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Review Paper

Generalized fractional integral formula Involving Generalized Bessel Functions

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ABSTRACT: *Some new results are obtained by the application of Fractional Integral formula on Generalized Bessel functions $I_v(x)$. We have also investigated the Saigo Maeda trnsforms of $I_v(x)$. Some results are established in terms of Lauricella Function $F_A^{(r)}$ defined by Srivastava & Daust. Applications are also reviewed.*
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I. INTRODUCTION

Many researchers in the field of Special Functions & their applications in other branches of Mathematics are applying Fractional Calculus/operators of many Special functions. In the recent past much work has been done on the application of Fractional Differential and Integral Operators on various known Special Functions. In this sequence Marichev along with Saigo & Maeda [1,2] defined and studied the following operators.

$$\left(I_{0,+}^{\alpha,\alpha^1,\beta,\beta^1,\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma\gamma}\int_0^x (x-t)^{\gamma-1} t^{-\gamma} f_3\left(\alpha,\alpha^1,\beta,\beta^1,\gamma;\frac{1-t}{x},\frac{1-x}{t}\right) f(t)\,dt \tag{1}$$

$$\left(I_{0,-}^{\alpha,\alpha^1,\beta,\beta^1,\gamma} f\right)(x) = \frac{x^{-\alpha^1}}{\Gamma\gamma}\int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} f_3\left(\alpha,\alpha^1,\beta,\beta^1,\gamma;\frac{1-t}{x},\frac{1-x}{t}\right) f(t)\,dt \tag{2}$$

In the present paper, we have applied these operators on $I_v(z)$ introduced by Watson, Deniz & Bricz [5] defined as :

$$I_v(z) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma k+v+1}\left(\frac{z}{2}\right)^{2k+v}, (z \in C) \tag{3}$$

We have obtained the main results in terms of ${}_p\psi_q(z)$ [1] & ${}_pF_q$ [10].

II. Preliminaries

2.1. LEMMA I [1] –

$$\left(I_{0,+}^{\alpha,\alpha^1,\beta,\beta^1,\gamma} t^{\rho-1}\right)(x) = \Gamma\begin{bmatrix} \rho, \rho+\gamma-\alpha-\alpha^1-\beta, \rho+\beta^1-\alpha^1, \\ \rho+\beta^1, \rho+\gamma-\alpha-\alpha^1, \rho+\gamma-\alpha^1-\beta \end{bmatrix}(x)^{\rho-\alpha-\alpha^1+\gamma^1} \tag{4}$$

where $\alpha, \alpha', \beta, \beta', \gamma, \rho \in C$, $R(\gamma) > 0$ & $R(\rho) > max\{0, R(\alpha-\alpha'-\beta-\gamma), R(\alpha'-\beta')\}$

$$\Gamma\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \frac{\Gamma a \quad \Gamma b \quad \Gamma c}{\Gamma d \quad \Gamma e \quad \Gamma f} \tag{5}$$

2.2. LEMMA II [2]-

$$I_{x,\infty}^{\alpha,\alpha^1,\beta,\beta^1,\gamma} x^{\rho-1} = x^{\rho-\alpha-\alpha^1+\gamma-1} \frac{\Gamma\left(1+\alpha+\alpha^1-\gamma-\rho\right)\Gamma\left(1+\alpha+\beta^1-\gamma-\rho\right)\Gamma\left(1-\beta-\gamma\right)}{\Gamma 1-\rho \Gamma 1+\alpha+\alpha^1+\beta^1-\gamma-\rho \Gamma 1+\alpha-\beta-\rho} \tag{6}$$

where $\alpha, \alpha', \beta, \beta', \gamma, \rho \in C$, $R(\gamma) > 0$ & $R(\rho) < 1 + \min\left[R(-\beta), R(1+\alpha'-\gamma)(\alpha+\beta'-\gamma)\right]$

2.3. Generalized Wright function $_p\Psi_q$ [1]:

$$_p\Psi_q(z) = {}_p\Psi_q\begin{bmatrix} (a_i, \alpha_i)1, p \\ (b_j \beta_j)1, q \end{bmatrix} | z \Bigg] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k) z^k}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k) k^i}$$

where

$$a_i, b_j \in C \;\&\; \alpha_i, \beta_j \in R \;\; (i = 1, 2, 3, \ldots.. p, \quad j = 1, 2, 3 \ldots..q) \tag{7}$$

2.4. Generalized Hypergeometric series $_pF_q$ [10]:

$$pFq\left(a_1, a_2 \ldots. a_p, c_1, c_2 \ldots \mathrm{c}_q : z\right)$$

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots\ldots (a_p)_k \; z^k}{(c_1)_k \ldots\ldots (c_q)_k} \tag{8}$$

2.5. Legendre duplication Formula [10]:

$$\Gamma 2z = \frac{2^{2z-1}}{\Gamma \pi} \Gamma z\, \Gamma\left(z + \frac{1}{2}\right) \tag{9}$$

2.6. $$(z)_{2k} = z^{2k} \left(\frac{z}{2}\right)_k \left(\frac{z+1}{2}\right)_k , \tag{10}$$

2.7. $$\Gamma z + k = \Gamma z (z)_k \tag{11}$$

III. Main Results

Theorem 3.1 :

$$\left(I_{0,+}^{\alpha,\alpha^1,\beta,\beta^1,\gamma} t^{\rho-1} I_v(t)\right)(x)$$

$$= \frac{x^{\rho+v-\alpha-\alpha'+\gamma-1}}{2^v} \times {}_3\Psi_4 \left[\begin{matrix} (\rho+v,2),(\rho+v+\gamma-\alpha-\alpha'-\beta,2),(\rho+v+\beta'-\alpha',2) \\ (\rho+v+\beta',2),(\rho+v+\gamma-\alpha-\alpha',2),(\rho+v+\gamma-\alpha'-\beta,2),(v+1,1) \end{matrix} \left(-\frac{x^2}{4}\right) \right]$$

(12)

where $\alpha, \alpha^{'}, \beta, \beta^{'}, \gamma, \rho, v, b, c \in C,\ \ R(\gamma) > 0$ & $R(\rho+v) > max\{0, R(\alpha+\alpha'+\beta-\gamma), R(\alpha'-\beta')\}$

Proof: Let us denote left hand side of equation (12) by A and then applying the equation (3) we get:

$$A = I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} \sum_{K=0}^{\infty} \left[\frac{1.\left(\frac{1}{2}\right)^{2k+v}}{k!\,\Gamma k+v+1} t^{\rho+v+2k-1} \right](x)$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{2k+v}}{\lceil k+v+1.\mathrm{k}!} \left[I_{0,+}^{\alpha_1\alpha^1,\beta_1\beta^1,\gamma} t^{\rho+v+2k-1} \right](x)$$

(using Eqn 1)

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{2k+v}}{\Gamma k+v+1.k!} \Gamma \left[\begin{matrix} \rho+v+2k, \rho+v+2k+\gamma-\alpha-\alpha'-\beta,\ \ \rho+v+2k+\beta'-\alpha', \\ \rho+v+2k+\beta',\ \ \rho+v+2k+\gamma-\alpha-\alpha', \rho+v+2k+\gamma-\alpha'-\beta \end{matrix} \left(-\frac{x^2}{4}\right)^k \right] x^{\rho+v+2k-\alpha-\alpha'+\gamma-1}$$

(using Eqn 4)

$$=\left(\frac{1}{2}\right)^{v} x^{\rho+v-\alpha-\alpha'+\gamma-1}\sum_{k=0}^{\infty}\frac{1}{\Gamma k+v+1.k!}\left(\frac{x}{2}\right)^{2k}\Gamma\left[\begin{matrix}\rho+v+2k,\rho+v+\gamma-\alpha-\alpha'-\beta+2k, & \rho+v+\beta'-\alpha'+2k, \\ \rho+v+2k+\beta', & \rho+v+\gamma-\alpha-\alpha'+2k,\rho+v+\gamma-\alpha'-\beta+2k\end{matrix}\left(-\frac{x^{2}}{4}\right)^{k}\right]$$

$$=\left(\frac{1}{2}\right)^{v} x^{\rho+v-\alpha-\alpha'+\gamma-1}\times {}_{3}\Psi_{4}\left[\begin{matrix}(\rho+v,2),(\rho+v+\gamma-\alpha-\alpha'-\beta,2),(\rho+v+\beta'-\alpha',2), \\ (\rho+v+\beta',2)(\rho+v+\gamma-\alpha-\alpha',2)(\rho+v+\gamma-\alpha'-\beta,2)\end{matrix}\left(-\frac{x^{2}}{4}\right)\right]$$

(13)

(using Eqn 7)

which proves the Theorem (3.1), i.e. desired result (12).

Corollary 3.1:

Substituting $\beta=\beta'=0$ in Eqn (12), we get the following result-

$$I_{0,+}^{\alpha,\alpha',\gamma}t^{\rho-1}I_{v}(t)(x)=\frac{x^{\rho+v-\alpha-\alpha'+\gamma-1}}{2^{v}}\,{}_{1}\Psi_{2}\left[\begin{matrix}(\rho+v,2) \\ (\rho+v+\gamma-\alpha-\alpha',2),(v+1,1)\end{matrix}\left(-\frac{x^{2}}{4}\right)\right] \tag{14}$$

Theorem 3.2:

$$\left(I_{0,}^{\alpha,\alpha^{1},\beta,\beta^{1},\gamma}t^{\rho-1}I_{v}\left(\frac{1}{t}\right)\right)(x)$$

$$=\frac{x^{\rho+v-\alpha-\alpha'+\gamma-1}}{2^{v}}\,{}_{3}\Psi_{4}\left[\begin{matrix}(1-\rho+v-\gamma+\alpha+\alpha',2),(1-\rho+v+\alpha-\beta'-\gamma,2),(1-\rho+v-\beta,2), \\ (1-\rho+v,2)(1-\rho+v-\gamma+\alpha+\alpha'+\beta',2)(1-\rho+v+\alpha-\beta,2)(v+1,k)\end{matrix}\left(-\frac{1}{16x^{2}}\right)\right]$$

(15)

where $\alpha,\alpha',\beta,\beta',\gamma,\rho,v,c\in C,\ R(\gamma)>0$ &

$R(\rho-v)<1+\min\left[R(-\beta),\ R(\alpha+\alpha'-\gamma)R(\alpha+\beta'-\gamma)\right]$

Proof: Let us denote left hand side of eqn (15) by B and then applying Eqn (3)we get-

$$B = I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} \sum_{K=0}^{\infty} \left[\frac{1.\left(\frac{1}{2}\right)^{2k+v}}{k!\Gamma k+v+1} t^{\rho-v-2k-1} \right](x)$$

$$\left(I_{0,-}^{\alpha_1\alpha^1,\beta_1\beta^1,\gamma} t^{\rho-1} I_v\left(\frac{1}{t}\right)\right)(x)$$

$$= x^{\rho-v-2k-\alpha-\alpha^1+\gamma-1} \sum_{k=0}^{\infty} \Gamma$$

$$\left[\begin{matrix} 1-\rho-v-\gamma+\alpha+\alpha^1+2k, 1-\rho+v+\alpha+\beta^1-\gamma+2k, 1-\rho+v-\beta+2k \\ 1-\rho+v+2k, 1-\rho+v-\gamma+\alpha+\alpha^1+\beta^1+2k, 1-\rho+v+\alpha-\beta+2k \end{matrix} \right] \frac{1}{k!}\left(-\frac{1}{4x^2}\right)^k$$

(using Eqn 2)

$$= \left(\frac{1}{2}\right)^v x^{\rho-v-\alpha-\alpha^1+\gamma-1} \sum_{k=0}^{\infty} \frac{1}{k!\Gamma k+v+1}\left(\frac{x}{2}\right)^{2k}$$

$$\Gamma\left[\begin{matrix} 1-\rho-v-\gamma+\alpha+\alpha^1+2k, 1-\rho+v+\alpha+\beta^1-\gamma+2k, 1-\rho+v-\beta+2k \\ 1-\rho+v+2k, 1-\rho+v-\gamma+\alpha+\alpha^1+\beta^1+2k, 1-\rho+v+\alpha-\beta+2k \end{matrix} \right]\left(-\frac{1}{4x^2}\right)^k$$

(using Eqn4)

$$= \left(\frac{x}{2}\right)^v \frac{x^{\rho-\alpha-\alpha'+\gamma-1}}{2^v} {}_3\Psi_4\left[\begin{matrix} (1-\rho+v-\gamma+\alpha+\alpha',2),(1-\rho+v+\alpha-\beta'-\gamma,2),(1-\rho+v-\beta',2), \\ (1-\rho+v,2)(1-\rho+v-\gamma+\alpha+\alpha'+\beta',2)(1-\rho+v+\alpha-\beta,2)(v+1,k) \end{matrix} \left(-\frac{1}{16x^2}\right)\right]$$

(using Eqn 7)

$$= \frac{x^{\rho+v-\alpha-\alpha'+\gamma-1}}{2^v} {}_3\Psi_4\left[\begin{matrix} (1-\rho+v-\gamma+\alpha+\alpha',2),(1-\rho+v+\alpha-\beta'-\gamma,2),(1-\rho+v-\beta',2), \\ (1-\rho+v,2)(1-\rho+v-\gamma+\alpha+\alpha'+\beta',2)(1-\rho+v+\alpha-\beta,2)(v+1,k) \end{matrix} \left(-\frac{1}{16x^2}\right)\right]$$

(16)

which proves the Theorem (3.2) i.e. the desired result(15).

Corollary 3.2:

Substituting $\beta = \beta' = 0$ in Eqn (15), we get the following result-

$$I_{0,-}^{\alpha,\alpha',\gamma} t^{\rho-1} I_v\left(\frac{1}{t}\right)(x) = \frac{x^{\rho+v-\alpha-\alpha'+\gamma-1}}{2^v} {}_1\Psi_2\left[\begin{matrix}(1-\rho+v-\gamma+\alpha+\alpha',2)\\(1-\rho+v,2),(v+1,k)\end{matrix}\left(\frac{-1}{16}\right)\right] \tag{17}$$

Theorem 3.3:

$$\left(I_{0,+}^{\alpha,\alpha^1,\beta,\beta^1,\gamma} t^{\rho-1} I_v(t)\right)(x) = \frac{x^{\rho+v-1}}{2^v} \frac{\Gamma\rho+v\Gamma\left(\rho+v+\gamma-\alpha-\alpha^1-\beta\right).\Gamma\rho+v+\beta^1-\alpha^1}{\Gamma\rho+v+\beta^1\ \Gamma\rho+v+\gamma-\alpha-\alpha^1\ \Gamma k+1\ \Gamma\rho+v+\gamma-\alpha^1-\beta} \times$$

$${}_6F_7\left[\begin{matrix}\frac{\rho+v}{2},\frac{\rho+v+1}{2},\frac{\rho+v+\gamma-\alpha-\alpha'-\beta}{2},\frac{\rho+v+\gamma-\alpha-\alpha'-\beta+1}{2},\frac{\rho+v+\beta'-\alpha'}{2},\frac{\rho+v+\beta'-\alpha'+1}{2}\\ v+1,\frac{\rho+v+\beta'}{2},\frac{\rho+v+\beta'+1}{2},\frac{\rho+v+\gamma-\alpha-\alpha'}{2},\frac{\rho+v+\gamma-\alpha-\alpha'+1}{2},\frac{\rho+v+\gamma-\alpha'-\beta}{2},\frac{\rho+v+\gamma-\alpha'-\beta+1}{2}\end{matrix} \middle| -\frac{x^2}{4}\right] \tag{18}$$

where $\alpha,\alpha',\beta,\beta',\gamma,\rho,v,c \in C,\ R(\gamma)>0$ &
$R(\rho+v) > max\{0,\ R(\alpha-\alpha'-\beta-\gamma),\ R(\alpha'-\beta')\}$

Proof: Let us denote the left hand side of Eqn (18) by C and by use of eqn (11) we get:

$$C = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)^{2k+v}}{\Gamma(v+k+1)k!} \times \frac{\Gamma(\rho+v)\Gamma(\rho+v+\gamma-\alpha-\alpha'-\beta)\Gamma\rho+v+\beta'-\alpha'}{\Gamma\rho+v+\beta'\Gamma(\rho+v+\gamma-\alpha-\alpha')\Gamma(\rho+v+\gamma-\alpha'-\beta)}$$

$$\times \frac{(\rho+v)_{2k}(\rho+v+\gamma-\alpha-\alpha'-\beta)_{2k}(\rho+v+\beta'-\alpha')_{2k}}{(\rho+v+\beta')_{2k}(\rho+v+\gamma-\alpha-\alpha')_{2k}(\rho+v+\gamma-\alpha'-\beta)_{2k}} \times x^{\rho+v+2k-1}$$

$$= \frac{x^{\rho+v-1}}{2^v} \frac{\Gamma(\rho+v)\Gamma(\rho+v+\gamma-\alpha-\alpha'-\beta)\Gamma\rho+v+\beta'-\alpha'}{\Gamma\rho+v+\beta'\Gamma(\rho+v+\gamma-\alpha-\alpha')\Gamma(\rho+v+\gamma-\alpha'-\beta)}$$

$$=\sum_{k=0}^{\infty}\left(\frac{\left(\frac{\rho+v}{2}\right)_k\left(\frac{\rho+v+1}{2}\right)_k\left(\frac{\rho+v+\gamma-\alpha-\alpha'-\beta}{2}\right)_k\left(\frac{\rho+v+\gamma-\alpha-\alpha'-\beta+1}{2}\right)_k\left(\frac{\rho+v+\beta'-\alpha'}{2}\right)_k\left(\frac{\rho+v+\beta'-\alpha'+1}{2}\right)_k}{(v+1)_k\left(\frac{\rho+v+\beta'}{2}\right)_k\left(\frac{\rho+v+\beta'+1}{2}\right)_k\left(\frac{\rho+v+\gamma-\alpha-\alpha'}{2}\right)_k\left(\frac{\rho+v+\gamma-\alpha-\alpha'+1}{2}\right)_k\left(\frac{\rho+v+\gamma-\alpha'-\beta}{2}\right)_k\left(\frac{\rho+v+\gamma-\alpha'-\beta+1}{2}\right)_k}\frac{\left(\frac{-x^2}{4}\right)^k}{K!}\right)$$

$$=\frac{x^{\rho+v-1}}{2^v}\frac{\Gamma\rho+v\ \Gamma\left(\rho+v+\gamma-\alpha-\alpha^1-\beta\right).\Gamma\rho+v+\beta^1-\alpha^1}{\Gamma\rho+v+\beta^1\ \Gamma\rho+v+\gamma-\alpha-\alpha^1\ \Gamma k+1\ \Gamma\rho+v+\gamma-\alpha^1-\beta}\times$$

$${}_6F_7\left[\frac{\frac{\rho+v}{2},\frac{\rho+v+1}{2},\frac{\rho+v+\gamma-\alpha-\alpha'-\beta}{2},\frac{\rho+v+\gamma-\alpha-\alpha'-\beta+1}{2},\frac{\rho+v+\beta'-\alpha'}{2},\frac{\rho+v+\beta'-\alpha'+1}{2}}{v+1,\frac{\rho+v+\beta'}{2},\frac{\rho+v+\beta'+1}{2},\frac{\rho+v+\gamma-\alpha-\alpha'}{2},\frac{\rho+v+\gamma-\alpha-\alpha'+1}{2},\frac{\rho+v+\gamma-\alpha'-\beta}{2}\frac{\rho+v+\gamma-\alpha'-\beta+1}{2}}\Big|-\frac{x^2}{4}\right] \quad (19)$$

(using Eqn 8)

which proves the Theorem (3.3),i.e. the desired result (18).

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