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Research Paper

CERTAIN NEW INTEGRAL FORMULAE INVOLVING THE BESSEL-WRIGHT FUNCTION AND TRICOMI FUNCTIONS

Priyanka Srivastava[1] and S. K. Raizada[2]

1. *(Research Scholar, Department of Mathematics and Statistics, Dr. Rammanohar Lohia Avadh University Ayodhya, U. P., India)*
2. *(Professor, Department of Mathematics and Statistics, Dr. Rammanohar Lohia Avadh University Ayodhya, U. P., India)*

Corresponding Author: Priyanka Srivastava

ABSTRACT: In this paper, using Oberhet-Tinger's integral formula. We have obtained two integral formulae involving the Bessel-Wright Function $J_n^{(m)}(x)$ introduced by Srivastava and Manocha in the year 1984. The results are given in the form of two Theorems. Two integral formulae for the Tricomi functions defined by Andrews have been obtained as special cases from our two main results.

KEYWORDS*:* Bessel-Wright function, Tricomi functions, Generalized Wright Hypergeometric function and Oberhet-Tinger integral formula.

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I. INTRODUCTION AND PRELIMINARIES

A remarkably large number of integral formulae involving a variety of Special functions have been developed by many authors [2], [3], [4], [5] and [7].

The Bessel-Wright function is defined by Srivastava and Manocha, [10] as:

$$J_n^m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!\ (n+mk)!} \tag{1.1}$$

For m=1, the function $J_n^m(x)$ given is (1.1) reduces to the n^{th}order Tricomi function $C_n(x)$ is defined by Andrews, [1] as:

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!\ (n+r)!} \tag{1.2}$$

The Generalized Wright Hypergeometric function [9, Eq^n no. 38, Pg. no. 21] ${}_p\Psi_q$ is given as:

$${}_p\Psi_q \left[\begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p); \\ \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{matrix} z \right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j k) z^k}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j k) k!}, \tag{1.3}$$

where the coefficients $A_1, \dots, A_p$ and $B_1, \dots, B_q$ are real positive numbers such that:

$$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geqq 0 \tag{1.4}$$

A special case of (1.3) is given by:

$${}_p\Psi_q \left[\begin{matrix} (\alpha_1, 1), \dots, (\alpha_p, 1); \\ \\ (\beta_1, 1), \dots, (\beta_q, 1); \end{matrix} \; z \right] = \sum_{l=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{q} \Gamma(\beta_j)} \; {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \\ \beta_1, \dots, \beta_q; \end{matrix} \; z \right], \tag{1.5}$$

where ${}_pF_q$ is the Generalized Hypergeometric Function [11] defined by:

$${}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \\ \beta_1, \dots, \beta_q; \end{matrix} \; z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n, \dots, (\alpha_p)_n z^n}{(\beta_1)_n, \dots, (\beta_q)_n n!} \tag{1.6}$$

where $(\lambda)_n$, $(\lambda \in \mathbb{C})$ is the Pochhammer symbol [8] defined:

$$(\lambda)_n = \begin{cases} \lambda(\lambda+1)\dots(\lambda+n+1) \end{cases} \quad n \geq 1, \quad (\lambda)_0 = 1 \tag{1.7}$$
$$= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$$

For our present investigation, we also need the following Oberhet-Tinger's integral formula [6]:

$$\int_0^{\infty} x^{\mu-1} \left(x + a + \sqrt{x^2 + 2ax}\right)^{-\lambda} dx = 2\,\lambda\, a^{-\lambda} \left(\frac{a}{2}\right)^{\mu} \frac{\Gamma(2\mu)\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu)},$$
$$\text{Provided}\big(0 < \Re(\mu) < R(\lambda)\big). \tag{1.8}$$

In the present note, we have established two Integral formulae for the Bessel-Wright function $J_n^{(m)}(x)$ which are represented in terms of the Generalized Wright Hypergeometric function ${}_p\Psi_q$. We have also obtained two integral formulae for Tricomi functions in terms of ${}_2\Psi_3$.

II. MAIN RESULTS

In this section, we will establish two generalized integral formulae for the Generalized Bessel-Wright function $J_n^{(m)}(x)$, which are expressed in terms of the Generalized Wright Hypergeometric function ${}_p\Psi_q$:

Theorem I. *The following result has been obtained*:

$$\int_0^{\infty} x^{\mu-1} \left(x + a + \sqrt{x^2 + 2ax}\right)^{-\lambda} J_{\nu}^{(m)} \left(\frac{y}{x + a + \sqrt{x^2 + 2ax}}\right) dx$$

$$= 2^{1-\mu} a^{-\lambda-\nu+\mu} \Gamma 2\mu$$
$$\times \; {}_2\Psi_3 \left[\begin{matrix} (\lambda+\nu-\mu, 2), (1+\lambda+\nu, 2); \\ \\ (\nu+1, m), (\lambda+\nu, 2), (1+\lambda+\nu+\mu, 2); \end{matrix} \; -\frac{y}{a^2} \right], \tag{2.1}$$

where:
for $(x > 0;\ \lambda,\ \mu,\ \nu \epsilon\, \mathbb{C}\ with\ \Re(\nu) > -1\ and\ 0 < \Re(\mu) < R(\lambda+\nu))$.

Proof: Denoting the left-hand side of equation (2.1) by I and replacing λ by $(\lambda+\nu+2k)$ we get:

$$I = \int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-(\lambda+\nu+2k)} J_\nu^{(m)}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) dx$$

$$= 2\,(\lambda+\nu+2\kappa)\, a^{-(\lambda+\nu+2\kappa)}\left(\frac{a}{2}\right)^{\mu}\frac{\Gamma 2\mu\, \Gamma(\lambda+\nu+2\kappa-\mu)}{\Gamma(1+\lambda+\nu+2\kappa+\mu)}$$

$$\times \sum_{\kappa=0}^{\infty}\frac{(-1)^k y^k}{k!\ \Gamma(\nu+mk+1)} \qquad \{using\ (1.1)\ \&\ (1.8)\}$$

$$= 2^{1-\mu}a^{-\lambda-\nu+\mu}\Gamma 2\mu \sum_{\kappa=0}^{\infty}\frac{(-1)^\kappa\Gamma(\lambda+\nu+2\kappa-\mu)\Gamma(\lambda+\nu+2\kappa+1)}{\kappa!\ \Gamma(\nu+m\kappa+1)\Gamma(1+\lambda+\nu+2\kappa+\mu)\Gamma(\lambda+\nu+2\kappa)}\left(\frac{y}{a^2}\right)^k,$$

$$= 2^{1-\mu}a^{-\lambda-\nu+\mu}\Gamma 2\mu$$

$$\times\ {}_2\Psi_3\left[\begin{matrix}(\lambda+\nu-\mu,2),(1+\lambda+\nu,2); \\ (\nu+1,m),(\lambda+\nu,2),(1+\lambda+\nu+\mu,2);\end{matrix}\ -\frac{y}{a^2}\right], \quad \{using\ (1.3)\}$$

which is the right-hand side of (2.1), i.e. the desired result.

Theorem II. The following result has been obtained:

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2}+2ax\right)^{-\lambda} J_\nu^{(m)}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= 2^{1-\mu-\nu}a^{\mu-\lambda}\Gamma(\lambda-\mu)$$

$$\times\ {}_2\Psi_3\left[\begin{matrix}(2\mu+2\nu,4),(\lambda+\nu+1,2); \\ (\nu+1,m),(\lambda+\mu+2\nu,4),(\lambda+\nu,2);\end{matrix}\ -\frac{y}{4}\right], \qquad (2.2)$$

where:

for (x> 0; $\lambda,\mu,\nu\, \epsilon\, \mathbb{C}\ with\ \Re(\nu) > -1\ and\ 0 < \Re(\mu) < R(\lambda+\nu)$).

Proof: Denoting the left-hand side of equation (2.3) by I_1 and replacing λ by $(\lambda+\nu+2k)$ and μ by $(\mu+\nu+2k)$ we get:

$$I_1 = \int_0^\infty x^{(\mu+\nu+2\kappa)-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-(\lambda+\nu+2\kappa)} J_\nu^{(m)}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right) dx$$

$$= 2\,(\lambda+\nu+2\kappa)\, a^{-(\lambda+\nu+2\kappa)}\left(\frac{a}{2}\right)^{\mu+\nu+2\kappa}\frac{\Gamma(2\mu+2\nu+4\kappa)\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu+2\nu+4\kappa)}$$

$$\times\sum_{\kappa=0}^{\infty}\frac{(-1)^\kappa y^\kappa}{\kappa!\ \Gamma(\nu+m\kappa+1)} \qquad \{using\ (1.1)\ \&\ (1.8)\}$$

$$= 2^{1-\mu-\nu} a^{\mu-\lambda} \Gamma(\lambda-\mu)$$

$$\times \sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa} \Gamma(2\mu+2\nu+4\kappa) \Gamma(\lambda+\nu+2\kappa+1)}{\Gamma(\nu+m\kappa+1) \Gamma(1+\lambda+2\nu+4\kappa) \Gamma(\lambda+\nu+2\kappa)} \left(\frac{y}{4}\right)^{\kappa}$$

$$= 2^{1-\mu-\nu} a^{\mu-\lambda} \Gamma(\lambda-\mu)$$

$$\times \; {}_2\Psi_3 \left[\begin{matrix} (2\mu+2\nu,4),(\lambda+\nu+1,2); \\ \\ (\nu+1,m),(\lambda+\mu+2\nu,4),(\lambda+\nu,2); \end{matrix} \; -\frac{y}{4} \right], \{using\ (1.3)\}$$

which is the right-hand side of (2.2), i.e. the desired result.

III. SPECIAL CASES

For m=1, Theorem I and Theorem II reduce to the integrals for Tricomi functions $C_{\nu}(x)$ [1] given respectively as:

I.

$$\int_0^{\infty} x^{\mu-1} \left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} C_{\nu}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) dx$$

$$= 2^{1-\mu} a^{\mu-\lambda-\nu} \Gamma 2\mu$$

$$\times \; {}_2\Psi_3 \left[\begin{matrix} (\lambda+\nu-\mu,2), \; (1+\lambda+\nu,2); \\ \\ (\nu+1,1),(1+\lambda+\nu+\mu,2), \; (\lambda+\nu,2); \end{matrix} \; -\frac{y}{a^2} \right] \tag{3.1}$$

II.

$$\int_0^{\infty} x^{\mu-1} \left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} C_{\nu}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right) dx$$

$$=2^{1-\mu-\nu} a^{\mu-\lambda} \; \Gamma(\lambda-\mu)$$

$$\times \; {}_2\Psi_3 \left[\begin{matrix} (2\mu+2\nu,4), \; (1+\lambda+\nu,2); \\ \\ (\nu+1,1),(1+\lambda+\mu+2\nu,4),(\lambda+\nu,2); \end{matrix} \; -\frac{y}{4} \right] \tag{3.2}$$

Proof of (3.1): Denoting the left-hand side of equation (3.1) by I_2 and replacing λ by $(\lambda+\nu+2k)$ we get:

$$I_2 = \int_0^{\infty} x^{\mu-1} \left(x+a+\sqrt{x^2+2ax}\right)^{-(\lambda+\nu+2k)} C_{\nu}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) dx$$

$$= 2\,(\lambda+\nu+2\kappa) \left(\frac{a}{2}\right)^{\mu} a^{-(\lambda+\nu+2\kappa)} \frac{\Gamma 2\mu \, \Gamma(\lambda+\nu+2\kappa-\mu)}{\Gamma(1+\lambda+\nu+2\kappa+\mu)}$$

$$\times \sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa} y^{\kappa}}{\kappa!\ \Gamma(\nu+\kappa+1)} \qquad \{using\ (1.2)\ \&\ (1.8)\}$$

$$= 2^{1-\mu} a^{\mu-\lambda-\nu} \Gamma 2\mu \sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa} \Gamma(\lambda+\nu+2\kappa+1)\Gamma(\lambda+\nu+2\kappa-\mu)}{\kappa!\ \Gamma(\nu+\kappa+1)\Gamma(\lambda+\nu+2\kappa)\Gamma(1+\lambda+\nu+2\kappa+\mu)} \left(\frac{y}{a^2}\right)^{\kappa}$$

$$= 2^{1-\mu} a^{\mu-\lambda-\nu} \Gamma 2\mu$$

$$\times\ {}_2\Psi_3 \left[\begin{matrix} (\lambda+\nu-\mu,2),(1+\lambda+\nu,2); \\ (\nu+1,1),(1+\lambda+\nu+\mu,2),(\lambda+\nu,2); \end{matrix} \; -\frac{y}{a^2} \right], \quad \{\text{using } (1.3)\}$$

which is the right-hand side of (3.1), i.e. the desired result.

Proof of (3.2): Using (1.2) and (1.8) and replacing λ by $(\lambda+\nu+2k)$ and μ by $(\mu+\nu+2k)$, on the left-hand side of (3.2) and then (1.3) and after simple calculation, as in the proof of (3.1) we get the right-hand side of (3.2), i.e. the desired result.

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