Quest Journals
Journal of Research in Applied Mathematics
Volume 9 ~ Issue 11 (2023) pp: 26-35
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

Convection Induced by Time Dependent Temperature Gradient and horizontal mass flow in a viscous fluid.

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ABSTRACT
This study investigated the conditions that led to the stability of thermal convection inclined by time dependent temperature gradient and horizontal mass flow in a viscous fluid on free boundaries. The energy method was employed for the actualization of eigen value for the non-linear stability analysis. Normal mode analysis was also employed which yield two differential equations in 4th order. The usual Bousinesq approximation was adopted for the density variation. The Rayleigh-Ritz method was used for the 4th order equations for the value of the velocity which yield a differential equation in order eight, coupled with the energy equation. The data were analyzed using Mathematica and found that Q_0 ranging from 0 - 10 has great stabilizing effect on the system, as the horizontal temperature gradient H_T increases as Q_0 (mass flow rate) increases from eleven (11) the flow becomes stable, increase in Prandtl number P_r stabilizes the system provided Q_0 does not exceed 4 .Suggestions based on the findings were equally made such as, the values of n in the wave number should not be greater than one (1) for the system to be stable among others.

Keywords: *Mass Flow, Temperature Gradient, Viscosity, Fluid, Convection*

R*eceived 12 Nov., 2023;* ***R****evised 26 Nov., 2023;* ***A****ccepted 28 Nov., 2023*
P*ublished with open access at* ***www.questjournals.org***

I. Introduction

The global incessant technological and climate changes is evidence that fluid dynamics no doubt has an essential role to play, which cannot be overemphasized. The principles of fluid dynamics is well understood in recent times and are adopted in many practical situations or problems, many of such problems if not all in fluid dynamics are mostly non-linear. Ordinary or partial differential equations are used in modelling these problems and phenomena. The mathematical solutions of these differential equations bring real life solutions to real life situations. It is important as it also helps in predicting the situation with regards to the study.

Convection due to temperature gradient study has involved so many researchers because of its importance in real life requests, like the underground energy transport system, geophysics, and nuclear reactors cooling.

Free convection flow has drawn the attention of numerous researchers due to its wide range of uses in geophysics, aerodynamics, energy producers, accelerators, astrophysics, aerodynamic heating, petroleum industry, crude oil purification, polymer technology, and materials processing such as continuous casting and metal forming, wire, and glass fibre drawing (Dagana & Amos, 2020).

The transport behaviour across the fluid barrier is affected by temperature changes, hence this is insufficient to accurately represent flow behaviour (Ajibade et al., 2021). Therefore, it is crucial to take temperature-dependent viscosity into account in any research of the flow of viscous fluids in order to effectively forecast the correct fluid flow characteristics.

Ram et al. (2016) tried to create a magnetic nanofluid with time-dependent flow through a stretched rotating plate and changing viscosity features.

The temperature difference is a phenomenon where the temperature in one region is higher than another region. The extent the temperature reduced and the distance in heat flow direction is termed the temperature gradient. The temperature gradient explains which direction and rate the temperature changes the highest at a particular area. Revnic, et. al, (2019) studied the impact of inclined boundary temperature variations of Nanofluid using Buogiorno’s model which they discovered that the control characteristics affected fluid circulation and energy movement coefficients and the Nusselt number as an increasing function of wave number, amplitude as well as Raleigh number.

Instability of natural convection in a vertical fluid layer with net horizontal through flow was studied by Shankar, et al (2019) by parameterizing the basic steady flow through the peclet and Prandtle number representing the liquid mercury air, water, and oil. A modal analysis was performed, and stability eigenvalue problem was solved numerically by the cheybshev method, the effect of the horizontal through flow is found to be dependent on the value of the prandtl number. While Kumar, et al (2020) made an inclusion of inclined porous medium combined with concentration on internal heat source in which they discovered that the system becomes stable at increased angle of inclination and the increased doubled convective term made the system to be more unstable. Maryshev and Klimenko (2021) investigated a solutal convection over a horizontal porous medium with clogging and immobilization and observed that immobilization made the system of study to be stable at a uniform filtration regime. Some values of porosity yield instability, Secondly, they discovered that the most favourable mode in the study of convection was the longitudinal mode, and the reduction of gravity variation stabilizes the system.

Upon all these investigations, discoveries and studies, none has considered the effects of the mass flow rate together with time dependent temperature gradient stability study, which is the centre of this research work.

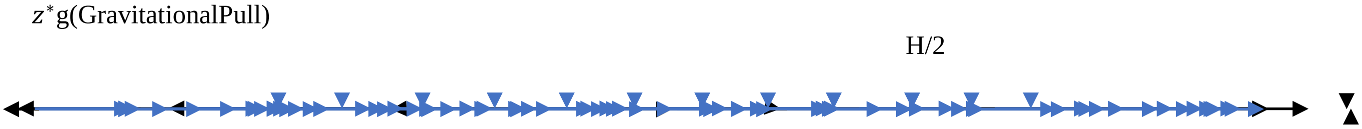
Mathematical Formulation

In the study of viscous fluid assumed to be confined between two horizontal plates with a distance apart, it is also assumed that the fluid domain is large enough for the convection to take place effectively. The origin is taken from the middle of the fluid layer and a three-dimensional coordinate system is applied, where z^* represent the vertical coordinate, x^* represents the horizontal and y^* the remaining coordinate. The corresponding velocity components of the various directions (x*, y*, z*) are (U*, V*, W*).

The sum horizontal mass flow is assumed as H_m* and a constant temperature difference T is fixed between the upper and lower plates; H_T is assumed to be the horizontal temperature gradient in the x^* coordinate which will be in the Rayleigh number.

Figure 1: The physical illustration of the above is depicted below

z^*g(GravitationalPull)



The fluid in study is an incompressible fluid; this implies that the density is constant in all areas at all giving time. The equation of flow is only affected by the varying density due to the applied heat below the bottom plate. The plates are held fixed so no external forces except the buoyancy force.

The equation of density is giving as:

$$\rho = \rho_0(1 - \alpha_T(T^* - T_0))$$

The only force acting on the fluid is the buoyancy force, which is in the opposite direction of the gravitational force acting on the fluid. The Bousinesq approximation was applied because it states that the density is assumed to be constant except on the buoyancy term and can be ignored in the Navier Stokes equation.

$$F_\nabla = -\rho_0 g = g\rho_0\alpha_T(T^* - T_0) \quad (1)$$

The governing equations, boundary and horizontal net mass flow conditions are as follow:

$$\nabla^*.V^* = 0 \quad (2)$$

$$\frac{\partial V^*}{\partial t^*} + (V^*.\nabla^*)V^* = \frac{-1}{\rho_0}\nabla^* P^* + \upsilon\nabla^{*2}V^* + F_\nabla \quad (3)$$

Substituting (1) in (3)

$$\frac{\partial V^*}{\partial t^*} + (V^*.\nabla^*)V^* = \frac{-1}{\rho_0}\nabla^* P^* + \upsilon\nabla^{*2}V^* + \frac{g\alpha_T\rho_0(T^* - T_0)}{\rho_0}$$

$$\frac{\partial V^*}{\partial t^*} + (V^*.\nabla^*)V^* = \frac{-1}{\rho_0}\nabla^* P^* + \upsilon\nabla^{*2}V^* + g\alpha_T(T^* - T_0) \quad (4)$$

Introducing unit vector $\hat{k}$, on the vertical axis

$$\frac{\partial V^*}{\partial t^*} + (V^*.\nabla^*)V^* = \frac{-1}{\rho_0}\nabla^* P^* + \upsilon\nabla^{*2}V^* + g\alpha_T(T^* - T_0)\hat{k} \quad (5)$$

$$\rho_0 C_p\left(\frac{\partial T^*}{\partial t^*} + (V^*.\nabla^*)T^*\right) = K\nabla^{*2}T^* \quad (6)$$

Where ∇^* is the gradient in three dimensional, V* is the velocity vector, ρ_0 is the reference density, t* is time, P is the pressure, F_∇ is the force, υ *is* the kinematic viscous, T* is the temperature, C_p is the specific heat

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capacity, α_T is the coefficient of volume expansion, $\widehat{k}$ represent the unit vector in the Z* direction and K is the thermal conductivity, where the asterisks indicate dimensional variables.
The basic concepts and the procedures for dimensional analysis were developed by Hydraulic engineers to determine the performance of a prototype (a full-scaled structure) from the data obtained by tests on a model (reduced scale structure). However, in this study we will apply Weber (1974) non-dimensional quantities as follows:

$$X = \frac{X^*}{H}, \quad t = \frac{\alpha t^*}{H^2}, \quad \nabla = \nabla^* H, \quad V = \frac{HV^*}{\alpha}, \quad T = R_z\left(\frac{T^* - T_0}{T_1 - T_2}\right), \quad P = \frac{H^2 P^*}{\rho_0 \alpha \upsilon},$$

$$Q_0 = \frac{H H_m{}^*}{\alpha}, \quad V^* = V^*(U^*, V^*, W^*) \qquad (7)$$

Substituting equ (7) into equations (2) - (6) and non-dimensionalizing yields:

$$\nabla . V = 0 \qquad (8)$$

$$\frac{\partial V}{\partial t} + (V.\nabla)V = P_r(\nabla P + \nabla^2 \mathrm{V} + \mathrm{T}\widehat{\mathrm{k}}) \qquad (9)$$

$$\frac{\partial T}{\partial t} + (V.\nabla)T = \nabla^2 T \qquad (10)$$

Where, V, t, P, T and ∇ are dimensionless variables.
where, $\upsilon = \frac{\mu}{\rho_0}$, μ is the dynamic viscosity, the vertical Rayleigh number

$R_z = \frac{\alpha_T (T_1 - T_2) g H^3}{\alpha \upsilon} = \frac{\alpha_T g H^3 \delta_T}{\alpha \upsilon}$,

The horizontal Rayleigh number $R_H = g\alpha_T H^4 H_T / \alpha\upsilon$ which is the measure of the temperature difference in the horizontal axis, $\widehat{k}$ is the unit vector in the horizontal plane. The Rayleigh number is the aggregate of Grashofnumber (which approximates the ratio of the buoyancy to viscosity force) and Prandtl number which defines the interaction between momentum diffusivity and be seen as the fraction between buoyancy and viscosity in product with fraction between momentum and heat dispersion.

The Basic Steady State Solution

We now find a solution to the governing equations where the fluid properties are assumed uniform as the time tends from zero to infinity. Uniformity implies that the fluid properties do not vary, and we assume the velocity V = (U(z), 0, 0) that is, the horizontal velocity component depends on Z alone. At steady state, the solutions of the velocity V, pressure P and temperature T are expressed as U_s, P_s, and T_s.
Using the boundary conditions:

V= W =0, $T = -(\pm \frac{R_Z}{2}) - R_H X$

$$\int_{\frac{-1}{2}}^{\frac{1}{2}} U dZ = Q_0, \qquad \int_{\frac{-1}{2}}^{\frac{1}{2}} V dz = 0, \;\; \mathrm{Z} = \pm \frac{1}{2} \quad (11)$$

Applying the above boundary conditions in equations (8) – (10)and at steady state, there is no change in fluid properties with time, we now have.

$\frac{\partial U_s}{\partial t} = \frac{\partial T_s}{\partial t} = 0$, the equations now become:

$$\nabla . U_s = 0 \qquad (12)$$

$$(U_s.\nabla)U_s = P_r(\nabla P_s + \nabla^2 \mathrm{U_s} + \mathrm{T_s}\widehat{\mathrm{k}}) \qquad (13)$$

$$(U_s.\nabla)T_s = \nabla^2 T_s \qquad (14)$$

Where U_s is the steady state velocity, P_s is the steady state pressure and Ts is the steady state temperature.
From equation (13), $(U_s.\nabla)U_s$ = 0 by applying the continuity equation (12),
therefore (13) becomes: $-\nabla P_s = \nabla^2 U_s + T_s \hat{k}$(15)
To solve for U_s we take the curl on (15)

$$0 = \nabla \times \nabla^2 U_s + \nabla \times T_s$$

$$\nabla^3 U_s = -\nabla T_s = -R_H \qquad (16)$$

Integrating (16)

$$\int \nabla^3 U_s dZ = -\int \nabla T_s dZ = -\int R_H dZ + \int \mathrm{U(z)dZ}$$

$$\int \nabla^3 U_s dZ = -\int \nabla T_s dZ = -\int R_H dZ + Q_0$$

Using the boundary conditions: $U_s = 0, T_s = \pm \frac{R_Z}{2} - R_H Z, Z = \mp \frac{1}{2}$ (17)

$$U_s = R_H F(Z) + Q_0 A(Z) \qquad (18)$$

$$F(Z) = \frac{1}{24}(Z - 4Z^3) \qquad (18a)$$

$$A(Z) = \frac{1}{8}(4Z^2 - 1) \qquad (18b)$$

To find the value of T_s we apply same boundary conditions in (17) in (14)

$$(V.\nabla)T_s = \nabla^2 T_s$$
$$(U_s.\nabla)T_s = \nabla^2 T_s$$
$$U_s.(-R_H) = \nabla^2 T_s$$

Integrating and substituting the value of U_s

$$T_s = -\frac{{R_H}^2 Z^3}{144} + \frac{{R_H}^2 Z^5}{120} - \frac{Q_0 R_H Z^4}{24} + \frac{Q_0 R_H Z^2}{16} + \frac{7{R_H}^2 Z}{5760} - 5\frac{R_H Q_0}{384} - R_z Z - R_H X$$

$$T_s = {R_H}^2 G(Z) - Q_0 R_H B(Z) - R_Z Z - R_H x \qquad (19)$$

where, $G(z) = \frac{48Z^5 - 40Z^3 + 7Z}{5760}$, $B(z) = \frac{16Z^4 - 24Z^2 + 5}{384}$ (20)

Perturbation Analysis

The basic steady solutions P_s, T_s, and U_s are solutions when the system is in its uniform state. That is, if the steady states are solutions, then the perturbed state are also solutions of the equations of motion. Substituting $P = P_s + p$, $V = U_s + U$, $T = T_s + \theta$ in to (8) – (10)

$$\nabla.(U_s + U) = 0 \qquad (21)$$

$$\frac{\partial(U_s+U)}{\partial t} + ((U_s + U).\nabla)(U_s + U) = P_r(\nabla(\mathrm{P_s} + P) + \nabla^2(U_s + U) + (\mathrm{T_s} + \theta)\widehat{k)} \qquad (22)$$

$$\frac{\partial(\mathrm{T_s} + \theta)}{\partial t} + (U_s + U).\nabla = \nabla^2\,(\mathrm{T_s} + \theta) \qquad (23)$$

Expanding and subtracting (21) - (23) from (11) – (14)

$$\nabla.U = 0 \qquad (24)$$

$${P_r}^{-1}\left(\frac{\partial U}{\partial t} + (U.\nabla)U\right) = -\nabla P + \nabla^2\mathrm{U} + \theta\hat{\mathrm{k}} - {P_r}^{-1}(U.\nabla U_s + U_s.\nabla U) \qquad (25)$$

$$\frac{\partial\theta}{\partial t} + U.\nabla\theta = \nabla^2\,\theta - U_s.\nabla\theta - U.\nabla T_s \qquad (26)$$

Where, $U = \theta = 0, Z = \mp\frac{1}{2}$

3.5 Energy Function

The energy stability analysis is very useful in convection problems as it usually guarantees exponential decay. In this method, a generalized functional is taken to create a differential inequality. The energy functional employed by Zafar et al (2021) has the form.

$E = E_0(t) + \lambda E_1(t)$ where, λ (>0) is a coupling parameter.

The two parts play different roles, $E_0(t)$ is the important one that yields the stability boundary while $E_1(t)$ plays the role of a piece to dominate the non-linearity. The application of this method leads to various problems and to the actualization of eigen-value problem, with this; we define an energy-functional:

$$E(t) = \frac{\|U\|^2}{2P_r} + \frac{\lambda\,\|\theta\|^2}{2}, \qquad (27)$$

Where λ is an interaction constant attached to the energy function which determines the strength of the force between particles, $\|.\|$ is the L^2 norm on the periodic cell.

Differentiating (27) with respect to "t"

$$\frac{dE(t)}{dt} = \frac{1}{2P_r}\frac{\partial}{\partial t}\int U^2 dV_0 + \frac{\lambda}{2}\frac{\partial}{\partial t}\int \theta^2 dV_0 \quad (28)$$

where $\|U\|^2 = \int U^2 dV_0$, $\|\theta\|^2 = \int \theta^2 dV_0$

$$\frac{dE(t)}{dt} = \frac{2}{2P_r}\int U\frac{\partial U}{\partial t}dV_0 + \frac{2\lambda}{2}\int \theta\frac{\partial\theta}{\partial t}dV_0$$

$$\frac{dE(t)}{dt} = \frac{1}{P_r}\int U\frac{\partial U}{\partial t}dV_0 + \lambda\int\theta\frac{\partial\theta}{\partial t}dV_0 \qquad (29)$$

Substituting the values of $\frac{1}{P_r}\frac{\partial u}{\partial t}$ and $\frac{\partial\theta}{\partial t}$ from (25) and (26):

$$\frac{dE(t)}{dt} = \int_{v_0} U(-\nabla P + \nabla^2 U + \theta\hat{k} - \frac{1}{P_r}(U.\nabla U_s) - \frac{1}{P_r}(U_s.\nabla U) - \frac{1}{P_r}\mathrm{U}.\nabla\mathrm{U})\,\mathrm{dU}) + \lambda(\int_{V_0}\theta(\nabla^2\theta - U_s.\nabla\theta - \mathrm{U}.\nabla T_s - U.\nabla\theta)\,d\theta)$$

Integrating and applying divergence theorem, since the velocity is solenoidal and v_0 is the domain of admissible function we look for a maximum.

$$\frac{dE(t)}{dt} = -\|\nabla U\|^2 + < W\theta\hat{k} > -\frac{1}{P_r} < U.\nabla U_s.U > -\lambda\|\nabla\theta\|^2 - \lambda < U.\nabla T_s.\theta >$$

$$\frac{dE(t)}{dt} = -(\|\nabla U\|^2 + \lambda\|\nabla\theta\|^2) + < W\theta\hat{k} > -\frac{1}{P_r} < U.\nabla U_s.U > -\lambda\|\nabla\theta\|^2 - \lambda < U.\nabla T_s.\theta \quad >$$

$$\frac{dE(t)}{dt} = -D + \eta$$

$$D = (\|\nabla U\|^2 + \lambda\|\nabla\theta\|^2),$$

$$\eta = < W\theta\hat{k} > -\frac{1}{P_r} < U.\nabla U_s.U > -\lambda\|\nabla\theta\|^2 - \lambda < U.\nabla T_s.\theta > \qquad (3.4.3 \qquad 30)$$

$$\frac{dE(t)}{dt} = -D\left(1 - \frac{\eta}{D}\right) = -D\gamma$$

$were\ \gamma = 1 - \frac{\eta}{D}$, if $\frac{1}{R_E} = \max_H \frac{\eta}{D}$, (31)

$$\frac{dE(t)}{dt} = -D\left(1 - \frac{1}{R_E}\right)$$

$If\ R_E > 1, It\ implies\ \gamma > 0$

Using point-care inequality on "D"$D \geq 2\pi^2(\|U\|^2 + \lambda\|\theta\|^2)\frac{1}{2}$

$$\frac{dE(t)}{dt} + 2\pi^2\,\gamma E(t) \leq 0$$

$$\frac{d}{dt}\left(E(t)e^{(2\pi^2\gamma t)}\right) \leq 0 \qquad (32)$$

Integrating (32) we will have

$$E(t) \leq e^{(-2\pi^2\gamma t)}E(0) \qquad (33)$$

As t increases to infinity E(t) approaches zero, that is:

$$\xrightarrow{E(t)} 0, \xrightarrow{t} \infty\ decays\ at\ least\ exponencially\ in\ time.$$

To solve forR_E, from (31)

$$\frac{1}{R_E} = \max_H \frac{\eta}{D}$$

$$\eta R_E - D = 0 \qquad (34)$$

To find R_E of the maximum problem, we let R_E=1 as the sharpest boundary condition and employ the method of calculus of variation to find the Euler and Lagrangian equation.

$$Let\ L_I = \eta, L_D = D$$

$$\frac{d}{d\epsilon}\left(\frac{L_I}{L_D}\right)|_{\epsilon=0} = \frac{d}{d\epsilon}\left(\frac{L_I\big((U+\epsilon\vartheta),(\theta+\epsilon\emptyset)\big)}{L_D\big((\nabla U+\epsilon\nabla\vartheta),(\nabla\theta+\epsilon\nabla\emptyset)\big)}\right) \qquad (35)$$

Therefore (34) can be written as:

$$d\eta R_E - dD = 0, \qquad (36)$$

$where\ "d"\ reprents\ differentiation$

But if the values of "D" are squared (35) becomes:

$$\frac{d}{d\epsilon}\left(\frac{L_I}{L_D}\right)|_{\epsilon=0} = \frac{d}{d\epsilon}\left(\frac{L_I\big((U+\epsilon\vartheta),(\theta+\epsilon\emptyset)\big)}{L_D((\nabla U+\epsilon\nabla\vartheta)^2,(\nabla\theta+\epsilon\nabla\emptyset)^2)}\right)$$

when R_E =1, from (36)

$dL_I\big((U+\epsilon\vartheta),(\theta+\epsilon\emptyset)\big) = dL_D((\nabla U+\epsilon\nabla\vartheta)^2,(\nabla\theta+\epsilon\nabla\emptyset)^2)$

$\frac{dL_D}{d\epsilon} = \frac{d}{d\epsilon}(\int(\nabla^2(U+\epsilon\vartheta)^2 + \lambda\nabla^2(\theta+\epsilon\emptyset)^2)dV_0$, solving the numerator first:

$$\frac{dL_I}{d\epsilon} = -\int \lambda\nabla T_s\frac{d}{d\epsilon}(U\theta)dV_0 + \int\frac{d}{d\epsilon}(W\theta)dV_0 - \frac{1}{P_r}\int \nabla U_s\frac{d}{d\epsilon}(U.U)dV_0$$

$$\frac{dL_I}{d\epsilon} = \int((-\lambda\nabla T_s\left(\theta\frac{d}{d\epsilon}(U+\epsilon\vartheta) + U\frac{d}{d\epsilon}(\theta+\epsilon\emptyset)\right) + \theta\frac{d}{d\epsilon}(w+\epsilon\vartheta) + W\frac{d}{d\epsilon}(\theta+\epsilon\emptyset)$$

$$-\frac{1}{P_r}\left(\nabla U_s\left(U\frac{d}{d\epsilon}(U+\epsilon\vartheta) + U\frac{d}{d\epsilon}(U+\epsilon\vartheta)\right)\right)dV_0$$

Differentiating gives:

$$-\int(\lambda\nabla T_s\theta\vartheta+\lambda\nabla \mathrm{T_s}.\mathrm{U}\emptyset-\mathrm{w}\emptyset-\theta\vartheta+\tfrac{1}{\mathrm{P_r}}\nabla\mathrm{U_s}.\mathrm{U}\vartheta+\tfrac{1}{\mathrm{P_r}}\nabla U_s.U\vartheta)$$

Solving for the denominator: $dL_D((\nabla U+\epsilon\nabla\vartheta)^2,(\nabla\theta+\epsilon\nabla\emptyset)^2)=\frac{dL_D}{d\epsilon}=\frac{d}{d\epsilon}(\int(\nabla(U+\epsilon\vartheta))^2+\lambda(\nabla(\theta+\epsilon\emptyset))^2)dV_0$

Differentiating yields:

$$\frac{dL_D}{d\epsilon}=\int(2(\nabla U+\epsilon\nabla\vartheta)\nabla\vartheta+2\ \lambda(\nabla\theta+\epsilon\emptyset)\nabla\emptyset)dV_0|_{\epsilon=0}$$

$$=\int(2\nabla U.\nabla\vartheta+2\nabla\theta.\lambda\nabla\emptyset)dV_0$$

Integrating by parts and applying continuity equation gives:

$$\frac{dL_D}{d\epsilon}=-2\vartheta\int\nabla^2U-2\,\lambda\emptyset\int\nabla^2\theta$$

Substituting the values of $\frac{dL_D}{d\epsilon}$ *and* $\frac{dL_I}{d\epsilon}$ *into* (36)
And equating terms in ∅ and ϑ to zero (0) yields:

$$-\int(\lambda\nabla T_s.\theta\vartheta-\theta\vartheta+\tfrac{1}{\mathrm{P_r}}\nabla\mathrm{U_s}.\mathrm{U}\vartheta+\tfrac{1}{\mathrm{P_r}}\nabla U_s.U\vartheta)+2\vartheta\int\nabla^2U=0$$

$$\int(2\lambda\emptyset\nabla^2\theta-\ \lambda\nabla\mathrm{T_s}.\mathrm{U}\emptyset+\mathrm{w}\emptyset)=0$$

$$\textit{that is, when}\int\emptyset=\int\vartheta\neq0$$

$$-\lambda\nabla T_s.\theta+\theta-\tfrac{1}{\mathrm{P_r}}\nabla\mathrm{U_s}.\mathrm{U}-\tfrac{1}{\mathrm{P_r}}\nabla U_s.U)+2\nabla^2U=\dot{\omega}\qquad(37)$$

$$2\lambda\nabla^2\theta-\ \lambda\nabla\mathrm{T_s}.\mathrm{U}+\mathrm{W}=0\qquad(38)$$

where $\dot{\omega}$ is a Lagrangian multiplier
From (22), $\nabla T_s=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)T_s$

$$\nabla T_s=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)({R_H}^2G(z)-QR_HB(z)-R_Zz-R_Hx\ )$$

$$\frac{\partial T_s}{\partial x}=-R_H,\quad\frac{\partial T_s}{\partial y}=0,\quad\frac{\partial T_s}{\partial z}={R_H}^2G'(z)-Q_0R_HB'(z)-R_Z$$

where B(z) and G(z) are given in (20)

$$\nabla U_s=\left(\frac{\partial U_s}{\partial x}+\frac{\partial U_s}{\partial y}+\frac{\partial U_s}{\partial z}\right)$$

$\frac{\partial U_s}{\partial x}=0=\frac{\partial U_s}{\partial y},\frac{\partial U_s}{\partial z}=\frac{R_H}{24}(1-12Z^2)+Q_0Z$

Substituting the above, (37) becomes:

$$\lambda R_H\theta+H_z\theta-R_HE_z(U+W)+2\nabla^2U=\nabla\omega\qquad(39)$$

$$H_z=\lambda\left[-{R_H}^2G_2+Q_0R_HB_2+R_z\right]+1$$

$$E_z=\frac{1}{P_r}\left(F_1(Z)+\frac{Q_0A_1(Z)}{R_H}\right)$$

$$G_2=\frac{1}{5760}[240Z^4-120Z^2+7]$$

$$B_2=\frac{1}{384}(64Z^3-48Z)$$

$$F_1=\frac{1}{24}(1-12Z^2)\qquad,\qquad A_1=z$$

Taking the double curl of (39) and taking the third component of the outcome, yields:

$$\lambda R_H(\nabla\times\nabla\times\theta)+H_z(\nabla\times\nabla\times\theta)-E_z\big(\nabla\times\nabla\times(w+u)\big)+2\nabla^2(\nabla\times\nabla\times U)=0$$

Therefore, the above equation becomes:

$$\lambda R_H\frac{\partial^2\theta}{\partial x\partial z}-H_z\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\theta-E_z\frac{\partial^2w}{\partial x\partial z}R_H+E_zR_H\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)U-2\nabla^4w=0$$

$$2\nabla^4 w + \nabla_h{}^2(E_z R_H U - H_z\theta) = R_H \frac{\partial^2}{\partial x \partial z}(\lambda\theta - E_z w) \qquad (40)$$

Also taking the horizontal components of (38)

$$\lambda R_H \theta - R_H E_z w\hat{\imath} + 2\nabla^2 U = \frac{\partial \omega}{\partial \mathrm{x}} \qquad (41)$$

And (38) yields:

$$\lambda R_H U + H_z w + 2\lambda \nabla^2\theta = 0 \qquad (42)$$

From (42), $U = -\frac{H_z}{\lambda R_H} w - \frac{2}{R_H}\nabla^2\theta$ (43)

3.6 Normal Mode Analysis

To continue with the non-linear stability analysis we employ the technique ofChandtrasekhar, (1961) in Orukari (2010), to search for the solutions of the above equations in the form: [U,W,θ,**ω**] = [U(z),W(z),θ(z),**ω**(z)]$e^{\mathrm{i}(a_x X + a_y Y)}$

Solving (40) – (42) by method of normal mode, and substituting (43)

where $D = \frac{\partial}{\partial z}, \partial H_Z/\partial z = \lambda\left[-R_H{}^2 \frac{\partial G(z)_2}{\partial z} + QR_H \frac{\partial B(Z)_2}{\partial Z}\right]$

$$2\Big(\frac{-2\big(a^4\theta(z)\big)}{R_H} + \frac{2a^2D^2\theta(z)}{R_H} + \frac{a^2H_ZW(z)}{\lambda R_H} + \frac{2a^2D^2\theta(z)}{R_H} - \frac{2D^4\theta(z)}{R_H} - \frac{H_ZD^2W(z) + 2DW(z)H'_Z + W(z)H''_z}{\lambda R_H} + \lambda R_H\theta(z) - R_HE_zW(z) = ia_x\omega(z)$$

$$\frac{-4a^4\theta(z)}{R_H} + \frac{8a^2D^2\theta(z)}{R_H} + \frac{2a^2H_ZW(z)}{\lambda R_H} - \frac{4D^4\theta(z)}{R_H} - \frac{2H_ZD^2W(z)}{\lambda R_H} - \frac{-4DW(Z)}{\lambda R_H}\Big(\lambda\left(-R_H{}^2\frac{\partial G(z)_2}{\partial z} + QR_H\frac{\partial B(Z)_2}{\partial Z}\right) - \frac{2W(z)\big(-R_H{}^2G(z)''_2 + QR_HG(z)''_2\big)}{\lambda R_H} + \lambda R_H\theta - R_HE_ZW(z) = ia_x\omega(z)$$

Taking the real part and simplifying yields:

$$D^4\theta(z) = -a^4\theta + 2a^2D^2\theta + W\left(\frac{R_H{}^2G''_2 - QR_HB''_2)}{2} + \frac{a^2H_Z}{2\lambda}\right) + DW\big(R_H{}^2G'_2 - QR_HB'_2\big) + D^2\left(\frac{-H_z}{2\lambda}\right) + \frac{\lambda R_H{}^2\theta}{4} - \frac{E_zR_H{}^2W}{4}$$

$$D^4\theta(z) = \Big(\frac{\lambda R_H{}^2}{4} - a^4\Big)\theta + 2a^2D^2\theta + W\left(\frac{L_0}{2} + \frac{a^2H_Z}{2\lambda} - \frac{E_zR_H{}^2W}{4}\right) + DW\big(R_H{}^2G'_2 - QR_HB'_2\big) + D^2W\left(\frac{-H_z}{2\lambda}\right)$$

Solving (41) and simplifying gives:

$$D^4W = \Big(\frac{a^2E_ZH_Z}{2\lambda} - a^4\Big)W + 2a^2D^2W + \Big(\frac{a^2H_z}{2} - a^4E_z\Big)\theta + a^2E_zD^2\theta$$

Therefore, the equations become:

$$D^4W = L_1W + L_2D^2W + L_3\theta + L_4D^2\theta \qquad (44)$$

$$D^4\theta = L_5W + L_6DW + L_7D^2W + L_8\theta + L_9D^2\theta \qquad (45)$$

where,

$$L_1 = \frac{a^2E_ZH_Z}{2\lambda} - a^4, L_2 = 2a^2, L_3 = \frac{a^2H_z}{2} - a^4E_z, L_4 = a^2E_z, L_5 = \frac{L_0}{2} + \frac{a^2H_Z}{2\lambda} - \frac{E_zR_H{}^2}{4}, L_6 = R_H{}^2G'_2 - QR_HB'_2, L_7 = \frac{-H_z}{2\lambda}, L_8 = \frac{\lambda R_H{}^2}{4} - a^4, L_9 = 2a^2, L_0 = \frac{R_H{}^2G''_2 - Q_0R_HB''_2)}{2} \qquad (46)$$

The vital boundary conditions are:

$$\theta = W = DW = D^2\theta = 0 \qquad (47)$$

We now fix R_z as eigenvalue with other variables as parameters. The energy critical Rayleigh number can then be written as

$$R_{ZE} = \max_{\lambda} \min_{a^2} R_z(R_H, a^2, \lambda, P_r, Q)$$

R_{ZE} *is the critical Rayleigh number on the vertical axis*

Solving (44) and (45) by dissolving θ *in* w *yields*

$$D^8 w - D^6 w(L_2 + L_9) - D^4 w(L_1 + L_8 + L_4 L_7 - L_2 L_9) - L_4 L_6 D^3 w - D^2 w(L_3 L_7 + L_4 L_5 - L_2 L_8 - L_1 L_9) - L_3 L_6 D w - (L_3 L_5 - L_1 L_8) w = 0 \quad (48)$$

Applying Rayleigh –Ritz method in (48) for a free- free surface

$$w = W_0 \sin \pi n z \qquad (49)$$

Where W_0 is a positive constant which is assumed as one (1) in this problem, n is any real number and Z is the vertical coordinate.

$W = \sin \pi n Z$, differentiating and substituting into (48)

$$\pi^8 n^8 + \pi^6 n^6 (L_2 + L_9) - \pi^4 n^4 (L_1 + L_8 + L_4 L_7 - L_2 L_9) + \pi^3 n^3 (L_4 L_6) + \pi^2 n^2 (L_3 L_7 + L_4 L_5 - L_2 L_8 - L_1 L_9) - \pi n \cot \pi n z\,(L_3 L_6) - L_3 L_5 + L_1 L_8 = 0$$

$$\pi^8 n^8 + 4a^2 \pi^6 n^6 - \pi^4 n^4 \left(\frac{\lambda {R_H}^2}{4} - 6a^4\right) + \pi^3 n^3 (a^2 E_z L_6) + \pi^2 n^2 \left(-\frac{a^2 {H_z}^2}{4\lambda} - \frac{a^2 {R_H}^2 {E_z}^2}{4} - \frac{a^2 \lambda {R_H}^2}{2} + \frac{a^2 E_Z L_0}{2} + 4a^6\right) - \pi n \left(\frac{L_6 a^2 H_z}{2} - a^4 L_6 E_z\right) \cot \pi n z - \frac{a^4 {H_z}^2}{4\lambda} + \frac{a^2 E_z H_z {R_H}^2}{4} - \frac{a^2 H_z L_0}{4} - \frac{a^4 {E_z}^2 {R_H}^2}{4} + \frac{a^4 E_z L_0}{2} - \frac{a^4 \lambda {R_H}^2}{4} + a^8 \qquad (50)$$

Where, L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8, L_9 are giving in equation (46).

$$\left(\frac{\pi^2 n^2 a^2}{4\lambda} + \frac{a^4}{4\lambda}\right) {H_z}^2 + \left(\frac{n \pi L_6 a^2 \cot \pi n z}{2} - \frac{a^2 E_z {R_H}^2}{4} + \frac{a^2 L_0}{4}\right) H_z - 4a^2 \pi^6 n^6 + \pi^4 n^4 \left(\frac{\lambda {R_H}^2}{4} - 6a^4\right) - \pi^3 n^3 (a^2 E_z L_6) - \pi^2 n^2 \left(-\frac{a^2 {R_H}^2 {E_z}^2}{4} - \frac{a^2 \lambda {R_H}^2}{2} + \frac{a^2 E_Z L_0}{2} + 4a^6\right) - \pi n (a^4 L_6 E_z \cot \pi n z) + \frac{a^4 {E_z}^2 {R_H}^2}{4} - \frac{a^4 E_z L_0}{2} + \frac{a^4 \lambda {R_H}^2}{4} - a^8 - \pi^8 n^8 = 0$$

The above equation is now a quadratic equation.

$$H_z = \frac{-b \pm \sqrt{b^2 - 4k_1 c}}{2k_1} \qquad (51)$$

where,

$$k_1 = \frac{a^4}{4\lambda} + \frac{a^2 \pi^2 n^2}{4\lambda}, b = \frac{L_6 a^2 \pi n \cot \pi n z}{2} - \frac{a^2 E_z {R_H}^2}{4} + \frac{a^2 L_0}{4}, c = -\pi^8 n^8 - 4a^2 \pi^6 n^6 + \pi^4 n^4 \left(\frac{\lambda {R_H}^2}{4} - 6a^4\right) - \pi^3 n^3 (a^2 E_z L_6) - \pi^2 n^2 \left(-\frac{a^2 {R_H}^2 {E_z}^2}{4} - \frac{a^2 \lambda {R_H}^2}{2} + \frac{a^2 E_Z L_0}{2} + 4a^6\right) - \pi n (a^4 L_6 E_z \cot \pi n z) + \frac{a^4 {E_z}^2 {R_H}^2}{4} - \frac{a^4 E_z L_0}{2} + \frac{a^4 \lambda {R_H}^2}{4} - a^8$$

For real roots: $b^2 - 4k_1 c \geq 0$, if $k_1 = 0$

$$\frac{a^2}{4\lambda} + \frac{a^4 \pi^2 n^2}{4\lambda} = 0, \frac{a^2 \pi^2 n^2}{4\lambda} = -\frac{a^4}{4\lambda},$$

$$a^2 = -\pi^2 n^2, \qquad (52)$$

where a^2 is the wave number.

$$Rz = -((3317760\pi^4 n^4 P_r + 5529600\pi^6 n^6 P_r - 5760\pi^2 n^2 (R_H + 24Q_0 z - 12R_H z^2) - 960\pi^3 n^3 (3Q_0 - R_H z)(-1 + 4z^2)(R_H + 24Q_0 z - 12R_H z^2) + R_H\big(-24Q_0 + R_H(-1 + 12z^2)\big)\big(-5760P_r + (1 + P_r)R_H\big(240Q_0 z(3 - 4z^2) + R_H(7 - 120z^2 + 240z^4)\big)\lambda\big) + \frac{2n\pi(3Q_0 - R_H z)(-1 + 4z^2)\Big(480\pi^2 n^2 (R_H + 24Q_0 z - 12R_H z^2) + P_r R_H\big(5760 + R_H\big(240Q_0 z(-3 + 4z^2) + R_H(-7 + 120z^2 - 240z^4)\big)\lambda\big)\Big)\cot n\pi z}{5760 R_H \lambda (1 + P_r)(R_H + 24Q_0 z - 12R_H z^2)} + 2n\pi P_r (3Q_0 - R_H z)(-1 + 4z^2)\cot n\pi z$$

II. RESULTS AND DISCUSSIONS

In this study we are considered the Prandtl number (Pr), Rayleigh number (Ra), (both the horizontal and vertical Raleigh numbers), the mass flow rate Q_0, the wave number "a" which is measured in the product of n and π which is giving in (52) and the coupling parameter λ, at a depth of 0.3 (z = 0.3).

The Influence of the Horizontal Mass Flow Rate Q_0

At various values of Q_0, when Pr and λ are fixed on varied n values, it was observed that as the value n increases, the vertical Rayleigh number converges faster at smaller values, it is also observed that when n is 13, Rz reduces at increasing R_H which stirs instability.

The Influence of the Strength of Coupling Parameter

At various values of λ in the variations of both Q_0, at constant value of n, it is observed the vertical Rayleigh number increases as the horizontal Rayleigh number for smaller numbers of λ and Q_0. While smaller values of λ caused Rz to converge quickly at R_H equals 150 at different values of Pr with constant n value and the system became stable as Rz increases with R_H .

The Influence of the Prandtl Number Pr

At constant λ and different values of n, when Q_0 is large, the vertical Rayleigh number Rz increases as R_H increases for smaller values of Prandtl number, but when n is 13, Rz decreases as R_H ranges from 100 – 150 negatively then increased positively from 160. When Q_0 is 10, at n=13, Rz converges faster at RH equals 100. Rz also increases as RH for smaller numbers of Q_0.

III. SUMMARY OF FINDINGS

The stability study of convection induced by time dependent temperature gradient and horizontal mass flow a viscous fluid was studied using the Energy Method with application of the Rayleigh Ritz method to obtain an Eigen value as Rz dependent on other parameters. We then solve for the energy stability of Rz by fixing values of some independent parameters of Rz.

The corresponding results for the non-linear stability results are as follows: it is discovered that for a constant Prandtl number Pr and constant coupling parameter λ with mass flow rate Q_0 at 5, 10, 15 and taken n as one (1), the horizontal Rayleigh number R_H is taken from 100 to 250. The vertical Rayleigh number Rz increases as the horizontal Rayleigh number R_H increases. This indicates the stability in all values of Q_0.

There is a little increase when n is three (3), Rz converges faster at $R_H$at Q_0 equals 15and 10, Rz converges faster only at RH equals 100 but Rz decreases as R_H increases, and this indicates instability. When n is four (4) the converging effect became stronger and Rz increases positively as R_H ranges from 100 to 130, then dropped to negative values. This shows stability occurs from 100 to 130 and became unstable just after 130 and became stable from 140 in negative values. It's clear that as n increases, stability occurs in the flow in measure of the vertical and Horizontal Rayleigh numbers. However, it was observed that when n was increased to 13, n=13 the flow becomes unstable as Rz decreases,R_H increases. .

The coupling parameter λ was also tested with same variations of Q_0 at constant n value. It is observed that λ has a strong converging effect on Rz at smaller values of Q_0. It also has a great stability effect on the flow.

Considering the effect of Prandtl number Pr, when we varied Pr between 10 and 100at constant Q_0 and n, with different values of λ and the horizontal Rayleigh number ranges from 100 to 300. It was observed that at R_H is 150, Rz converges faster, also the Prandtl number has a stabilizing effect on Rz and $R_{H.}$ Different values of the coupling parameter λ never affects stability attribute on Pr.

Variations of n at constant λ, it is observed that Rz converges faster at Pr equals 100. When n is one (1) the flow becomes stable in measure of the vertical and horizontal Rayleigh numbers. When n is two (2), lower Prandtl numbers destabilizes the system, this instability begins at R_H = (160 to 300) when Pr is 10, 50 and R_H is 150. But Pr becomes a stabilizing factor when it is at 100. When n is 4, all values of Pr stand as good stabilizing values on Rz. When n is 13 the system became unstable.

In conclusion, the study found the effects of the Prandtl number Pr which is a ratio of the Kinematic viscosity and thermal diffusivity, the horizontal temperature gradient H_T and the mass flow rate Q_0 stands as a stabilizing factor to convection induced by inclined time dependent temperature gradient and horizontal mass flow. It was also seen that Q_0 ranging from 0 - 10 has great stabilizing effect on the system. Finally, as the horizontal temperature gradient H_T increases as Q_0 increases from eleven (11), the flow becomes stable. These findings summarize that:

The system was stable at constant Prandtl number regardless of the rate of mass flow at the initial study. If mass flow rate and Prandtl number does not change, the system is stable regardless of the time. Instability occurs as time grows with increasing prandtle number when mass flow rate was small but when the mass flow rate was increases it became stable.

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