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Research Paper

Where are the Integers?

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Abstract

Various sequences such as $S_n = \frac{n}{1}, \frac{n-1}{2}, \frac{n-2}{3}, \ldots$ *and* $T_n = \frac{n}{1}, \frac{n+1}{2}, \frac{n+2}{3}, \frac{n+3}{4}, \cdots$ *are studied. Which terms are integers? Is 1 a term? How many terms are integers?*

Key terms: *divisor;integer; mod; sequence*

Let n be a positive integer such that $n > 1$. Consider the sequence, $S_n = \frac{n}{1}, \frac{n-1}{2}, \frac{n-2}{3}, \ldots$ The sequence decreases, and weonly include terms ≥ 1.(If we did not terminate the sequence, then the (n+1)-st term of $S_n$would be 0.)

Example: $S_{11} = \frac{11}{1}, \frac{10}{2}, \frac{9}{3}, \frac{8}{4}, \frac{7}{5}, \frac{6}{6}$, all integers, except for $\frac{7}{5}$. In stark contrast, $S_{12} = \frac{12}{1}, \frac{11}{2}, \frac{10}{3}, \frac{9}{4}, \frac{8}{5}, \frac{7}{6}$, containing no integers except for $\frac{12}{1}$.

Fact:1 is a term of $S_n$if and only if n is odd.

Proof: When n is odd, the difference between the numerator, n, and denominator, 1, of the initial term is even. Since these differences decrease by 2 each time we subtract 1 from the numerator and add 1 to the denominator, the difference between the numerator and denominator remains even till it reaches 0, at which point the numerator equals the denominator, thereby yielding 1. When n is even, the differences between numerators and denominators are odd, in which case no term can equal 1. ■

For a given integer, m, satisfying $1 \leq m < n$, when does one of the members of S_n equal m? To answer this question, represent the sequence members by $\frac{n-x+1}{x} = \frac{n+1-x}{x} = \frac{n+1}{x} - 1$, where x is a positive integer, starting with 1, that identifies the location of the term, $\frac{n-x+1}{x}$, in S_n. That is, $\frac{n-x+1}{x}$ is the x-th term of S_n. If a sequence member equals m, there will be a corresponding x, such that $m = \frac{n+1}{x} - 1$is an integer, or, equivalently, $\frac{n+1}{x}$is an integer, that is, x must be a divisor of n+ 1.Noting that $x = \frac{n+1}{m+1}$, we have the following fact.

Fact 1: A sequence member of $S_n$equals the positive integer, m, if and only if $m + 1 | n + 1$, or, equivalently,$n + 1 = 0 \bmod (m + 1)$.■

Examples: When $m = 1$, we have$n + 1 = 0 \bmod 2$, implying that n is odd. When $m = 2$, we have $n + 1 = 0 \bmod 3$, implying that $n = 2 \bmod 3$. When $m = 3$, this becomes $n + 1 = 0 \bmod 4$, implying that $n = 3 \bmod 4$, etc.

Remark 1: $\frac{n-x+1}{x}$, is an integer if and only if $x|n-x+1$, that is, if and only if $x|\ n+1$. Then x is not a divisor of n. Note, further, that if $\frac{n-x+1}{x}$ can be reduced by cancelling a common divisor, d, then d does not divide n.

Remark 2: Let $n = p - 1$, where p is an odd prime. Then the only integer-valued term of $S_n$is the initial term, $\frac{n}{1}$, as is exemplified by $S_{100} = \frac{100}{1},\frac{99}{2},\frac{98}{3},\frac{97}{4},\cdots,\frac{52}{49},\frac{51}{50}$, where$p = 101$.

Remark 3: Since $\lim_{x\to\infty}\frac{n-x+1}{x}=-1$ and the terms decrease monotonically, no term of S_n, if we extend it indefinitely, equals -1.

Fact2: Consider the sequence that begins with

$$S_{n!-1}=\frac{n!-1}{1},\frac{n!-2}{2},\frac{n!-3}{3},\ldots,\frac{n!-n}{n},\ldots \qquad (*)$$

Then if $k = 1, 2, 3, \ldots$, or n, then the k-th term is $\frac{n!-k}{k}=\frac{n!}{k}-1$ which is an integer. Then for any positive integer, n, (*) is a sequence whose first n members are integers. ■

Example: $6! - 1 = 719$. The first six members of S_{719} are $\frac{719}{1},\frac{718}{2},\frac{717}{3},\frac{716}{4},\frac{715}{5},\frac{714}{6}$.

The sequence, $S_{101} = \frac{101}{1},\frac{100}{2},\frac{99}{3},\frac{98}{4},\cdots,\frac{52}{50},\frac{51}{51}$, contains the integers,

$$\frac{101}{1}=101,\frac{100}{2}=50,\frac{99}{3}=33,\frac{96}{6}=16,\frac{85}{17}=5,\frac{68}{34}=2,\frac{51}{51}=1.$$

Let us analyze S_{101} be rewriting the terms as

$$\frac{102-1}{1},\frac{102-2}{2},\frac{102-3}{3},\frac{102-4}{4},\cdots,\frac{102-50}{50},\frac{102-51}{51}$$

which become

$$\frac{102}{1}-1,\frac{102}{2}-1,\frac{102}{3}-1,\frac{102}{4}-1,\cdots,\frac{102}{50}-1,\frac{102}{51}-1$$

These terms will be integers if and only if the fractions are integers. Recalling (see [1]) that the number of divisors of n is given by $\tau(n)$, the number of integers in S_{101} is $\tau(102) - 1 = 8 - 1 = 7$. We subtract 1 since the denominator cannot be102. More generally, we have the following.

Fact 3: The number of integer-valued terms in $S_n$is $\tau(n + 1) - 1$.■

Examples: The number of integers in $S_{100}$is $\tau(101) - 1 = 2 - 1 = 1$. The number of integers in $S_7$is $\tau(8) - 1 = 4 - 1 = 3$. The integers in S_7 are $\frac{7}{1},\frac{6}{2},\frac{4}{4}$.The number of integers in $S_{32}$is $\tau(33) - 1 = 4 - 1 = 3$. The integers in S_{32} are $\frac{32}{1},\frac{30}{3},\frac{22}{11}$. Their denominators are precisely the proper divisors of 33.

Fact 4: Let the odd integer $n = 2k - 1$. Then $S_n=\frac{2k-1}{1},\frac{2k-2}{2},\frac{2k-3}{3},\cdots,\frac{2k-k}{k}=1$. Let $f(k)$ be the number of terms that are not fully reduced. Then $f(k)\geq\left\lceil\frac{k}{2}\right\rceil$. ■

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Examples: When n = 11, k = 6. Then S_{11} = $\frac{11}{1},\frac{10}{2},\frac{9}{3},\frac{8}{4},\frac{7}{5},\frac{6}{6}$ and $\left\lceil\frac{6}{2}\right\rceil = 3$. Since four terms of S_{11}, that is, $\frac{10}{2},\frac{9}{3},\frac{8}{4},\frac{6}{6}$, are not fully reduced, we have $f(6) \geq \left\lceil\frac{6}{2}\right\rceil$. When n = 13, k = 7. Then S_{13} = $\frac{13}{1},\frac{12}{2},\frac{11}{3},\frac{10}{4},\frac{9}{5},\frac{8}{6},\frac{7}{7}$ and $\left\lceil\frac{7}{2}\right\rceil = 4$. Since four terms of S_{13}, that is, $\frac{12}{2},\frac{10}{4},\frac{8}{6},\frac{7}{7}$, are not fully reduced, we have $f(7) \geq \left\lceil\frac{7}{2}\right\rceil$.

We turn our attention to the sequence, T_n = $\frac{n}{1},\frac{n+1}{2},\frac{n+2}{3},\frac{n+3}{4},\cdots$, for $n > 1$. Letting $f(x) = \frac{n+x-1}{x}$, we have $f'(x) = \frac{(x)(1)-(n+x-1)(1)}{x^2} = -\frac{n-1}{x^2}$, implying that T_n is a decreasing sequence. Furthermore, $\lim_{x\to\infty}\frac{n+x-1}{x} = 1$, so 1 is never a term of T_n.

Example: T_7 = $\frac{7}{1},\frac{8}{2},\frac{9}{3},\frac{10}{4},\frac{11}{5},\frac{12}{6},\cdots$ yielding the integers, 2, 3, 4, and 7. Since 1 is not in T_n(for any n), the search for integers terminates after its first six terms.

For a given integer, m, satisfying $1 \leq m<n$, when does one of the members of T_n equal m? To answer this question, represent the sequence members by $\frac{n+x-1}{x} = \frac{n-1+x}{x} = \frac{n-1}{x}+1$, where x is a positive integer. If a sequence member equals m, there will be a corresponding x, such that $m = \frac{n-1}{x}+1$ must be an integer. Equivalently, $\frac{n-1}{x}$ must be an integer, that is, x must be a divisor of n– 1.

Fact 5: The number of integer-valued terms of T_n is $\tau(n-1) - 1$.■

It follows that if n – 1 is an odd prime, then the number of integer-valued terms of T_n is 1. Moreover, noting that $x = \frac{n-1}{m-1}$, we have the following fact.

Fact 6: A sequence member of $T_n$equals the positive integer, m, if and only if $m - 1 \mid n - 1$, or, equivalently, $n - 1 = 0 \bmod (m-1)$. ■

Fact7: Given the positive integers a and k, where $k<a$, let $P = a(a-1)(a-2)\ldots(a-k) + a$. Then we have the sequence of k + 1 *integers*, $\frac{P}{a},\frac{P-1}{a-1},\frac{P-2}{a-2},\frac{P-3}{a-3},\cdots,\frac{P-k}{a-k}$. ■ (**)

Example: Let a = 7 and let k = 3. Then $P = 7\cdot6\cdot5\cdot4 + 7 = 847$. We have the sequence of integers

$$\frac{847}{7}=121,\quad \frac{846}{6}=141,\quad \frac{845}{5}=169,\quad \frac{844}{4}=211$$

This sequence can be extended to include integers $\frac{843}{3}=281,\ \frac{842}{2}=421,\ \frac{841}{1}=841$, and we obtain the first seven members of T_{841} listed backwards. In fact, (**) may be continued to yield the first a members of T_{P-a+1}, listed backwards. The terms with which we augment our sequence may or not be integers, except for the last term, P–a+1.

Let $\mathbf{S}_{n-1} = \left\langle \frac{n-1}{1},\frac{n-2}{2},\ \ldots,\frac{n-n}{n}\right\rangle$ and $\mathbf{T}_{n-1} = \left\langle \frac{n+1}{1},\frac{n+2}{2},\ \ldots,\frac{n+n}{n}\right\rangle$. Then $\mathbf{S}_{n-1}\cdot\mathbf{T}_{n-1}$=

$$\left(\frac{n-1}{1}\right)\left(\frac{n+1}{1}\right)+\left(\frac{n-2}{2}\right)\left(\frac{n+2}{2}\right)+\ \ldots+\left(\frac{n-n}{n}\right)\left(\frac{n+n}{n}\right) = \frac{n^2-1}{1}+\frac{n^2-2^2}{2^2}+\ldots+\frac{n^2-n^2}{n^2} =$$

$$n^2\left(1+\frac{1}{2^2}+...+\frac{1}{n^2}\right)-n$$

Fact: $\mathbf{S}_{n-1}\cdot\mathbf{T}_{n-1}\sim\frac{\pi n^2}{6}$. (Two functions are asymptotic (~), if their limiting ratio is 1.)

Proof: $\mathbf{S}_{n-1}\cdot\mathbf{T}_{n-1}=n^2\left(1+\frac{1}{2^2}+...+\frac{1}{n^2}\right)-n$. Since $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$, the Fact follows. ■

Let n and d be a positive integers such that $n > 1$. Consider the sequence, $D_{n,d}=\frac{n}{1},\frac{n-d}{1+d},\frac{n-2d}{1+2d},\frac{n-3d}{1+3d},...$ The sequence clearly decreases, and we only include terms ≥ 1.

Examples: $D_{13,2}=\frac{13}{1},\frac{11}{3},\frac{9}{5},\frac{7}{7}$.$D_{15,2}=\frac{15}{1},\frac{13}{3},\frac{11}{5},\frac{9}{7}$. $D_{31,3}=\frac{31}{1},\frac{28}{4},\frac{25}{7},\frac{22}{10},\frac{19}{13},\frac{16}{16}$.

Fact 7: Since the k-th term of $D_{n,d}$ is $\frac{n-(k-1)d}{1+(k-1)d}$, 1 will be a term of the sequence if and only if

$$\frac{n-(k-1)d}{1+(k-1)d}=1\Rightarrow n-(k-1)d=1+(k-1)d\Rightarrow\boxed{n=1+2(k-1)d}\ ■$$

Example: When $n = 31$ and $d = 3$, we have $31 = 1 + 6(k - 1)$, so $k = 6$. Indeed, the 6-th term of $D_{31,3}=\frac{16}{16}=1$.

Let $d = 2$. Then $D_{n,2}=\frac{n}{1},\frac{n-2}{3},\frac{n-4}{5},\frac{n-6}{7},...$ If n is even, 1 will not be a term of $D_{n,2}$. Let us examine a few cases where n assumes the even values, 4, 6, 8, and 10.

$D_{4,2}=\frac{4}{1}$. The length, $l(D_{4,2}) = 1$, and the last term is $\frac{4}{1}$.

$D_{6,2}=\frac{6}{1},\frac{4}{3}$. The length, $l(D_{6,2}) = 2$, and the last term is $\frac{4}{3}$.

$D_{8,2}=\frac{8}{1},\frac{6}{3}$. The length, $l(D_{8,2}) = 2$, and the last term is $\frac{6}{3}$.

$D_{10,2}=\frac{10}{1},\frac{8}{3},\frac{6}{5}$. The length, $l(D_{10,2}) = 3$, and the last term is $\frac{6}{5}$.

Fact: Given $D_{n,2}$ where n is even, $l(D_{n,2})=\left\lceil\frac{n}{4}\right\rceil$ and the last term is $\frac{2\left\lfloor\frac{n}{4}\right\rfloor+2}{2\left\lfloor\frac{n-1}{4}\right\rfloor+1}$. ■

Reference

[1] M. Lewinter, J. Meyer, *Elementary Number Theory with Programming*, Wiley & Sons. 2015.

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