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Research Paper

Degree of Homogeneity of Primitive Permutation Groups Based On the Bounds of the Socle of the Groups.

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ABSTRACT
Bounding the order of finite primitive permutation groups were carried out through the construction of the base size of the socle of the groups. The idea of a better bounds for primitive groups provided themost essential background to determine the base size of the rank of these groups, which served as bounds to the groups. The bounds of the socle of the groups were used to determine the degreeof homogeneity of finite primitive permutation groups.
***Key words** 1: Bounds of the socle of a primitive groups; Degree of homogeneiy of the groups.*
***Key words** 2: Transitive groups; Rank of the primitive groups.*

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I. Background

The idea of bounding primitive groups comes as a result of the work of Bochert, He first used the concept to bound finite primitive groups which was first described in [2]. It was estimated that the bound for $|S_n: G|$ is$\frac{(n-1)}{2}$ which was called a better bound. Wielandt in[17] extended the work where he proved that the order of a simply primitive group of degree k is at most $c^n$for c=24.Further, Praeger and Saxl in [15] improved this bound to $c^n$for c=4 and showed that it holds for all proper primitive groups of degree n. The bounds on finite primitive groups are widely known as Praeger –Saxl bounds. The most recent work on the bound of a primitive permutation groups is the work of Babai as captured in [9] where he improve the bound to$|G| < n^{4\sqrt{n}logn}$ which is more better than the better bound. These vital contributions promted the quest to work on how to bound primitive permutation group based on the socle of the group. Further wetry to determine the degree of homogeneity of the k- transitive permutation groups using the concept of socle of the groups. We impose the idea of the base size of the primitive so as to bound thesocle of the groups, with the view to determine the degree of homogeneity bearing in mind the rank of the group in relation to the base size of the socle of the groups.

II. Prelimiary results

We provide basic definitions and theorems as essential tools for the attainment of our main results.
1.1 Definition: Let$\alpha \in G$, then the stabiliser of α is given by $G_\alpha = \{g \in G | \alpha^g = \alpha\}$.
1.2 Definition: Let $\alpha \in G$, orbit of G containing α is given by $\alpha^G = \{\alpha^g | g \in G. = \{g^{-1}\alpha g | g \in G\}$

1.3 Definition Let $G \leq sym(\Omega)$. A subset $\Delta \subseteq \Omega$ is a base for G if $G_{(\Delta)} = 1$.
We denote the base size of$G$ by $\partial = \partial(G)$ throughout, in order to bound finite primitive groups.
To explore the relation between the stabilizer of a group and the socle of a group of finite primitive permutation groups, we take the following definitions.
1.4 Definition: Let G be a transitive group with a point stabilizer, then the rank of G is said to be the size of the point stabilizer of G.
Thus we clearly state that, if G is transitive of size m and suppose the point stabilizer G_α is of size t, then the rank for G is t, which could be the bound for the point stabilizer.

1.5 Definition: The sub rank of a transitive group G is the maximum rank of the transitive constituents of the point stabilizer.

1.5 Definition; The socle of G of a finite group is the subgroup generated by the minimal normal subgroups of G denoted by $soc(G)$.
The next theorem gives condition on the existence of minimal order of an element in a finite primitive group.
1.7 Theorem: Let G be a primitive permutation group of degree n and a minimal degree m. Then each element of G has order at most $n^{n/m}$.
Proof
Let $x\epsilon G$ have order h. suppose that p is a prime and that p^e (e≥ 1) is the largest power of p dividing h. Then $x^{h/p}$is a product of p-cycles, and α ∈supp($x^{h/p}$) if and only the cycle of the lengths of p-cycles of x whose length are divisible by p^e is at least m. Suppose we factor h=$q_{i,\ldots,}$ q_s where q_i are not trivial powers of distinct primes and let $h_1,\ldots h_i$ denote the length of the disjoint cycles for (i=1,2….s and j=1,….t,), then it implies q_i/h_j. and so we have m≤ $\sum_{qi/hj} hj$ for each i, and evidently we also have $\sum \log q_i \ \leq \log h_i$ for each j therefore, h≤ $n^{n/m}$ as required.
We now take two proposition which has the order of Sylow p-subgroups being defined, in such a way that the degree of transitivity is determine.
1.8 Proposition: Let G be a primitive group of degree which is t −transitive but not $(t+1)$ −transitive, with $1 \leq t \leq n-2$. Let p be a prime number with$p^2 \leq n$ and let Q be a p-subgroup of G all whose orbit have length 1 or p. If Q p-subgroups has order p^f then $f \leq \left\{\frac{t-1}{p}\right\} + \frac{n}{p^2}$
1.9 Proposition: Let G be primitive and p be a prime number with $p^2 \leq n$. If the Sylowp-subgroups have orderp^e. Then $e \leq \left\{\frac{t-1}{p}\right\} + \frac{n}{p^2} + \frac{n}{p(p-1)}$
Further we will try to provide few results which will give us idea on the base size of primitive groups.

1.10Theorem: Let G be a transitive group of degree n, and if $\partial = \partial(G)$. Then G has a base of size at most$s < \frac{n(2\log n - \log 2)}{\partial}$ and is it s-transitive.
Proof:
It is enough to show if s is an integer such that $0 < s \leq n$ and no s-subset of Ω is a base for G, then $s < n|2|logn - log2)/\partial$. If G is transitive on s-subset Ω^s and $X_{\alpha\beta}(\Sigma) = 1$ if $\Sigma \cap \Psi = 0$ and $X_{\alpha\beta}(\Sigma) = 0$ otherwise. Then for each $\alpha, \beta \in \Omega^s$we have$X_{\alpha\beta}(\Sigma) = 1$ if $\Sigma \cap \Psi = 0$ and $X_{\alpha\beta}(\Sigma) = 0$ otherwise.
Now define $m = \Sigma\Psi_{\alpha\beta}(\Sigma)$. Then sum over all $\Sigma \in \Omega^s$ and all $\alpha, \beta \in \Omega^2$ Thus considering some condition for minimality we see that| G|≤ $n(n-1)\binom{n-d}{s}$. By fixing ε and summing over $(\alpha, \beta) \in \Omega^s$. Since ε is not a base for G we have $\Sigma \cap \Psi_{\alpha\beta} = 0$. But for the case $\alpha, \beta \in \Omega^2$ we have $\Psi_{\alpha\beta}(\Sigma) = \Psi_{\beta\alpha}(\Sigma) = 1$ and is also true for $\Sigma \in \Omega^s$ in which ε is s-transitive, hence
$m \geq 2|\Omega^s| = 2\binom{n}{s}$. Thus if there is no base for G of size s then G is s- transitive.
1.11Theorem: Let$G$ be a primitive group of degree n not containing A_n. Then the base is $\partial(G) \leq \frac{n}{2}$rem: Let
1.12Theorem: Let G be a primitive permutation group of degree n not containingA_n.
i. If

1.14Theorem:Let G be a primitive group of degree n not containingA_n. Then either
i. $\partial(G) < 9logn$ or
ii. $G \wr S_k$is product- type group, where $H \leq sym(\Gamma)$ is a primitive group with socle A_n and Γ is the set of t element subset of $\{1,2,\ldots\ldots,m\}$. Thus in particular, $\partial(G) < c\sqrt{n}$ for some absolute constant c.
We try to impose a referral condition and try to overlook the previous conditions in Theorem 1.11 and Theorem 1.12 and Theorem 1.14 to give a wide range of view by stating the next theorem which further capture the assertion ofTheorem 1.8.
1.15 Theorem: Let G be a group acting primitively on a finite set Ω of size n, and suppose that G has a Jordan complement of size m where $m \geq n/2$ then G is 3-transitive, more over if $m > n/2$ then $G \geq Alt(\Omega)$.
Next we state a lemma without a proof

1.16 Lemma: If A_n ≥ 7 then for any real number z, $\theta^8 > \frac{z}{2}$ for all $z > 11$.
1.17 Remark:
In this case we let $\sigma(p)logp = m$, where $\sigma(p) = \sum_{\substack{i \geq 1 \\ 0<j<n}} \{\frac{2p}{p^i} - 2[\frac{n}{p^i}]\}$ to obtain the upper and the lower bound forA_n

1.18 Theorem: Let G be as in Theorem 1.14 such that $A_n \geq 7$ then A_n contains an element of order greater than $e^{\sqrt{\frac{1}{4}nlogn}}$.

MAIN RESULT

2.1 Theorem

Let $G \leq sym(\Omega)$ be a primitive group of degree n ≥ 11. Then ifA_n has a lower bound $2^{2n}/2n$ and an upper bound$\sum_{p\leq 2n} \sigma(p)logp$, itimply$A_n$ is k-homogeneous.

Proof:

Suppose $G$is finite primitive group and $n \geq 11$ then by Theorem 1.15 implies that $G \geq A_n$ and soA_n is section isomorphic to A_k. It follows that A_n has a minimal degree by Theorem 1.7 and so by Theorem 1.12 G is not A_n but has a lower bound of G is $2^{2n}/2n$, so we conclude that$G$ is 3-transitive.Also since G has complement of size greater than $\frac{n}{2}$ it follows from Theorem 1.14 that $G \geq A_n$ which is the socle of the group by Theorem 1.14. Moreover A_n permutation isomorphic to A_n where k positive integer acting on the k-elements subset ofΩ^k. Hence A_n has an upper bound $\sum_{p\leq 2n} \sigma(p)logp$ as defined in Remark 1.17, and consequently by Theorem 1.10 we infer that$G$ is$k -$ homogeneous.

2.2 Theorem: Let G be a primitive group of degreen and H a subgroup such that the rank of G is r, then$G$is $r -$homogeneous.

Proof:

Since G is primitive of degree n, let H be a subgroup of order r by Theorem1.14,G is 3-transitive if $\geq \frac{n}{2}$, or G contain A_n for$r > \frac{n}{2}$. Suppose $H = soc(G)$ and that H is maximal in G such that G has rank r, say, moreover if $= G_\alpha$, then by Definition 1.4 it imply $|H| = r$ which is a bound for$H$. Next, we invoke that $r < n$ and conclude that G is r-transitive if and only if G has a base size as define in Theorem 1.10. Hence G is r-transitive.

2.3 Theorem: Let G be a primitive group of degree n of order p^a, for $a \geq 1$ and H a subgroup of$G$. Then G is t-transitive.

Proof:

Suppose that p is prime as in Proposition 1.8 then G has rank t. Let Hbe a maximal subgroup of G, if H is a sylow $p -$subgroup of G then $soc(G) = H$ and so $H = G_\alpha$ of rank$t$. Therefore by Theorem 1.10, it imply G is t-homogeneous.

Conversely, if G is t-transitive then by Theorem 1.12 it shows that$G$ has a base size$\partial(G) < 4\sqrt{n}logn$ for $k = 2$, for if $\partial(G) < c^{\sqrt{logn}}$, for some constant c, then G is not $2 -$transitive and does not contain A_n. However H is a sylow p-subgroup of G which is a direct consequence of Proposition 1.8, and so $H = G_\alpha$. Also it follows from Theorem 1.13 (ii), and in line with Theorem 1.14that A_n is the $soc(H)$, thereforeby Theorem 1.10 G hasa base size of $t < \frac{n(2logn - log2)}{\partial}$. Thus we conclude that$G$ is t-transitive.

III. Conclusion

The base size of the stabilizer of finite primitive groupsplayed a pivotal role in determining the bounds of the socle of finite primitive groups, in order to determine the degree of homogeneiy of the groups. The case in which G contains the alternating groups was considered, where the rank of the point stabilizer was used to determine the bounds of the socle of the groups, where$H = G_\alpha$. The idea of the bounds for the socle of a group is very useful in the determination of degree of homogeneity in a finite primitive permutation group.

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