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Research Paper

An Optimal Class of Eighth-Order Iterative Methods for Solving Nonlinear Equations

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ABSTRACT: *This paper represents a new class of optimal eighth-order iterative methods for finding the roots of nonlinear equations. The class is designed to enhance the order of convergence by multiplying the Thongmoon method by one weighted function. The efficiency index (EI) of this method is $8^{\frac{1}{4}} \approx 1.682$. Some numerical comparisons have been considered to demonstrate the performance of the proposed method.*
KEYWORDS: *Nonlinear equations, Iterative methods Order of convergence, Optimal method*

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I. INTRODUCTION

Finding a simple root of a nonlinear equation is one of the most significant and oldest numerical analys isproblems. In this paper, we developed an iterative method to find the simple root of a nonlinear equation

$$f(x) = 0, \text{where } f: D \subset \mathrm{R} \to \mathrm{R} \text{ for an open interval D.} \tag{1}$$

Newton's method (NM) is a well-known method for solving nonlinear equations, see for example [1-10]. The method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0,1,2, \dots \, and \, f'(x_n) \neq 0. \tag{2}$$

Newton's method is optimal quadratic convergence [1-10]. The efficiency index (EI) can be defined by $p^{\frac{1}{n}}$, where p is the order of convergence and n is the number of total function and its derivative evaluations per iteration [9]. According to the optimality principle, the optimal order can be calculated by 2^{n-1}[2]. The efficiency index of the optimal method (2), EI = $2^{\frac{1}{2}} \approx 1.4142$.

In recent years, there have been many methods of the optimal three-step eighth-order for solving nonlinear equations which depend onNewton's method in the first step [3-9]. A family of three-step methods is proposed by Thongmoon [1], given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \left(\frac{f(x_n)}{f'(x_n)}\right)^5\right],$$

$$z_n = y_n - \frac{f(y_n)}{f'(x_n)}\left[\frac{1+\beta\left(\frac{f(y_n)}{f(x_n)}\right)^2}{1-2\frac{f(y_n)}{f(x_n)}}\right],$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}\left[\left(\frac{f(y_n)}{f(x_n)}\right)^4 + \left(\frac{f(y_n)}{f(x_n)}\right)^5\right] - \frac{\left[1+\frac{f(z_n)}{f(x_n)}+\left(\frac{f(z_n)}{f(x_n)}\right)^2\right] f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]}. \tag{3}$$

Where $\beta \in R$, this family is seventh–order of convergence, and has efficiency index $7^{\frac{1}{4}} \approx 1.2627$.

If $\beta = 0$, then the algorithm defined by equation (3) has eighth order of convergence.

II. THE CLASS OF METHODS AND CONVERGENCE ANALYSIS

To increase the convergence of method (3) so we can have an optimal eighth order, we will multiply the third step by H(v).

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \left(\frac{f(x_n)}{f'(x_n)}\right)^5\right],$$

$$z_n = y_n - \frac{f(y_n)}{f'(x_n)}\left[\frac{1+\beta\left(\frac{f(y_n)}{f(x_n)}\right)^2}{1-2\frac{f(y_n)}{f(x_n)}}\right],$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}\left[\left(\frac{f(y_n)}{f(x_n)}\right)^4 + \left(\frac{f(y_n)}{f(x_n)}\right)^5\right] - \frac{\left[1+\frac{f(z_n)}{f(x_n)}+\left(\frac{f(z_n)}{f(x_n)}\right)^2\right]f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]}\cdot\{H(v)\}. \quad (4)$$

where $H(v)$ is a real valued weighted function and $v = \frac{f(y_n)}{f(x_n)}$.

The next theorem we prove that the method (4) has an optimal eighth order of convergence when the condition of the weighted function is applied.

Theorem 2.1 Let α be a simple zero of sufficient differentiable function $f:D\subseteq R \to R$ for an open interval D. If x_0 is sufficient closed to α, then the three-step iterative method defined by Algorithm (4) gives an optimal eighth order of convergence if satisfies the following conditions:

$$H(0) = 1,\ H'(0) = 0,\ H''(0) = 0,\ H'''(0) = 6\beta,\ |H^{(4)}(0)| < \infty.$$

Proof: Let $e_n = x_n - \alpha$ *be the error. Expanding* $f(x_n)$ *about* α *by Taylor expansion, we get*

$$f \quad (5)$$

where $c_k = \dfrac{f^{(k)}(\alpha)}{k!\,f'(\alpha)}$ *and* $k = 2,3,\ldots$

Calculating the first derivative of $f(x_n)$ *with respect to* x_n*, we have*

$$f'(x_n) = f'(\alpha)\big(1 + 2c_2e + 3c_3e^2 + 4c_4e^3 + 5c_5e^4 + 6c_6e^5 + 7c_7e^6 + 8c_8e^7 + 9c_9e^8 + O(e^9)\big) \quad (6)$$

Substituting equation (5) *and* (6)*, into the first step of* (4)*, we have*

$$y_n = \alpha + c_2e^2 + (-2c_2^2 + 2c_3)e^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e^4 + \cdots + (64c_2^7 - 304c_2^5c_3 + 176c_2^4c_4 + \cdots + 12c_3 + 7c_8)e^8 + O(e^9) \quad (7)$$

Expanding $f(y_n)$ *about* α *to get*

$$f(y_n) = f'(\alpha)\big(c_2e^2 + (-2c_2^2 + 2c_3)e^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e^4 + \cdots + (144c_2^7 - 552c_2^5c_3 \qquad + \cdots - 31c_4c_5 + 12c_3 + 7c_8)e^8 + O(e^9)\big) \quad (8)$$

In view of (5), (6), (7) *and* (8)*, we get*

$$z_n = \alpha + (-\beta c_2^3 + c_2^3 - c_2c_3)e^4 + \cdots + (-561\beta c_2^7 + 1562\beta c_2^5c_3 + \cdots - 13c_3c_6 - 17c_4c_5 \qquad + 5c_3)e^8 + O(e^9) \quad (9)$$

Expanding $f(z_n)$ *about* α *to get*

$$f(z_n) = f'(\alpha)((-c_2^3\beta + c_2^3 - c_2c_3)e^4 + \cdots + (-180c_2^5c_3 + 101c_2^4c_4 + \cdots - 48\beta c_2c_3c_5)e^8 + O(e^9)) \quad (10)$$

since $f[x_n, y_n] = \dfrac{f(y_n) - f(x_n)}{y_n - x_n}$, *we have*

$$f[x_n, y_n] = f'(\alpha)\Big(1 + c_2e + (c_2^2 + c_3)e^2 + \cdots + (-256c_2^6c_3 + 184c_2^5c_4 + \cdots + 116c_2^2c_3c_5)e^8 + O(e^9)\Big) \quad (11)$$

since $f[y_n, z_n] = \dfrac{f(z_n) - f(y_n)}{z_n - y_n}$ *and* $f[x_n, z_n] = \dfrac{f(z_n) - f(x_n)}{z_n - x_n}$, *we have*

$$f[y_n, z_n] = f'(\alpha)\Big(1 + c_2^2e^2 + \cdots + (170c_2^5c_3 - 118c_2^4c_4 + \cdots + 160\beta c_2^7 - 36\beta c_2^2c_3c_4)e^7 + O(e^8)\Big) \quad (12)$$

and

$$f[x_n, z_n] = f'(\alpha)\Big(1 + c_2e + c_3e^2 + c_4e^3 + \cdots + (-197c_2^6c_3 + 111c_2^5c_4 + \cdots - 84\beta c_2c_3^2c_4)e^8 + O(e^9)\Big) \quad (13)$$

respectivly.

By expanding $H(v)$ *using Taylor expansion, we have*

$$H(v) = H(0) + H'(0)v + \frac{1}{2}H''(0)v^2 + H'''(0)\frac{1}{3!}v^3 + H''''(0)\frac{1}{4!}v^4 + \cdots + O(e^9) \quad (14)$$

Finally, using equations (5) − (6), (8) − (14) *and the conditions*

$$H(0) = 1, \qquad H'(0) = 0, \qquad H''(0) = 0, \qquad H'''(0) = 6\beta, \qquad |H^{(4)}(0)| < \infty \quad (15)$$

we have

$$x_{n+1} = \alpha + \Big(5\beta c_2^5c_3 - \beta c_2^4c_4 + \frac{1}{24}H^{(4)}(0)c_2^5c_3 - \frac{1}{24}H^{(4)}(0)c_2^7 - c_2^2c_3c_4 + 2c_2^7 - 6c_2^5c_3 + c_2^4c_4 + \beta^2c_2^7 + 4c_2^3c_3^2 + \frac{1}{24}\beta c_2^7H^{(4)}(0) - 3\beta c_2^7\Big)e^8 + O(e^9) \quad (16)$$

from (16) *and* $e_{n+1} = x_{n+1} - \alpha$, *then we have the error equation*

$$e_{n+1} = \Big(5\beta c_2^5c_3 - \beta c_2^4c_4 + \frac{1}{24}H^{(4)}(0)c_2^5c_3 - \frac{1}{24}H^{(4)}(0)c_2^7 - c_2^2c_3c_4 + 2c_2^7 - 6c_2^5c_3 + c_2^4c_4 + \beta^2c_2^7 + 4c_2^3c_3^2 + \frac{1}{24}\beta c_2^7H^{(4)}(0) - 3\beta c_2^7\Big)e^8 + O(e^9) \quad (17)$$

which indicates that the new algorithm defined by equation (4) *has an optimal eighth order of convergence for any value of* $\beta \in R$ *and this completes the proof.*

III. THE CONCRETE ITERATIVE METHODS

In the following we present some iterative forms of scheme (4)

Method1 (ZSM1): Let

$$H(v) = -6\beta \sin(v) + 6\beta v + 1, \qquad where\ \beta \in R.$$

It can easily be seen that the function satisfies conditions of Theorem2. Hence, we get a family of eighth order methods (ZSM1), given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \left(\frac{f(x_n)}{f'(x_n)}\right)^5\right],$$

$$z_n = y_n - \frac{f(y_n)}{f'(x_n)}\left[\frac{1+\beta\left(\frac{f(y_n)}{f(x_n)}\right)^2}{1-2\frac{f(y_n)}{f(x_n)}}\right],$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}\left[\left(\frac{f(y_n)}{f(x_n)}\right)^4 + \left(\frac{f(y_n)}{f(x_n)}\right)^5\right] - \frac{\left[1+\frac{f(z_n)}{f(x_n)}+\left(\frac{f(z_n)}{f(x_n)}\right)^2\right]f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]}\{-6\beta \sin(v) + 6\beta v + 1\}. \quad (18)$$

Method2 (ZSM2): Choosing

$$H(v) = \beta v^3 + 1, \qquad where\ \beta \in R.$$

We get another eighth order methods (ZSM2), given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \left(\frac{f(x_n)}{f'(x_n)}\right)^5\right],$$

$$z_n = y_n - \frac{f(y_n)}{f'(x_n)}\left[\frac{1+\beta\left(\frac{f(y_n)}{f(x_n)}\right)^2}{1-2\frac{f(y_n)}{f(x_n)}}\right],$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}\left[\left(\frac{f(y_n)}{f(x_n)}\right)^4 + \left(\frac{f(y_n)}{f(x_n)}\right)^5\right] - \frac{\left[1+\frac{f(z_n)}{f(x_n)}+\left(\frac{f(z_n)}{f(x_n)}\right)^2\right]f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]}\{\beta v^3 + 1\}. \quad (19)$$

Method3 (ZSM3): For the function H defined by

$$H(v) = \cos(v) + \frac{v^2}{2} + \beta v^3, \qquad where\ \beta \in R.$$

Another new eighth order method (ZSM3), given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \left(\frac{f(x_n)}{f'(x_n)}\right)^5\right],$$

$$z_n = y_n - \frac{f(y_n)}{f'(x_n)}\left[\frac{1+\beta\left(\frac{f(y_n)}{f(x_n)}\right)^2}{1-2\frac{f(y_n)}{f(x_n)}}\right],$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}\left[\left(\frac{f(y_n)}{f(x_n)}\right)^4 + \left(\frac{f(y_n)}{f(x_n)}\right)^5\right] - \frac{\left[1+\frac{f(z_n)}{f(x_n)}+\left(\frac{f(z_n)}{f(x_n)}\right)^2\right]f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]}\left\{\cos(v) + \frac{v^2}{2} + \beta v^3\right\}. \quad (20)$$

Method4 (ZSM4): Choosing

$$H(v) = 1 + \beta v^3 e^v, \qquad where\ \beta \in R.$$

We get another eighth order methods (ZSM4), given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \left(\frac{f(x_n)}{f'(x_n)}\right)^5\right],$$

$$z_n = y_n - \frac{f(y_n)}{f'(x_n)}\left[\frac{1+\beta\left(\frac{f(y_n)}{f(x_n)}\right)^2}{1-2\frac{f(y_n)}{f(x_n)}}\right],$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}\left[\left(\frac{f(y_n)}{f(x_n)}\right)^4 + \left(\frac{f(y_n)}{f(x_n)}\right)^5\right] - \frac{\left[1+\frac{f(z_n)}{f(x_n)}+\left(\frac{f(z_n)}{f(x_n)}\right)^2\right]f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]}\{1+\beta v^3 e^v\}. \quad (21)$$

IV. NUMERICAL RESULTS

In this section, wepresent some nonlinear equations to investigate the validity and efficiency of the new methods.We specifically take $\beta = 1$ for the methods (ZSM1-ZSM4) except the first one (ZSM1) which had been tested at 3 different values of beta ($\beta = 0, -2, 1$). We compared the performance of (ZSM1-ZSM4) of the new optimal eighth-order methods, with Newton's method (NM), (2), and some optimal eighth-order methods, for example, (SAM), proposed by Sharma et al. [5], given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \left(3 - 2\frac{f[y_n,x_n]}{f'(x_n)}\right)\frac{f(y_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \left(\frac{f'(x_n)-f[y_n,x_n]+f[z_n,y_n]}{2f[z_n,y_n]-f[z_n,x_n]}\right)\frac{f(z_n)}{f'(x_n)}. \quad (22)$$

Method (LWM) proposed by Liu et al. [4], given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \left(\frac{f(x_n)}{f(x_n)-2f(y_n)}\right)\frac{f(y_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \left(\left(\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}\right)^2 + \frac{f(z_n)}{f(y_n)-f(z_n)} + \frac{4f(z_n)}{f(x_n)+f(z_n)}\right)\frac{f(z_n)}{f'(x_n)}. \quad (23)$$

Method (ASM) proposed byAl-Luhaybi et al. [7], given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \left(\frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}\right)\frac{f(y_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \left(\frac{f(z_n)f[x_n,y_n]}{f[x_n,z_n]f[y_n,z_n]}\right)(1+\sin(v) - 2\beta u^3 e^u), \quad (24)$$

Where $\beta = 1, v = \frac{f(z_n)}{f(x_n)}$ and $u = \frac{f(y_n)}{f(x_n)}$.

Method (TSM) proposed byAl-Harbi et al. [3], given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \left(\frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}\right)\frac{f(y_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \{-2\beta s_1^3 e^{s_1}(s_2^4 + s_2^4 \sin(s_3) + \sin(s_3) + 1) + (s_2^4 + 1)(\sin(s_3) + 1)\}\left(\frac{f(z_n) f[x_n, y_n]}{f[y_n, z_n] f[x_n, z_n]}\right). \quad (25)$$

Where $\beta = 0, s_1 = \frac{f(y_n)}{f(x_n)}, s_2 = \frac{f(z_n)}{f(y_n)}$ and $s_3 = \frac{f(z_n)}{f(x_n)}$.

Method (HSM) proposed byAbbas et al. [6], given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n + (\beta - 1)c_1 - \beta\left(\frac{f(x_n)}{f'(x_n)} + \frac{f(y_n)\left(f(x_n)^3 + f(y_n)^2 f(x_n) + \frac{1}{2} f(y_n)^3\right)\left(f(x_n) + f(y_n)\right)^2}{f'(x_n) f(x_n)^5}\right),$$

$$x_{n+1} = z_n - \frac{f(z_n)}{a_1 + 2a_2(z_n - x_n) + 3a_3(z_n - x_n)^2}. \quad (26)$$

Where $\beta = 2,$ $a_1 = f'(x_n), a_2 = \frac{f[y_n, x_n, x_n](z_n - x_n) - f[z_n, x_n, x_n](y_n - x_n)}{z_n - y_n}, a_3 = \frac{f[z_n, x_n, x_n] - f[y_n, x_n, x_n]}{z_n - y_n}, c_1 = \frac{f(x_n)\left(f(x_n) - f(y_n)\right)}{f'(x_n)\left(f(x_n) - 2f(y_n)\right)},$ $f[y_n, x_n, x_n] = \frac{f[y_n, x_n] - f'(x_n)}{y_n - x_n}$ and $f[z_n, x_n, x_n] = \frac{f[z_n, x_n] - f'(x_n)}{z_n - x_n}$.

Table 1 shows the test functions and their exact roots (α) with only 15 decimal digits.

All calculations were done by MATLAB (R2018a) software, using 1000 digits. We use the following stopping criteria for computer programs:

i. $|x_n - \sigma| \leq 10^{-300}$,

ii. $|f(x_n)| \leq 10^{-300}$.

As displayed in Table 2, it shows a comparison of the number of iterations (IT), the absolute value of the function $|f(x_n)|$, the absolute error $|x_n - \sigma|$, and computational order of convergence (COC) for the methods in NM (2), SAM (22), LWM (23), ASM (24), TSM (25), and HSM (26) provided some initial values to solve the functions in Table 1. Note that the sign (-) in Table 2 shows that the method could not find the root of the function.Moreover, the computational order of convergence (COC) [10] was approximated by

$$\rho \approx \frac{ln|(x_{n+1} - \alpha)/(x_n - \alpha)|}{ln|(x_n - \alpha)/(x_{n-1} - \alpha)|}. \quad (27)$$

Table 1. Test functions and their exact root

Functions	Roots (α)
$3f_1(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5$	-1.207647827130919
$9f_1(x) = cos(x) - x$	0.739085133215161
$15f_2(x) = xe^x - log(1 + x + x^4)$	0
$21f_3(x) = x^3 + 4x^2 - 15$	1.631980805566064
$22f_4(x) = sin^2(x) - x^2 + 1$	1.404491648215341

Table 2. Numerical Comparison

Method	IT	$\lvert f(x_n)\rvert$	$\lvert x_n - \sigma\rvert$	COC
$f_1(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5$, $x_0 = -1.1$				
NM	9	1.02348e-401	5.03992e-403	2
SAM	-	-	-	-

LWM	3	8.19115e-479	4.03358e-480	8
ASM4	3	1.45254e-345	7.15277e-347	8
TSM3	-	-	-	-
HSM1	-	-	-	-
ZSM1 beta = 0	3	1.22858e-456	6.0499e-458	8
ZSM1 beta = -2	3	2.70818e-420	1.33359e-421	8
ZSM1 beta = 1	3	9.77378e-356	4.81291e-357	8
ZSM2	3	6.50726e-356	3.20438e-357	8
ZSM3	3	4.25995e-355	2.09773e-356	8
ZSM4	3	1.5671e-341	7.71687e-343	8
$f_2(x) = cos(x) - x, \quad x_0 = 0.6$				
NM	8	3.00558e-379	1.79587e-379	2
SAM	3	9.31564e-622	5.56619e-622	8
LWM	3	2.73271e-626	1.63282e-626	8
ASM	3	2.5518e-641	1.52473e-641	8
TSM	3	1.01004e-673	6.03511e-674	8
HSM	3	4.28048e-644	2.55763e-644	8
ZSM1,$\beta = 0$	3	2.70227e-725	1.61464e-725	8
ZSM1, $\beta = -2$	3	3.09453e-668	1.84901e-668	8
ZSM1, $\beta = 1$	3	3.77837e-830	2.25761e-830	8
ZSM2	3	2.84063e-831	1.6973e-831	8
ZSM3	3	6.75273e-813	4.03482e-813	8
ZSM4	3	7.17799e-745	4.28892e-745	8
$f_3(x) = xe^x - log(1 + x + x^4), \quad x_0 = 0.25$				
NM	9	5.41196e-552	2.70598e-552	2
SAM	3	1.50377e-441	7.51884e-442	8
LWM	3	9.9968e-389	4.9984e-389	8
ASM	3	2.17545e-533	1.08772e-533	8
TSM	3	4.91806e-441	2.45903e-441	8
HSM	3	2.86805e-436	1.43403e-436	8
ZSM1, $\beta = 0$	3	2.53856e-434	1.26928e-434	8
ZSM1, $\beta = -2$	3	5.56384e-503	2.78192e-503	8
ZSM1, $\beta = 1$	3	2.54883e-421	1.27441e-421	8
ZSM2	3	2.64781e-421	1.32391e-421	8
ZSM3	3	3.29645e-421	1.64822e-421	8
ZSM4	3	3.21412e-419	1.60706e-419	8

Method	IT	$\|f(x_n)\|$	$\|x_n - \sigma\|$	COC
$f_4(x) = x^3 + 4x^2 - 15, \quad x_0 = 2$				
NM	9	2.77652e-437	1.31927e-438	2
SAM	3	1.55077e-365	7.36848e-367	8
LWM	3	3.95747e-386	1.88039e-387	8
ASM	3	6.31234e-373	2.99931e-374	8

TSM	3	1.20424e-426	5.72198e-428	8
HSM	3	1.64273e-354	7.80544e-356	8
ZSM1, $\beta = 0$	3	1.19156e-477	5.6617e-479	8
ZSM1, $\beta = -2$	3	1.11793e-404	5.31184e-406	8
ZSM1, $\beta = 1$	3	1.90995e-475	9.07515e-477	8
ZSM2	3	2.2118e-475	1.05094e-476	8
ZSM3	3	6.24043e-473	2.96515e-474	8
ZSM4	3	3.99572e-454	1.89857e-455	8
$f_5(x) = sin^2(x) - x^2 + 1, \quad x_0 = 1.6$				
NM	9	1.17422e-446	4.73003e-447	2
SAM	3	2.56403e-370	1.03285e-370	8
LWM	3	2.02765e-382	8.16787e-383	8
ASM	3	9.22215e-378	3.71491e-378	8
TSM	3	1.45168e-423	5.84773e-424	8
HSM	3	3.23698e-365	1.30394e-365	8
ZSM1, $\beta = 0$	3	2.76026e-431	1.1119e-431	8
ZSM1, $\beta = -2$	3	6.01381e-385	2.42251e-385	8
ZSM1, $\beta = 1$	3	4.18107e-486	1.68424e-486	8
ZSM2	3	3.74384e-486	1.50811e-486	8
ZSM3	3	1.12888e-487	4.5474e-488	8
ZSM4	3	1.26825e-491	5.10881e-492	8

V. CONCLUSION

In this paper, we have developed a new class of optimal eighth-order iterative methods for solving nonlinear equations. The new proposed class isobtained by multiplying the Thongmoon method by one weighted function. The new class of methods has EI = $8^{\frac{1}{4}} \approx 1.682$, and four functions evaluation. Numerical results were presented to explain the performance and efficiency of the newly proposed methods.According to the results, the iterative methods presented in this paper perform effectively like its counterparts from theeighth-order.

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