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Research Paper

On Geodetic Sets and Geodetic Polynomials of Fan Graphs

Dr.K.Vijila Dafini, Dr.A.Vijayan
(Assistant Professor, Department of Mathemetics,Malankara Catholic college, Mariagiri,Kaliakkavilai, Kanyakumari District, Tamilnadu, India)
(Associate Professor, Department of Mathemetics,Nesamony memorial Christian college, Kanyakumari District, Tamilnadu, India)
Corresponding author: Dr.K.Vijila Dafini

ABSTRACT: *Let G = (V,E) be a simple graph. A set of vertices S of a graph G is geodetic, if every vertex of G lies on a shortest path between two vertices in S. The geodetic number of G is the minimum cardinality of all geodetic sets of G, and is denoted by g (G). In (8), the concept of geodetic polynomial is defined as* $g(G,x)=\sum_{i=g(G)}^{n} g_e(G,i)x^i$ *where* $g_e(G,i)$ *is the number of geodetic sets of cardinality i. In this paper, we obtain the geodetic sets and geodetic polynomials of the Fan graph. Also, we study some properties of geodetic sets and the coefficients of the polynomials. It is also derived that the geodetic polynomial of the centipede* P_n^* *is* $x^n(1+x)^n$.

KEYWORDS: *Geodetic sets, Geodetic number, Geodetic polynomial, Recursive formula, Fan graph.*

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I. INTRODUCTION

Let G=(V, E) be a simple graph of order |V| = n. The distance d(u, v) between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. A u-v path of length d(u, v) is called u-v geodesic. The closed interval I[u, v] consists of all vertices lying on some u-v geodesic of G, while for S $\subseteq$ V, I [S] = $\bigcup_{u,v}$ I [u, v]. A set S of vertices is a geodetic set if I [S] = V, and the minimum cardinality of a geodetic set is the geodetic number g(G). The geodetic number of a graph was introduced in [4,5]. In [I], the domination polynomial was introduced and some properties have been derived. In [8], the concept of geodetic polynomial was introduced. It is defined as $g(G,x)=\sum_{i=g(G)}^{n} g_e(G,i)x^i$ where G is a graph of order n and $g_e(G,i)$ is the number of geodetic sets of G of cardinality i. Let $G_e(F_{1,n},i)$ be the family of geodetic sets of $F_{1,n}$ with cardinality *i*. Let $g_e(F_{1,n},i)=|G_e(F_{1,n},i)|$. The geodetic polynomial, $g(F_{1,n},x)$ of $F_{1,n}$ is defined as $g(F_{1,n},x)=\sum_{i=g(F_{1,n})}^{n+1} g_e(F_{1,n},i)x^i$.

In the next section, we construct the families of the geodetic sets of fan graphs by a recursive method. In section 3, we use the results obtained in section 2 to study the geodetic polynomial of Fan graphs.

As usual we use $\lceil x \rceil$ for the smallest integer greater than or equal to x. In this paper, we denote the set $\{1,2,...n\}$ simply by *[n]*.

II. GEODETIC SETS OF FAN GRAPH

Let $F_{1,n}$ be the fan graph with $n+1$ vertices $v_0, v_1, v_2, v_3, \ldots v_{n-1}, v_n$ with v_0 having degree *n*, v_1 and v_n having degree two and all the remaining vertices having degree three. Let $\mathcal{G}_e(F_{1,n}, i)$ be the family of geodetic sets of $F_{1,n}$ with cardinality *i*. Let $g_e(F_{1,n}, i) = |\mathcal{G}_e(F_{1,n}, i)|$. We investigate the geodetic sets of the Fan graph $F_{1,n}$. We need the following lemma to prove our main results.

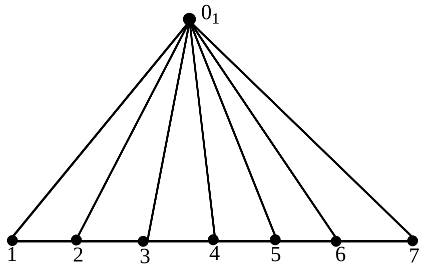

Figure1: Labeled Fan graph, $F_{1,7}$

Lemma 2.1. Let $F_{1,n}$ be the fan graph. Then, the following hold for $F_{1,n}$:

(i) $g(F_{1,n}) = \left\lceil \frac{n+1}{2} \right\rceil$

(ii) $\mathcal{G}_e(F_{1,n}, i) = \phi$ if and only if $i < \left\lceil \frac{n+1}{2} \right\rceil$ or $i > n+1$

Proof: Proof is obvious.

Remark 2.2. Let $V = \{0_1, v_1, v_2, \ldots, v_n\}$ be the vertex set of the fan graph $F_{1,n}$. Let *S* be a subset of *V*. If $v_i, v_{i+1} \notin S$ for some *i*, then *S* is not a geodetic set.

Proof. We have $V = \{0_1, v_1, v_2, \ldots, v_n\}$. Let *S* be a subset of *V*. The shortest distance between any two vertices of *S* is at most two. If two consecutive vertices say v_i, v_{i+1} among $v_1, v_2, \ldots, v_n$ is missing in the set *S*, then any path joining two elements of *S* and covering v_i, v_{i+1} is of distance greater than two. Therefore, this path is not a geodetic path. Thus, the vertices of *S* do not cover v_i and v_{i+1}. Therefore, *S* is not a geodetic set.

Lemma 2.3. If $\mathcal{G}_e(F_{1,n-2}, i-1) = \phi$ and $\mathcal{G}_e(F_{1,n-1}, i-1) = \phi$, then $\mathcal{G}_e(F_{1,n}, i) = \phi$

Proof. Since $\mathcal{G}_e(F_{1,n-2}, i-1) = \phi$ and $\mathcal{G}_e(F_{1,n-1}, i-1) = \phi$, by lemma 2.1(ii), we have $i-1 < \left\lceil \frac{n-1}{2} \right\rceil$ or $i-1 > n-1$ and $i-1 < \left\lceil \frac{n}{2} \right\rceil$ or $i-1 > n$. From these, we obtain, $i-1 < \left\lceil \frac{n-1}{2} \right\rceil$ or $i-1 > n$. If $i-1 < \left\lceil \frac{n-1}{2} \right\rceil$, then $i < \left\lceil \frac{n+1}{2} \right\rceil$. Therefore, by lemma 2.1(ii), we have $\mathcal{G}_e(F_{1,n}, i) = \phi$. Also, if $i-1 > n$, then $i > n+1$; and by lemma 2.1(ii), we have $\mathcal{G}_e(F_{1,n}, i) = \phi$.

Lemma 2.4. Suppose $\mathcal{G}_e(F_{1,n}, i) \neq \phi$, then

(i) For odd *n*, $\mathcal{G}_e(F_{1,n-2}, i-1) \neq \phi$ and $\mathcal{G}_e(F_{1,n-1}, i-1) = \phi$ if and only if $i = \frac{n+1}{2}$.

(ii) $\mathcal{G}_e(F_{1,n-2}, i-1) = \phi$ and $\mathcal{G}_e(F_{1,n-1}, i-1) \neq \phi$ if and only if $i = n+1$.

(iii) $\mathcal{G}_e(F_{1,n-2}, i-1) \neq \phi$ and $\mathcal{G}_e(F_{1,n-1}, i-1) \neq \phi$ if and only if $\left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n$.

Proof of (i). ($\Rightarrow$) Since $G_e(F_{1,n-1}, i-1) = \phi$, by lemma 2.1(ii), we have $i-1 < \left\lceil \frac{n}{2} \right\rceil$ or $i-1 > n$. If $i-1 > n$, then $i > n+1$, by lemma 2.1(ii), $G_e(F_{1,n}, i) = \phi$, which is a contradiction. So, we must have $i-1 < \left\lceil \frac{n}{2} \right\rceil$. Therefore $i < \left\lceil \frac{n}{2} \right\rceil + 1$. Also, since $G_e(F_{1,n-2}, i-1) \neq \phi$, by lemma 2.1(ii), we have $\left\lceil \frac{n-1}{2} \right\rceil \leq i-1 \leq n-1$. Together, we have $\left\lceil \frac{n-1}{2} \right\rceil + 1 \leq i < \left\lceil \frac{n}{2} \right\rceil + 1$. ---(1)

When n is odd, $\left\lceil \frac{n-1}{2} \right\rceil = \frac{n-1}{2}$ and $\left\lceil \frac{n}{2} \right\rceil = \frac{n+1}{2}$. Therefore, from (1), we have $\frac{n-1}{2} + 1 \leq i < \frac{n+1}{2} + 1$.

Therefore, $\frac{n+1}{2} \leq i < \frac{n+3}{2}$. Hence $i = \frac{n+1}{2}$.

($\Leftarrow$) If $i = \frac{n+1}{2}$, then by lemma 2.1(ii), we have

$$\begin{aligned} G_e\left(F_{1,n-1}, i-1\right) &= G_e\left(F_{1,n-1}, \frac{n+1}{2} - 1\right) \\ &= G_e\left(F_{1,n-1}, \frac{n-1}{2}\right) \\ &= \phi, \text{ Since } \frac{n-1}{2} < \left\lceil \frac{n}{2} \right\rceil. \end{aligned}$$

Also, $$\begin{aligned} G_e\left(F_{1,n-2}, i\text{-}1\right) &= G_e\left(F_{1,n-2}, \frac{n+1}{2} - 1\right) \\ &= G_e\left(F_{1,n-2}, \frac{n-1}{2}\right) \\ &\neq \phi, \text{ Since } \frac{n-1}{2} = \left\lceil \frac{n-1}{2} \right\rceil. \end{aligned}$$

Proof of (ii). ($\Rightarrow$) Since $G_e(F_{1,n-2}, i-1) = \phi$, by lemma 2.1(ii), we have $i-1 < \left\lceil \frac{n-1}{2} \right\rceil$ or $i-1 > n-1$. If $i-1 < \left\lceil \frac{n-1}{2} \right\rceil$, then $i < \left\lceil \frac{n+1}{2} \right\rceil$ and by

lemma 2.1(ii), $G_e(F_{1,n}, i) = \phi$, which is a contradiction. So, we must have $i-1 > n-1$. Also, since $G_e(F_{1,n-1}, i-1) \neq \phi$, we have $\left\lceil \frac{n}{2} \right\rceil \leq i-1 \leq n$. Together, we have $n < i \leq n+1$. Therefore $i = n+1$.

($\Leftarrow$) If $i = n+1$, then by lemma 2.1(ii), we have $G_e(F_{1,n-2}, i-1) = G_e(F_{1,n-2}, n) = \phi$ and $G_e(F_{1,n-1}, i-1) = G_e(F_{1,n-1}, n) \neq \phi$. Since $G_e(F_{1,n-1}, i) \neq \phi$ for $\left\lceil \frac{n}{2} \right\rceil \leq i \leq n$.

Proof of (iii). ($\Rightarrow$) Since $G_e(F_{1,n-2}, i-1) \neq \phi$ and $G_e(F_{1,n-1}, i-1) \neq \phi$,

by lemma 2.1(ii), we have $\left\lceil \frac{n-1}{2} \right\rceil \leq i-1 \leq n-1$ and $\left\lceil \frac{n}{2} \right\rceil \leq i-1 \leq n$. From these, we obtain, $\left\lceil \frac{n}{2} \right\rceil \leq i-1 \leq n-1$ and hence $\left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n$.

($\Leftarrow$) If $\left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n$, then we have $\left\lceil \frac{n}{2} \right\rceil \leq i-1 \leq n-1$ and hence $\left\lceil \frac{n-1}{2} \right\rceil \leq i-1 \leq n-1$. Therefore, by lemma 2.1(ii), we have $G_e(F_{1,n-1}, i-1) \neq \phi$ and

$G_e(F_{1,n-2}, i-1) \neq \phi$.

Theorem 2.5. Let $n \geq 5$ and $i \geq \frac{n+1}{2}$.

(i) If $G_e(F_{1,n-2}, i-1) \neq \phi$ and $G_e(F_{1,n-1}, i-1) = \phi$, then

$$G_e(F_{1,n}, i) = G_e\left(F_{1,n}, \frac{n+1}{2}\right) = \{1, 3, 5, ..., n\}$$

(ii) If $G_e(F_{1,n-2}, i-1) = \phi$ and $G_e(F_{1,n-1}, i-1) \neq \phi$, then

$$G_e(F_{1,n}, i) = G_e\left(F_{1,n}, n+1\right) = \{0_1, 1, 2, 3, ..., n\}$$

(iii) If $G_e(F_{1,n-2}, i-1) \neq \phi$ and $G_e(F_{1,n-1}, i-1) \neq \phi$, then

$$G_e(F_{1,n}, i) = \left\{X_1 \cup \{n\} / X_1 \in G_e(F_{1,n-2}, i-1)\right\}$$
$$\cup \left\{X_2 \cup \{n\} / X_2 \in G_e(F_{1,n-1}, i-1)\right\}$$

Proof of (i). Since $G_e(F_{1,n-2}, i-1) \neq \phi$ and $G_e(F_{1,n-1}, i-1) = \phi$, and *n* is odd, by lemma 2.4(i), $i = \frac{n+1}{2}$. Clearly $\{1, 3, 5, ..., n\}$ is geodetic sets of $F_{1,n}$ of cardinality $\frac{n+1}{2}$. Therefore, $\{1, 3, 5, ..., n\} \subseteq G_e\left(F_{1,n}, \frac{n+1}{2}\right)$. If *X* is a subset of *V* other than $\{1, 3, 5, ..., n\}$ of cardinality $\frac{n+1}{2}$, then *X* will miss at least two consecutive numbers from $1, 2, 3, ..., n$. By remark 2.2, *X* is a not a geodetic set. Therefore, $G_e(F_{1,n}, i) = G_e\left(F_{1,n}, \frac{n+1}{2}\right) = \{1, 3, 5, ..., n\}$.

Proof of (ii). Since $G_e(F_{1,n-2}, i-1) = \phi$ and $G_e(F_{1,n-1}, i-1) \neq \phi$, by lemma 2.4(ii), $i = n+1$. Therefore, $G_e(F_{1,n}, i) = G_e(F_{1,n}, n+1) = \{0_1, 1, 2, 3, ..., n\}$.

Proof of (iii). Denote the families $\left\{X_1 \cup \{n\} / X_1 \in G_e(F_{1,n-2}, i-1)\right\}$ and $\left\{X_2 \cup \{n\} / X_2 \in G_e(F_{1,n-1}, i-1)\right\}$ simply by F_1 and F_2 respectively. Any geodetic set in $F_{1,n-2}$ contains the extreme vertex $n-2$. If we add *n* with $X_1 \in G_e(F_{1,n-2}, i-1)$, then $X_1 \cup \{n\}$ is of cardinality *i*. The shortest path between $n-2$ and *n* contains $n-1$. Therefore, $X_1 \cup \{n\}$ is a geodetic set in $F_{1,n}$ with cardinality *i*. Therefore $\mathsf{F}_1 \subseteq G_e(F_{1,n}, i)$ and $\mathsf{F}_2 \subseteq G_e(F_{1,n}, i)$. That is,

$\mathsf{F}_1 \cup \mathsf{F}_2 \subseteq G_e(F_{1,n}, i)$. Now, let $Y \in G_e(F_{1,n}, i)$. If $n \in Y$, then, at least one of the vertices labeled $n-1$ or $n-2$ is in Y. If $n-1 \in Y$ then $Y = X_2 \cup \{n\}$ for some $X_2 \in G_e(F_{1,n-1}, i-1)$. Therefore, $Y \in \mathsf{F}_2$. If $n-1 \notin Y$ and $n-2 \in Y$ then $Y = X_1 \cup \{n\}$ for some $X_1 \in G_e(F_{1,n-2}, i-1)$, that is $Y \in \mathsf{F}_1$. Therefore, $G_e(F_{1,n}, i) \subseteq \mathsf{F}_1 \cup \mathsf{F}_2$. Hence, we have $G_e(F_{1,n}, i) = \{X_1 \cup \{n\} / X_1 \in G_e(F_{1,n-2}, i-1)\} \cup \{X_2 \cup \{n\} / X_2 \in G_e(F_{1,n-1}, i-1)\}$.

Example 2.6. Consider $F_{1,7}$ with $V(F_{1,7}) = \{0_1, 1, 2, 3, 4, 5, 6, 7\}$. We use theorem 2.5 to construct $G_e(F_{1,7}, i)$, for $4 \leq i \leq 8$.

Since, $G_e(F_{1,6}, 3) = \phi$ and $G_e(F_{1,5}, 3) = \{1, 3, 5\}$, by theorem 2.5(i), we have $G_e(F_{1,7}, 4) = \{1, 3, 5, 7\}$.

Since, $G_e(F_{1,5}, 7) = \phi$ and $G_e(F_{1,6}, 7) = \{0_1, 1, 2, 3, 4, 5, 6\}$, by theorem 2.5(ii), we have $G_e(F_{1,7}, 8) = \{0_1, 1, 2, 3, 4, 5, 6, 7\}$.

Since, $\mathcal{G}_e(F_{1,5},4)=\{\{0_1,1,3,5\},\{1,2,3,5\},\{1,3,4,5\},\{1,2,4,5\}\}$, and

$\mathcal{G}_e(F_{1,6},4)=\{\{1,3,5,6\},\{1,2,4,6\},\{1,3,4,6\}\}$, by theorem 2.5(iii), we have

$$\mathcal{G}_e(F_{1,7},5)=\{X_1\cup\{7\}/X_1\in\mathcal{G}_e(F_{1,5},4)\}\cup\{X_2\cup\{7\}/X_2\in\mathcal{G}_e(F_{1,6},4)\}$$
$$=\{\{0_1,1,3,5,7\},\{1,2,3,5,7\},\{1,3,4,5,7\},\{1,2,4,5,7\},$$
$$\{1,3,5,6,7\},\{1,2,4,6,7\},\{1,3,4,6,7\}\}.$$

Since, $\mathcal{G}_e(F_{1,5},5)=\{\{0_1,1,2,3,5\},\{0_1,1,3,4,5\},\{0_1,1,2,4,5\},\{1,2,3,4,5\}\}$

and $\mathcal{G}_e(F_{1,6},5)=\{\{0_1,1,3,5,6\},\{0_1,1,2,4,6\},\{0_1,1,3,4,6\},\{1,2,3,5,6\},$

$\{1,3,4,5,6\},\{1,2,3,4,6\},\{1,2,4,5,6\}\}$

by theorem 2.5(iii), we have

$$\mathcal{G}_e(F_{1,7},6)=\{X_1\cup\{7\}/X_1\in\mathcal{G}_e(F_{1,5},5)\}\cup\{X_2\cup\{7\}/X_2\in\mathcal{G}_e(F_{1,6},5)\}$$
$$=\{\{0_1,1,2,3,5,7\},\{0_1,1,3,4,5,7\},\{0_1,1,2,4,5,7\},\{1,2,3,4,5,7\},$$
$$\{0_1,1,3,5,6,7\},\{0_1,1,2,4,6,7\},\{0_1,1,3,4,6,7\},\{1,2,3,5,6,7\},$$
$$\{1,3,4,5,6,7\},\{1,2,3,4,6,7\},\{1,2,4,5,6,7\}\}.$$

Since, $\mathcal{G}_e(F_{1,5},6)=\{0_1,1,2,3,4,5\}$ and

$\mathcal{G}_e(F_{1,6},6)=\{\{0_1,1,2,3,4,6\},\{0_1,1,2,3,5,6\},\{0_1,1,2,4,5,6\},$

$\{0_1,1,3,4,5,6\},\{1,2,3,4,5,6\}\}$

by theorem 2.5 (iii), we have

$$\mathcal{G}_e(F_{1,7},7)=\{X_1\cup\{7\}/X_1\in\mathcal{G}_e(F_{1,5},6)\}\cup\{X_2\cup\{7\}/X_2\in\mathcal{G}_e(F_{1,6},6)\}$$
$$=\{\{0_1,1,2,3,4,5,7\},\{0_1,1,2,3,4,6,7\},\{0_1,1,2,3,5,6,7\},$$
$$\{0_1,1,2,4,5,6,7\},\{0_1,1,3,4,5,6,7\},\{1,2,3,4,5,6,7\}\}.$$

Theorem 2.7. For every $n\geq 5$, $g_e(F_{1,n},i)=g_e(F_{1,n-2},i-1)+g_e(F_{1,n-1},i-1)$

Proof. Case(i). If $\mathcal{G}_e(F_{1,n-2},i-1)\neq\phi$ and $\mathcal{G}_e(F_{1,n-1},i-1)=\phi$, then,

by theorem 2.5(i), we have $\mathcal{G}_e(F_{1,n},i)=\mathcal{G}_e\left(F_{1,n},\frac{n+1}{2}\right)=\{1,3,5,...,n\}$. Therefore, $g_e(F_{1,n},i)=1$. Also,

$\mathcal{G}_e(F_{1,n-2},i-1)=\mathcal{G}_e\left(F_{1,n-2},\frac{n-1}{2}\right)=\{1,3,5,...,n-2\}$.

Therefore, $g_e(F_{1,n-2},i-1)=1$ and $g_e(F_{1,n-1},i-1)=0$. Hence the theorem holds.

Case(ii). If $\mathcal{G}_e(F_{1,n-2},i-1)=\phi$ and $\mathcal{G}_e(F_{1,n-1},i-1)\neq\phi$, then, by theorem 2.5(ii), we have $\mathcal{G}_e(F_{1,n},i)=\mathcal{G}_e(F_{1,n},n+1)=\{0_1,1,2,3,...,n\}$. Therefore, $g_e(F_{1,n},i)=1$. Also, $\mathcal{G}_e(F_{1,n-1},i-1)=\mathcal{G}_e(F_{1,n-1},n)=\{0_1,1,2,3,...,n-1\}$. Therefore, $g_e(F_{1,n-1},i-1)=1$ and $g_e(F_{1,n-2},i-1)=0$. Hence the theorem holds.

Case(iii). If $\mathcal{G}_e(F_{1,n-2},i-1)\neq\phi$ and $\mathcal{G}_e(F_{1,n-1},i-1)\neq\phi$, then, by theorem 2.5(iii), we have $\mathcal{G}_e(F_{1,n},i)=\mathsf{F}_1\cup\mathsf{F}_2$.

Where $\mathsf{F}_1=\{X_1\cup\{n\}/X_1\in\mathcal{G}_e(F_{1,n-2},i-1)\}$ and

$\mathsf{F}_2=\{X_2\cup\{n\}/X_2\in\mathcal{G}_e(F_{1,n-1},i-1)\}$. Therefore, $|\mathsf{F}_1|=g_e(F_{1,n-2},i-1)$ and

$|\mathsf{F}_2|=g_e(F_{1,n-1},i-1)$. Since for every $X_1\in\mathsf{F}_1$ and $X_2\in\mathsf{F}_2$, we have $n-1\in X_2$ but $n-1\notin X_1$, we have $\mathsf{F}_1\cap\mathsf{F}_2=\phi$. Therefore,

$$|\mathcal{G}_e(F_{1,n},i)|=|\{X_1\cup\{n\}/X_1\in\mathcal{G}_e(F_{1,n-2},i-1)\}|$$

$$+\left|\left\{X_2 \cup \{n\} \,/\, X_2 \in G_e(F_{1,n-1}, i-1)\right\}\right|.$$

Hence , $g_e(F_{1,n}, i) = g_e(F_{1,n-2}, i-1) + g_e(F_{1,n-1}, i-1).$

$1,n$ \ i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$F_{1,2}$	0	0	1											
$F_{1,3}$	0	1	2	1										
$F_{1,4}$	0	0	2	3	1									
$F_{1,5}$	0	0	1	4	4	1								
$F_{1,6}$	0	0	0	3	7	5	1							
$F_{1,7}$	0	0	0	1	7	11	6	1						
$F_{1,8}$	0	0	0	0	4	14	16	7	1					
$F_{1,9}$	0	0	0	0	1	11	25	22	8	1				
$F_{1,10}$	0	0	0	0	0	5	25	41	29	9	1			
$F_{1,11}$	0	0	0	0	0	1	16	50	63	37	10	1		
$F_{1,12}$	0	0	0	0	0	0	6	41	91	92	46	11	1	
$F_{1,13}$	0	0	0	0	0	0	1	22	91	154	129	56	12	1

Table 1: $g_e\left(F_{1,n}, i\right)$, the number of geodetic set of $F_{1,n}$ with cardinality *i*.

Theorem 2.8. For every $n \geq 5$, $g(F_{1,n}, x) = x\,[g(F_{1,n-2}, x) + g(F_{1,n-1}, x)]$ with the initial values $g(F_{1,3}, x) = x^2 + 2x^3 + x^4$ and $g(F_{1,4}, x) = 2x^3 + 3x^4 + x^5$.

Proof: By theorem 2.7,

$$g_e(F_{1,n}, i) = g_e(F_{1,n-2}, i-1) + g_e(F_{1,n-1}, i-1)$$

$$\sum_{i=\left\lceil \frac{n+1}{2} \right\rceil}^{n+1} g_e(F_{1,n}, i)\, x^i = \sum_{i=\left\lceil \frac{n+1}{2} \right\rceil}^{n+1} g_e(F_{1,n-2}, i-1)\, x^i + \sum_{i=\left\lceil \frac{n+1}{2} \right\rceil}^{n+1} g_e(F_{1,n-1}, i-1)\, x^i$$

$$g(F_{1,n}, x) = x\left[\sum_{i=\left\lceil \frac{n+1}{2} \right\rceil}^{n+1} g_e(F_{1,n-2}, i-1)\, x^{i-1} + \sum_{i=\left\lceil \frac{n+1}{2} \right\rceil}^{n+1} g_e(F_{1,n-1}, i-1)\, x^{i-1}\right]$$

$$g(F_{1,n}, x) = x\,[g(F_{1,n-2}, x) + g(F_{1,n-1}, x)].$$

Example 2.9. In this example, we have to find the geodetic polynomial of $F_{1,10}$ from the geodetic polynomial of $F_{1,8}$ and $F_{1,9}$ by using theorem 2.8.

From table 1, we have,

$g(F_{1,8}, x) = 4x^5 + 14x^6 + 16x^7 + 7x^8 + x^9$ and

$g(F_{1,9}, x) = x^5 + 11x^6 + 25x^7 + 22x^8 + 8x^9 + x^{10}$

By theorem 2.8, we have,

$g(F_{1,n}, x) = x\,[g(F_{1,n-2}, x) + g(F_{1,n-1}, x)]$

When $n = 10$,

$$g(F_{1,10}, x) = x\,[g(F_{1,8}, x) + g(F_{1,9}, x)]$$
$$= x\,[(4x^5 + 14x^6 + 16x^7 + 7x^8 + x^9)$$

*Corresponding Author: Dr.K.Vijila Dafini

$$+(x^5+11x^6+25x^7+22x^8+8x^9+x^{10})]$$

$$g(F_{1,10},x)=5x^6+25x^7+41x^8+29x^9+9x^{10}+x^{11}$$

Corollary 2.10. The following properties hold for the coefficients of $g(F_{1,n},x)$:

(i) $g_e(F_{1,n},n+1)=1,$ for every $n\geq 3.$

(ii) $g_e(F_{1,n},n)=n-1,$ for every $n\geq 3.$

(iii) $g_e(F_{1,n},n-1)=\dfrac{(n-1)(n-4)}{2}+2,$ for every $n\geq 3.$

(iv) $\sum_{j=n+1}^{2n+3} g_e(F_{1,j},n+2)=2\sum_{j=n}^{2n+1} g_e(F_{1,j},n+1),$ for every $n\geq 2.$

(v) If $S_n=\sum_{j=\lceil\frac{n+1}{2}\rceil}^{n+1} g_e(F_{1,n},j),$ then, for every $n\geq 5,$ $S_n=S_{n-1}+S_{n-2}$ with the initial values $S_3 = 4$, $S_4 = 6$.

(vi) $g_e(F_{1,2n},n+1)=n,$ for every $n\geq 3.$

(vii) $g_e(F_{1,2n+1},n+1)=1,$ for every $n\geq 3.$

Proof of (i). Since $G_e\left(F_{1,n},n+1\right)=\{0_1,1,2,3,...,n\}$, we have the result.

Proof of (ii). By induction on *n*. The result is true for all natural numbers less than *n*, and we prove it for *n*. By theorem 2.7, induction hypothesis and part(i), we have $g_e(F_{1,n},n)=g_e(F_{1,n-2},n-1)+g_e(F_{1,n-1},n-1)$

$$=1+n-2$$
$$=n-1.$$

Proof of (iii). By induction on *n*. The result is true for all natural numbers less than *n*, and we prove it for *n*. By theorem 2.7, the induction hypothesis and part(ii), we have

$$g_e(F_{1,n},n-1)=g_e(F_{1,n-2},n-2)+g_e(F_{1,n-1},n-2)$$
$$=n-3+\frac{(n-2)(n-5)}{2}+2$$
$$=\frac{2(n-3)+(n-2)(n-5)}{2}+2$$
$$=\frac{(n-1)(n-4)}{2}+2.$$

Proof of (iv). By induction on *n*. The result is true for $n=2,$ because $\sum_{j=3}^{7} g_e(F_{1,j},4)=12=2\sum_{j=2}^{5} g_e(F_{1,j},3).$

Now suppose that the result is true for all numbers less than $n+2$, and we prove it for $n+2$. By theorem 2.7, and the induction hypothesis, we have

$$\sum_{j=n+1}^{2n+3} g_e(F_{1,j},n+2) = \sum_{j=n+1}^{2n+3} g_e(F_{1,j-2},n+1) + \sum_{j=n+1}^{2n+3} g_e(F_{1,j-1},n+1)$$
$$=2\sum_{j=n}^{2n+1} g_e(F_{1,j-2},n)+2\sum_{j=n}^{2n+1} g_e(F_{1,j-1},n)$$
$$=2\sum_{j=n}^{2n+1}\left[g_e(F_{1,j-2},n)+g_e(F_{1,j-1},n)\right]$$
$$=2\sum_{j=n}^{2n+1} g_e(F_{1,j},n+1).$$

Proof of (v). $S_n=\sum_{j=\lceil\frac{n+1}{2}\rceil}^{n+1} g_e(F_{1,n},j)$

$$= \sum_{j=\left\lceil \frac{n+1}{2} \right\rceil}^{n+1} \left[g_e(F_{1,n-2}, j-1) + g_e(F_{1,n-1}, j-1) \right]$$

$$= \sum_{j=\left\lceil \frac{n+1}{2} \right\rceil-1}^{n} g_e(F_{1,n-2}, j) + \sum_{j=\left\lceil \frac{n+1}{2} \right\rceil-1}^{n} g_e(F_{1,n-1}, j)$$

$$= \sum_{j=\left\lceil \frac{n-1}{2} \right\rceil}^{n-1} g_e(F_{1,n-2}, j) + \sum_{j=\left\lceil \frac{n}{2} \right\rceil}^{n} g_e(F_{1,n-1}, j)$$

$$S_n = S_{n-2} + S_{n-1}.$$

Proof of (vi) and (vii) is obvious.

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