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Research Paper

Three Related Problems Involving Pythagorean Triples and Triangular Numbers

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I. Introduction:

The n-th *triangular number* is defined by $t_n = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$. The m-th *oblong number* is defined by $O_m = m(m + 1)$. A *Pythagorean triple* is a set of positive integers, (a, b, c), that satisfies $a^2 + b^2 = c^2$. See [1].We show that a solution to any one of the following three problems yields solutions to the other two.

1. Find Pythagorean triples of the form (t_{2x}, t_{2x+1}, z). That is, the legs are consecutive triangular numbers.
2. Find Pythagorean triple of the form $(x, x + 1, w)$.
3. Find triangular number that are oblong.

Problem 1

Let's examine the first problem. Starting with the Pythagorean Theorem, we have

$$t_{2x}{}^2 + t_{2x+1}{}^2 = \left(\frac{2x(2x+1)}{2}\right)^2 + \left(\frac{(2x+1)(2x+2)}{2}\right)^2 = z^2 \Rightarrow$$

$$\left(x(2x+1)\right)^2 + \left((2x+1)(x+1)\right)^2 = z^2 \Rightarrow x^2(2x+1)^2 + (x+1)^2(2x+1)^2 = z^2 \Rightarrow$$

$$\left(x^2 + (x+1)^2\right)(2x+1)^2 = z^2 \Rightarrow \left(2x^2 + 2x + 1\right)(2x+1)^2 = z^2 \Rightarrow$$

$$z = (2x+1)\sqrt{2x^2 + 2x + 1}$$

Then solving problem #1 depends on finding values of x for which $2x^2 + 2x + 1$ is a square. Put this aside to dry, and turn our attention to problem #2.

Problems 2 and 3

The Pythagorean Theorem yields

$$x^2 + (x + 1)^2 = w^2 \Rightarrow \qquad 2x^2 + 2x + 1 = w^2$$

Such an x would also solve problem #1, with $z = (2x + 1)w$.

To tackle problem #3, we shall start with the equation $2x^2 + 2x + 1 = w^2$. It follows that w^2 must be odd, implying that w must be odd. Letting $w = 2k + 1$ yields

$$2x^2 + 2x + 1 = w^2 = (2k + 1)^2 = 4k^2 + 4k + 1 \quad \Rightarrow \quad 2x^2 + 2x = 4k^2 + 4k \Rightarrow$$

$$\frac{x^2 + x}{2} = k^2 + k \Rightarrow t_x = \frac{x(x+1)}{2} = k(k+1) = O_k$$

That is, we have a triangular number, t_x, that equals an oblong number, O_k.

Example: $t_{20} = 210 = O_{14}$ solves problem #3. So $x = 20$ and $k = 14$. Since $w = 2k + 1$, we get $w = 2\cdot 14 + 1 = 29$, giving us the Pythagorean triple (20, 21, 29), a solution to problem #2, that is, we found a Pythagorean triple of the form (x, $x + 1$, w). We also obtain a solution to problem #1. $x = 20$ implies the Pythagorean triple (t_{40}, t_{41}, $41\cdot 29$) = (820, 861, 1189).

Consider the *coupled sequence*:

1, 1

2, 3

5, 7

12, 17

29, 41

⋮

Given any row a, b, the next row is $a + b$, $2a + b$. We will use this chart to solve problem #3, and find infinitely many pairs x and k, for which

$$t_x = \frac{x(x+1)}{2} = k(k+1) = O_k$$

which we write as $x(x + 1) = 2k(k + 1)$. We have the following sequence of equations.

$$x(x + 1) = 2k(k + 1) \quad \Longrightarrow \quad x^2 + x = 2(k^2 + k)$$

Completing the squares yields $$x^2 + x + \frac{1}{4} = 2\left(k^2 + k + \frac{1}{4}\right) - \frac{1}{4}$$

Multiplying by 4 yields $$4x^2 + 4x + 1 = 2(4k^2 + 4k + 1) - 1 \quad \Longrightarrow$$

$$(2x + 1)^2 = 2(2k + 1)^2 - 1 \quad \Longrightarrow \quad (2x + 1)^2 - 2(2k + 1)^2 = -1$$

Now let $a = 2k + 1$ and $b = 2x + 1$

yielding $$\boxed{b^2 - 2a^2 = -1}$$

which is satisfied by every odd row of the coupled sequence

1, 1

2, 3

5, 7

12, 17

29, 41

⋮

Example: The fifth row in the coupled sequence is $a = 29$ and $b = 41$. Then $x = 20$ and $k = 14$, implying that $t_{20} = O_{14}$, as we saw in the previous example.

Now $b^2 - 2a^2 = -1$ is a *Pell equation*. See [1]. It has infinitely many integer solutions, implying that all three problems have infinitely many integer solutions.

References

[1]. M.Lewinter, J.Meyer, *Elementary Number Theory with Programming*, Wiley & Sons. 2015.