Quest Journals
Journal of Research in Applied Mathematics
Volume 8 ~ Issue 9 (2022) pp: 44-48
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

Closeness centrality of complete, fan and wheel graphs

Shaoli Nandi
Department of Mathematics, Govt. General Degree College, Salboni
Paschim Medinipur – 721516, India

Abstract
Centrality is a good idea to know the important vertices in a graph. Closeness is one of the oldest centralities used to measure important vertices in social networks, biological networks, transportation networks, etc. Closeness centrality detects the vertices that can spread information through graphs. The closeness centrality of a vertex is the average farness to all other vertices. It measures how fast information flow is done via a given vertex. i.e., closeness centrality measures how short the shortest paths are from a vertex to all other vertices. This paper finds the closeness centrality of complete, fan and wheel graphs.
Keywords: *Closeness centrality, normalized closeness centrality, complete graph, fan graph, wheel graph.*
AMS Mathematics Subject Classification(2020): 05C30, 0562, 05C38

***R**eceived 12 Sep., 2022; **R**evised 25 Sep., 2022; **A**ccepted 28 Sep., 2022 © The author(s) 2022.*
P**ublished with open access at **www.questjournals.org

1. Introduction

There are different centrality measures for network analysis. Closeness centrality is one of the oldest centrality measures. It is used to analyze the social networks [26, 27], biological networks [33], transportation networks [9] etc. It is also used in bibliometrics [21] and to select potential leads in customer data.
Closeness centrality is the average shortest distance from each vertex to another vertex. Itis the inverse of the average shortest distance between the vertex and all other vertices in the network. Mathematically, the closeness centrality $C_C(x)$ is defined by $C_C(x) = \frac{1}{\sum_{y\in V} d(x,y)}$ where V is the set of vertices in the network, $d(x,y)$ is the distance between the vertices x and y. For normalization, closeness centrality $C'_C(x) = \frac{n-1}{\sum_{y\in V} d(x,y)}$ where n is the number of nodes in the network. This measure is more acceptable than degree centrality, as it counts indirect connections also. This measure aimed to recognize the nodes that could attain others morequickly.

The closeness centrality determines the location of the facility location problem and the indirect influence of a brain region on another brain region in the brain network. Closeness centrality represents the connectivity between a street and all other neighborhood streets in the network and measures their accessibility.

1.1 Review of the related works

In 1948, Bavelas [2] first introduced the concept of centrality and applied it to human communication. In 1948, Bavelas [2] first gave the idea of closeness centrality, and in 1966, Sabidussi [31] defined closeness centrality as the inverse of the sum of distances between every pair of vertices in the network. In 1978, Freeman [16] gave the mathematical formula of closeness centrality. In 2001, Newman [24] generalized the closeness centrality on weighted graphs by Dijkstras shortest paths algorithm. In 2001, Brandes [7] presented an $O(mn)$ time algorithm to measure the closeness centrality of all vertices in a network.

1.2 Result

Here, I study the closeness centrality of the complete graph, fan and wheel graphs.

1.3 Arrangement of the paper

In the next section, I state and prove a theorem related to the closeness centrality of the vertices of the complete graph. In Section 3, I find the closeness centrality of fan the graph. Section 4 gives the closeness centrality of the wheel graph. The last section provides the paper's conclusion.

2. Closeness centrality of complete graph

A simple graph G is called a complete graph if a unique edge joins every pair of distinct vertices in G. The complete graph of order n is denoted by K_n. The number of edges of K_n is $\frac{n(n-1)}{2}$.
A complete graph K_5 is shown in Figure 1.

Theorem 1: The closeness centrality $C_C(v)$ of each vertex v of K_n is $\frac{1}{n-1}$.

Proof. Let $V = \{v_1, v_2, \cdots, v_n\}$ be the set of vertices of the complete graph K_n. Again let, v be any vertex of K_n. The shortest distance between v and a vertex of $V - v$ is 1. There are $n-1$ pairs of vertices. Therefore, $\sum_{y\in V} d(v,y) = 1 + 1 + \cdots + 1 = n-1$. Hence the closeness centrality $C_C(v) = \frac{1}{\sum_{y\in V} d(v,y)} = \frac{1}{n-1}$.
Note1: The normalized closeness centrality $C'_C(v)$ of each vertex v of K_n is $C'_C(v) = \frac{n-1}{n-1} = 1$.

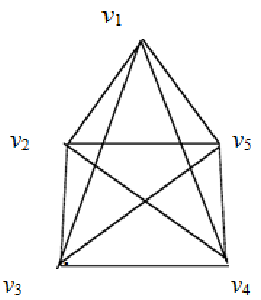

Figure 1: Complete graph K_5

3. Closeness centrality of fan graph

A fan graph is a graph joining the empty graph and the path graph. The fan graph $F(1, n)$ is defined by $F(1, n) = \overline{K_1} + P_n$ where $n > 1$, $\overline{K_1}$ is the empty graph of 1 vertex and P_n is the path graph of n vertices. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of the fan graph $F(1, n)$. The total number of vertices and edges of the $F(1, n)$ are, respectively, $1 + n$ and $n + n - 1 = 2n - 1$. A fan graph $F(1, 4)$ is shown in Figure 2.

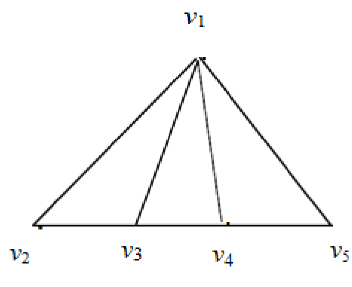

Figure 2: Fan graph $F(1, n)$

*Corresponding Author: Shaoli Nandi

Theorem 2. *The closeness centrality $C_C(v)$ of any vertex v of $F\ (1, n)$ is*

$$C_C(v) = \begin{cases} \frac{1}{n}, & \text{if v is the central vertex of fan graph,} \\ \frac{1}{2n-2}, & \text{if v is an extrimity of path graph } P_n, \\ \frac{1}{2n-3}, & \text{if v is an intermediate vertex of } P_n. \end{cases}$$

Proof. Let $\{v_1, v_2, \cdots, v_n, v_{n+1}\}$ be the set of vertices of the fan graph $F\ (1, n)$ where v_1 is the vertex of empty graph $\overline{K_1}$ and $\{v_2, \cdots, v_n, v_{n+1}\}$ is the set of vertices of the path graph P_n. The vertex v_1 is the central vertex of the fan graph $F\ (1, n)$. If $v = v_1$ then the shortest distance between v and a vertex of $F\ (1, n)$ except v is 1. There are n pairs of vertices. Therefore,
$\sum_{y \in V} d(v, y) = 1 + 1 + \cdots + 1 = n$. Hence, the closeness centrality of v is $C_C(v) = \frac{1}{\sum_{y \in V} d(v,y)} = \frac{1}{n}$.
If $v = v_2$ then the shortest distance between v and v_1 and also between v and v_3 is 1. The shortest distance between v and a vertex of $\{v_4, \cdots, v_n, v_{n+1}\}$ is 2. There are $n - 2$ pairs of vertices. Therefore, $\sum_{y \in V} d(v, y) = 1 + 1 + 2 \ldots + 2 + 2 = 1 + 1 + 2(n-2) = 2n - 2$. Hence, the closeness centrality of v is $C_C(v) = \frac{1}{\sum_{y \in V} d(v,y)} = \frac{1}{2n-2}$. If $v = v_{n+1}$ then similarly we can write $C_C(v) = \frac{1}{\sum_{y \in V} d(v,y)} = \frac{1}{2n-2}$.
If $v = v_p, p = 3,4, \ldots, n$ then the shortest distance between v and v_1 and between v and v_{p-1} and also between v and v_{p+1} is 1. The shortest distance between v and a vertex of $V - \{v, v_1, v_{p-1}, v_{p+1}\}$ is 2. There are $n - 3$ pairs of vertices. Therefore, $\sum_{y \in V} d(v, y) = 1 + 1 + 1 + 2 \ldots + 2 + 2 = 1 + 1 + 1 + 2(n - 3) = 2n - 3$. Hence, the closeness centrality of v is $C_C(v) = \frac{1}{\sum_{y \in V} d(v,y)} = \frac{1}{2n-3}$.

Not**e 2:** The normalized closeness centrality is

$$C'_C(v) = \begin{cases} \frac{n+1-1}{n} = 1, & \text{if v is the central vertex of fan graph,} \\ \frac{n+1-1}{2n-2} = \frac{n}{2n-2}, & \text{if v is an extrimity of path graph } P_n, \\ \frac{n+1-1}{2n-3} = \frac{n}{2n-3}, & \text{if v is an intermediate vertex of } P_n. \end{cases}$$

4. Closeness centrality of wheel graph

A wheel graph W_n is obtained by joining a vertex to each vertex of cycle graph C_{n-1}. The number of edges of W_n is $2n - 2$. A complete graph W_5 is shown in Figure 3.

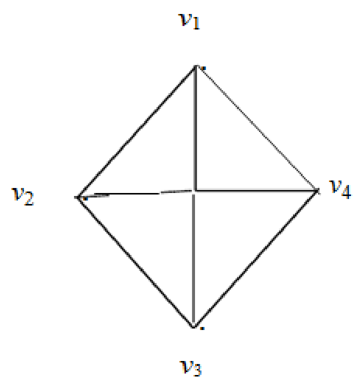

Figure 3: wheel graph W_5

Theorem 3. *The closeness centrality $C_C(v)$ of any vertex v of W_n is*

$$C_C(v) = \frac{1}{n-1}, \quad \text{if } v \text{ is central vertex of wheel graph,}$$
$$= \frac{1}{2n-5}, \quad \text{otherwise.}$$

Proof. Let $\{v_1, v_2, \cdots, v_n\}$ be the set of vertices of the wheel graph W_n where v_n is the central vertex of wheel graph W_n. If v is the central vertex of W_n, then the shortest distance between v and a vertex of $V-\{v\}$ is 1. There are $n-1$ pairs of vertices. Therefore, $\sum_{y\in V} d(v,y) = 1+1+\cdots+1 = n-1$. Hence, the closeness centrality of v is $C_C(v) = \frac{1}{\sum_{y\in V} d(v,y)} = \frac{1}{n-1}$.

If v is any vertex of W_n except the central vertex, then the shortest distance between v and the three vertices (central vertex and two adjacent vertices of v) of W_n are separately 1, and the shortest distance between v and others $n-4$ vertices is 2. So, $\sum_{y\in V} d(v,y) = 1+1+1+2(n-4) = 2n-5$. Hence, the closeness centrality of v is $C_C(v) = \frac{1}{\sum_{y\in V} d(v,y)} = \frac{1}{2n-5}$.

Note3: The normalized closeness centrality $C'_C(v)$ of any vertex $v\ of\ W_n$ is

$$C'_C(v) = \frac{n-1}{n-1} = 1, \qquad if\ v\ is\ central\ vertex\ of\ wheel\ graph,$$
$$= \frac{n-1}{2n-5}, \qquad otherwise.$$

5. Conclusion

To analyze network problems, closeness centrality has an important role. It is used in social networks, biological networks and transportation networks. In this paper, I state and prove three theorems related to the closeness centrality of the complete graph, fan graph and wheel graph and I also determine the normalized closeness centrality of these graphs.

References

[1]. Barrat, A., Barthelemy, M., Vespignai, A.: The architecture of complex weighted networks. Proceedings of the National Academy Sciences of the United States of America. **101(11)**, 3747-3752 (2004). https://doi.org/ 10.1073/pnas.0400087101

[2]. Bavelas, A.: A mathematical model for group structures. Applied Anthropology. **7(3)**, 16-30 (1948). https://doi.org/10.17730/humo.7.3.f4033344851gl053

[3]. Bavelas, A.: Communication patterns in task oriented groups. Journal of the Acoustical Society America. **22**, 725-730 (1950). https://dx.doi.org/10.1121/1.1906679

[4]. Beauchamp, M.: An improved index of centrality. Behavioral Science. **10**, 161-163 (1965). https://doi.org/10.1002/bs.3830100205

[5]. Bergamini, E., Crescenzi, P., D'angelo, G., Meyerhenke, H., Severini, L., Velaj, Y.: Improving the betweenness centrality of a node by adding links. ACM Journal of Experimental Algorithmics. **23**, 1-32 (2018). DOI: https://doi.org/10.1145/3166071

[6]. Borgatti, S.: Centrality and network flow. Social Networks. **27(1)**, 55-7 (2005). https://doi.org/10.1016/j.socnet.2004.11.008

[7]. Brandes, U.: A faster algorithm for betweenness centrality, J Math Sociol, **25(2)**, 163177 (2001). https://doi.org/10.1080/0022250X.2001.9990249

[8]. Buckley, N., Alstyne, M.: Does email make white collar workers more productive? Technical report, University of Michigan (2004).

[9]. Chaoqun et al.: Degree, Closeness and betweenness: Application of group centrality measurements to explore macro-disciplinary evolution diachronically, 605 (2011)

[10]. Coffman, T., Greenblatt, S., Marcus, S.: Graph-based technologies for intelligence analysis.

[11]. Communications of the ACM. **47(3)**, 45-47 (2004). https://doi.org/10.1145/971617.971643

[12]. Das, K., Samanta, S., Pal, M.: Study on centrality measures in social networks: a survey. Social Network Analysis and Mining. **8**, 13 (2018). https://doi.org/10.1007/s13278-018-0493-2

[13]. del Sol, A., Fujihashi, H., OMeara, P.: Topology of small-world networksof protein-protein complex structures. Bioinformatics. **21(8)**, 1311-1315 (2005). https://doi.org/10.1093/bioinformatics/bti167

[14]. Estrada, E., Rodriguez-Velazquez, J.: Subgraph centrality in complex networks. Physical Review. **71**, 056103 (2005). https://doi.org/10.1103/PhysRevE.71.056103

[15]. Estrada, E., Higham, D. J., Hatano, N.: Communicability betweenness in complex net- works. Physica A: Statistical Mechanics and its Applications. **388(5)**, 764-774 (2009). https://doi.org/10.1016/J.Physa.2008.11.011

[16]. Freeman, L.: A set of measures of centrality based on betweenness. Sociometry. **40**, 35-41 (1977). https://doi.org/10.2307/303543

[17]. Freeman, L.: Centrality in social networks conceptual clarification. Social Networks. **1**, 215-239 (1978). https://doi.org/10.1016/0378-8733(78)90021-7

[18]. Golumbic, M. C.: Algorithmic Graph Theory and Perfect Graphs. Elsevier B.V. (2004).

[19]. Guimera, R., Mossa, S., Turtschi, A., Amaral, L.A.N.: The worldwide air transportation network: Anomalous centrality. community structure, and cities global roles. **102(22)**, 7794-7799 (2005). https://doi.org/10.1073/pans.0407994102

[20]. Hadwiger, H., Debrunner, H., Klee, V.: Combinatorial Geometry in the plane. New York: Holt Rinehardt and Winston (1964).

[21]. Jeong, H., Mason, S.P., Barabasi, A. L., Oltvai, Z.N.: Lethality and centrality in protein networks. Nature, **411**, 4142 (2001). https://doi.org/10.1038/35075138

[22]. Kiss, C. and Bichler, M.: Identification of influencers- Measuring influence in customer networks, Decision Support Systems, **46(1)**, 233-253 (2008).

[23]. Krebs, V.E.: Mapping networks of terrorist cells. Connections. **24(3)**, 4352 (2002). ResearchGate

[24]. Latora, V., Marchiori, M.: A measure of centrality based on network efficiency. New Journal of Physics. **9(6)**, 188 (2007). ResearchGate

[25]. Newman, M., Scientific collaboration networks I, Network construction and fundamental results. Phys Rev E., **64**, 016131(2001).

[26]. Ore, O.: Theory of Graphs. American Mathematical Society Colloquium Publications. **38** 206 - 212 (1962).

[27]. Otte, E., Rousseau, R.: Social network analysis: a powerful strategy, also for the information sciences. Journal of Information Science. **28(6)**, 441-453 (2002). https://doi.org/10.1177/016555150202800601

[28]. Park, K., Yilmaz, A.: A social network an analysis approach to analyze road networks. in Proceedings of the ASPRS Annual Conference, San Diego, Calif, USA (2010). ResearchGate

[29]. McPherson, M., Smith-Lovin, L., Cook, J. M.: Birds of a feather: Homophily in social networks. Annual Review of Sociology. **27**, 415-444 (2001). https://doi.org/10.1146/annurev.soc.27.1.415

[30]. Pinney, J.W., McConkey, G.A., Westhead, D.R.: Decomposition of biological networks using betweenness centrality. In Proceeding 9th Annual Intl Conference on Research in Computational Molecular Biology (RECOMB 2005).

[31]. Puzis, R., Zilberman, P., Elovic, Y., Dolev, S., Brandes, U.: Heuristics for speeding up Betweenness centrality computation. IEEE International Conference on Social Computing (2012). https://doi.org/10.1109/SocialCom-PASSAT.2012.66

[32]. Sabidussi, G.: The centrality index of a graph, Psychometrika, **31(4)**, 581603 (1966).

[33]. Shaw, M.: Group structure and the behavior of individuals in small groups. The Journal Psychology. **38**, 139-149 (1954). https://doi.org/10.1080/00223980.1954.9712925

[34]. Wuchty, S. and Sadler P.: Centers of complex network, Journal of Theo. Bio. **223(1)**, 45-53 (2003).