Lagrange's method to solve (15). Lagrange's auxiliary equations are

$$\frac{dt}{1} = \frac{dx}{2x + tx^2} = \frac{dX}{2(1 + tx)X} \tag{16}$$

$(i) \qquad (ii) \qquad (iii)$

Using (ii) and (iii), we get –

$$\frac{dx}{x(2 + tx)} = \frac{dX}{2(1 + tx)X} \tag{17}$$

Take $(\frac{2}{x})$ and $(\frac{-1}{X})$ as multipliers in (17). Then each ratio in (9)

$$= \frac{\frac{2}{x}dx - \frac{1}{X}dX}{4 + 2tx - 2 - 2tx} = \frac{\frac{2}{x}dx - \frac{1}{X}dX}{2}$$

Combining (16) and (17), we get –

$$\frac{\frac{2}{x}dx - \frac{1}{X}dX}{2} = \frac{dt}{1} \tag{18}$$

$$\frac{2}{x}dx - \frac{1}{X}dX = 2dt$$

Integrating (18), we get –

$$2\log x - \log X + \log c_1 = 2t$$

$$\log\frac{c_1x^2}{X} = 2t$$

$$\frac{c_1x^2}{X} = e^{2t}$$

$$X = c_1x^2e^{-2t} \tag{19}$$

Taking $c_1=1$, the infinitesimals X and T are given by

$$(T, X) = (0, x^2e^{-2t}) \tag{20}$$

Now we find canonical co-ordinates r and s .

We use following equations to find r and s

$$r_tT + r_xX = 0 \tag{21}$$

$$s_tT + s_xX = 1 \tag{22}$$

Since $T = 0$, therefore, we have from (21) -

$$r_x = 0 \Rightarrow r = t \tag{23}$$

From equation (22), we get

$$s_t.0 + s_x(x^2e^{-2t}) = 1$$

$$s_x = \frac{e^{2t}}{x^2} \tag{24}$$

Integrating (24) w.r.t. '*x'* treating '*t'* as constant, we get -

$$s = e^{2t}\int\frac{1}{x^2}dx$$

$$s = -\frac{e^{2t}}{x} \tag{25}$$

Therefore, canonical co-ordinates are –

$$(r, s) = (t, \frac{-e^{2t}}{x}) \tag{26}$$

Now substituting $r \ and \ s$ into

$$\frac{ds}{dr} = \frac{s_t + s_x \frac{dx}{dt}}{r_t + r_x \frac{dx}{dt}}$$

We get

$$\frac{ds}{dr} = \frac{-\frac{2}{x}e^{2t} + \frac{1}{x^2}e^{2t}(2x + tx^2)}{1 + 0}$$

$$= -\frac{2}{x}e^{2t} + \frac{2}{x}e^{2t} + te^{2t}$$

$$\frac{ds}{dr} = te^{2t} \tag{27}$$

Since from equation (23), $t = r$
Therefore, from (27), we get

$$\frac{ds}{dr} = re^{2r}$$

$$ds = re^{2r}dr$$

Integrating

$$s = \int re^{2r}dr + c$$

where c is the constant of integration.

$$s = \frac{re^{2r}}{2} - 1.\frac{e^{2r}}{4} + c$$

$$s = \frac{1}{4}e^{2r}(2r - 1) + c \tag{28}$$

Using equation (23) and (25), we get

$$r = t, \ s = -\frac{e^{2t}}{x}$$

Substituting in equation (28), we get

$$\frac{-1}{x}e^{2t} = \frac{1}{4}(2t - 1)e^{2t} + c$$

$$\frac{-1}{x} = \frac{1}{4}(2t - 1) + ce^{-2t}$$

$$\frac{1}{x} = \frac{1}{4}(1 - 2t) - ce^{-2t} \tag{29}$$

which is the general solution of given equation (9).

Example 4.2 Consider first order ODE -

$$\frac{dx}{dt} = x - tx^3 \tag{30}$$

Here x is dependent and t is independent variable. Equation (30) is Bernoulli equation in dependent variable x.

Comparing equation (30) with $\frac{dx}{dt} = G(t, x)$, we get

$$G(t, x) = x - tx^3 \tag{31}$$

Lie invariance condition is given by

$$X_t + (X_x - T_t)G - T_x G^2 = TG_t + XG_x \tag{32}$$

where X and T are infinitesimals to be found.

From equation (31), we get

$$G_x = 1 - 3tx^2, G_t = -x^3 \tag{33}$$

Putting these values in Lie's invariance condition (32), we get -

$$X_t + (X_x - T_t)(x - tx^3) - T_x(x^2 + t^2x^6 - 4x^4t) = -Tx^3 + X(1 - 3tx^2) \tag{34}$$

Here we assume:

$$T = 0 \tag{35}$$

Putting this in equation (34), we get -

$$X_t + X_x(x - tx^3) = X(1 - 3tx^2) \tag{36}$$

We shall use Lagrange's method to solve (36). Lagrange's auxiliary equations are

$$\frac{dt}{1} = \frac{dx}{(x - tx^3)} = \frac{dX}{(1 - 3tx^2)X} \tag{37}$$

(i) (ii) (iii)

Using (ii) and (iii), we get –

$$\frac{dx}{x(1 - tx^2)} = \frac{dX}{(1 - 3tx^2)X} \tag{38}$$

Take $(\frac{3}{x})$ and $(\frac{-1}{X})$ as multipliers in (38). Then each ratio in (37)

$$= \frac{\frac{3}{x}dx - \frac{1}{X}dX}{3 - 3tx^2 - 1 + 3tx^2} = \frac{\frac{3}{x}dx - \frac{1}{X}dX}{2} \tag{39}$$

Combining (37) and (39), we get –

$$\frac{\frac{3}{x}dx - \frac{1}{X}dX}{2} = \frac{dt}{1} \tag{40}$$

$$\frac{3}{x}dx - \frac{1}{X}dX = 2dt$$

Integrating (41), we get –

$$3\log x - \log X + \log c_1 = 2t$$

$$\log \frac{c_1x^3}{X} = 2t$$

$$\frac{c_1x^3}{X} = e^{2t}$$

$$X = c_1x^3e^{-2t} \tag{42}$$

Taking $c_1=1$, the infinitesimals X and T are given by

$$(T, X) = (0, x^3e^{-2t}) \tag{43}$$

Now to find canonical co-ordinates r and s we use following equations

$$r_tT + r_xX = 0 \tag{44}$$

$$s_tT + s_xX = 1 \tag{45}$$

Since $T = 0$, therefore, we have from (44) -

$$r_x = 0 \quad \Rightarrow \quad r = t \tag{46}$$

From equation (45), we get

$$s_t.0 + s_x(x^3 e^{-2t}) = 1$$

$$s_x = \frac{e^{2t}}{x^3} \tag{47}$$

Integrating (47) w.r.t. '*x*' treating '*t*' as constant, we get -

$$s = e^{2t}\int \frac{1}{x^3}dx$$

$$s = -\frac{e^{2t}}{2x^2} \tag{48}$$

Therefore, canonical co-ordinates are –

$$(r, s) = (t, \frac{-e^{2t}}{2x^2}) \tag{49}$$

Now substituting $r \ and \ s$ into

$$\frac{ds}{dr} = \frac{s_t + s_x \frac{dx}{dt}}{r_t + r_x \frac{dx}{dt}}$$

We get

$$\frac{ds}{dr} = \frac{-\frac{1}{x^2}e^{2t} + \frac{1}{x^3}e^{2t}(x - tx^3)}{1 + 0}$$

$$= -\frac{1}{x^2}e^{2t} + \frac{1}{x^2}e^{2t} - te^{2t}$$

$$\frac{ds}{dr} = -te^{2t} \tag{50}$$

Since from equation (46), $t = r$

Therefore, from (50), we get

$$\frac{ds}{dr} = -re^{2r}$$

$$ds = -re^{2r}dr$$

Integrating

$$s = -\int re^{2r}dr + c$$

where *c* is the constant of integration.

$$s = -\frac{re^{2r}}{2} + 1.\frac{e^{2r}}{4} + c$$

$$s = \frac{1}{4}e^{2r}(1 - 2r) + c \tag{51}$$

Using equation (46) and (48), we get

$$r = t, \ s = -\frac{e^{2t}}{2x^2}$$

Substituting in equation (51), we get

$$\frac{-1}{2x^2}e^{2t} = \frac{1}{4}(1-2t)e^{2t} + c$$

$$\frac{-1}{2x^2} = \frac{1}{4}(1-2t) + ce^{-2t}$$

$$\frac{1}{x^2} = \frac{1}{2}(2t-1) - 2ce^{-2t} \qquad (29)$$

which is the general solution of given equation (30).

V. CONCLUSION

Lie symmetry method solves differential equation of any order with a systematic algorithm. This is an advantage of this method over other methods. In case of higher order differential equations whether ordinary or partial, this method helps to reduce the order of the differential equation so that the resultant equation can be easily integrated. In this work, this method is shown to find general solution of Bernoulli differential equation with the help of examples. The method finds exact general solution of the same.

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