Quest Journals
Journal of Research in Applied Mathematics
Volume 9 ~ Issue 12 (2023) pp: 23-32
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

Existence the locally attractive solution for second order nonlinear quadratic differential equation

Mr. Pravin M. More[1], Mr. S. P. Shinde[2], Mr. B.D. Karande[1],
Mr. Dr. S. D. Agalave[3]
Department of Mathematics
1BSS Arts Science and College, Makani-413604, Maharashtra, India
Corresponding Autor Email: pravinmorepravin@gmail.com
1Maharashtra Udayagiri Mahavidyalaya, Udgir-413517, Maharashtra, India
2Vasantrao Naik College Nanded, Maharashtra, India
3Jawhar CollegeAndur 413604, Maharashtra, India

Abstract; *In this chapter, we study the existence the solution and existence the locally attractive solution for an arbitrary order quadratic differential equation in Banach algebras under Lipschitz and Caratheodory conditions using a hybrid fixed point theorem.*

***R**eceived 07 Dec., 2023; **R**evised 19 Dec., 2023; **A**ccepted 21 Dec., 2023*

I. Introduction:

The theory of fractional calculus (that is fractional order differential and integral equation) has newly received a lot of attention and establishes a meaningful branch of nonlinear analysis [9,10,14,15] Number of research monographs and research papers has appeared to contribute to integrals and differential equation of fractional order which now constitutes a significant branch of nonlinear analysis [8, 10, 20, 3]. In the last few decades, however fractional differentiation proved very useful in various fields of applied sciences and engineering [11,17,22,25] and [5]. Fractional differential equation rise in the mathematical modelling of system and process occurring scientific disciplines such as physics, chemistry, biology, economics, signal and image processing, feedback amplifier and electric circuits [6.7.23,24]. Numerous research papers and monographs devoted to differentialand integral equation of fractional order have.

In this chapter we study the existence of locally attractive solution and existence of extremal solution of the following fractional order quadratic differential equation.

$$D^2\left(\frac{x(t)-h(t,x(t))}{f(t,x(t))}\right) = g(t,x(t))\ a.e.\ t \in \mathbb{R}_+ \qquad (1)$$

$$x(0) = x_0 \ \in \mathbb{R}_+$$

where $t \in \mathbb{R}_+ = [0,\infty)$ and $0 < \xi < 1$, $g: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $f: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are functions which satisfy special assumptions.

II. Preliminaries:

In this section we give the definitions, notation, hypothesis and preliminary tools, which will be needed in the sequel.

Let $\mathbb{X} = AC(\mathbb{R}_+, \mathbb{R})$ be the space of absolutely continuous function on $\mathbb{R}_+$ and S be a subset of $\mathbb{X}$. Let a mapping $\mathbb{A}: \mathbb{X} \to \mathbb{X}$ be an operator and consider the following operator equation in $\mathbb{X}$, namely,

$$x(t) = (\mathbb{A}x)(t), \text{ for all } t \in \mathbb{R}_+ \qquad (2.1)$$

Below we give some different characterization of the solutions for operator equation (2.1) on $\mathbb{R}_+$. We need the following definitions.

Definition2.1[12, 13, 22, 23, 25]: We say that solution of the equation (2.1) is locally attractive if there exists a closed ball $\overline{B_r(0)}$ in the space $AC(\mathbb{R}_+, \mathbb{R})$for some $x_0 \in AC(\mathbb{R}_+, \mathbb{R})$and for some real number $r > 0$ such that for arbitrary solution $x = x(t)$ and $y = y(t)$ of equation (2.1) belonging to $\overline{B_r(0)} \cap S$ we have that, $\lim_{t\to\infty}\big(x(t) - y(t)\big) = 0$

Theorem2.2: [**4,16,18,23,25]:** Let S be a non-empty, convex, closed and bounded subset of the Banach space $\mathbb{X}$ and let $\mathbb{A}, \mathbb{C}: \mathbb{X} \to \mathbb{X}$ and $\mathbb{B}: S \to \mathbb{X}$ are two operators satisfying:

a) $\mathbb{A}$ and $\mathbb{C}$ are Lipschitzian with lipschitz constants ζ, η respectively.
b) $\mathbb{B}$ is completely continuous, and
c) $x = \mathbb{A}x\mathbb{B}y + \mathbb{C}, x \in S$ for all $y \in S$
d) $\zeta M + \eta < 1$ where $M = \|\mathbb{B}(s)\| = sup\{\|\mathbb{B}x\|: x \in S\}$

Then the operator equation $x = \mathbb{A}x\mathbb{B}x + \mathbb{C}x$ has a solution in S

III. Existence Theory

Definition 3.1[22, 23, 24]: A mapping $\sigma: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Caratheodory if:

i) $t \to \sigma(t, x, y)$ is measurable for each $x, y \in \mathbb{R}$ and
ii) $(x, y) \to \sigma(t, x, y)$is continuous almost everywhere for $t \in \mathbb{R}_+$.

Furthermore, a Caratheodary function σ is $\mathcal{L}^1$ −Caratheodary if:

iii) For each real number $r > 0$ there exists a function $h_r \in \mathcal{L}^1(\mathbb{R}_+, \mathbb{R})$ such that$|\sigma(t, x, y)| \le h_r(t)$ $a.e.$ $t \in \mathbb{R}_+$ for all $x \in \mathbb{R}$ with $|x|_r \le r$ and $|y|_r \le$ r.

Finally, a caratheodary function σ is $\mathcal{L}^1_{\mathbb{X}}$ −caratheodary if:

iv) There exists a function $h \in \forall\mathcal{L}^1(\mathbb{R}_+, \mathbb{R})$ such that $|\sigma(t, x, y)| \le h(t)$, $a.e.$ $t \in \mathbb{R}_+$ for all $x, y \in \mathbb{R}$

For convenience, the function h is referred to as a bound function for σ.

IV. Main Result

4.1 Existence the solution of SNQDE (1)

Lemma 4.1: Suppose that $\xi \in (0,1)$ and the function f, g satisfying SNQDE (1.1) then x is the solution of the SNQDE (1.1) if and only if it is the solution of integral equation

$x(t) = h(t, x(t)) + f(t, x(t)) \left[\frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t (t - s) g(s, x(s)) ds\right], t \in \mathbb{R}_+$

(4.1)

Proof: Integrating equation (1.1) to second order, we get

$$I^2 D^2 \left(\frac{x(t) - h(t, x(t))}{f(t, x(t))}\right) = I^2 g(t, x(t))$$

$$\left[\frac{x(t) - h(t, x(t))}{f(t, x(t))}\right]_0^t = \int_0^t \int_0^t g(s, x(s)) ds\, ds$$

$$\frac{x(t) - h(t, x(t))}{f(t, x(t))} - \frac{x(0) - h(0, x(0))}{f(0, x(0))} = \int_0^t \int_0^t g(s, x(s)) ds\, ds$$

Since $\int_0^t f(t) dt^n = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s) ds$, Where $n = 0,1,2,3, \dots \dots \dots$

$$\frac{x(t) - h(t, x(t))}{f(t, x(t))} - \frac{x(0) - h(0, x(0))}{f(0, x(0))} = \frac{1}{(2-1)!} \int_0^t (t - s) g(s, x(s)) ds$$

$$\frac{x(t) - h(t, x(t))}{f(t, x(t))} = \frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t (t - s) g(s, x(s)) ds$$

$$x(t) = h(t, x(t)) + f(t, x(t)) \left(\frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t (t - s) g(s, x(s)) ds\right)$$

Conversely differentiate (4.1) twice w.r.to t, we get,

$$x(t) = h(t, x(t)) + f(t, x(t)) \left[\frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t (t - s) g(s, x(s)) ds\right]$$

Can be written as,

$$\frac{x(t) - h(t, x(t))}{f(t, x(t))} = \left[\frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t (t - s) g(s, x(s)) ds\right]$$

Differentiating this equation

$$D\left[\frac{x(t) - h(t, x(t))}{f(t, x(t))}\right] = D\left[\frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t \int_0^t g(s, x(s)) ds\right]$$

$$D\left[\frac{x(t) - h(t, x(t))}{f(t, x(t))}\right] = \int_0^t g(s, x(s)) ds$$

Again, differentiating above equation

$$D^2\left[\frac{x(t) - h(t, x(t))}{f(t, x(t))}\right] = g(s, x(t))$$

hence x is the solution of the SNQDE (1.1) if and only if it is the solution of integral equation $x(t) = h(t, x(t)) + f(t, x(t)) \left[\frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t (t - s) g(s, x(s)) ds\right] t \in \mathbb{R}_+$

We shall study the existence of solution for the (SNQDE) (1.1) under the following general assumptions:

($\mathcal{H}_1$) The function $f: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} - \{0\}$ is continuous and bounded with bound $\mathbb{F} = sup_{(t,x)}|f(t, x(t))|$, there exist a bounded function $\zeta: \mathbb{R}_+ \to \mathbb{R}$ with bound $\|\zeta\|$ such that, $|f(t, x(t)) - f(t, y(t))| \leq \xi(t)|x(t) - y(t)|, \forall x, y \in \mathbb{R}$

($\mathcal{H}_2$) The function $h: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is measurable in (t, s) for any $x \in \mathbb{R}$ and continuous in x for almost all (t, s) and $\mathbb{H} = sup_{(t,x)}|h(t, x)|$, there exist a bounded function $\eta: \mathbb{R}_+ \to \mathbb{R}$ with bound $\|\eta\|$ such that,

$|h(t, x(t)) - h(t, y(t))| = \eta(t)|x(t) - y(t)|, \forall x, y \in \mathbb{R}$ and it vanishes at infinity.

($\mathcal{H}_3$) The function $g: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ continuous and Caratheodory then there exist a function $p: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ with bound $\|p\|$ such that $|g(t, x(t))| \leq p(t, s)\ \forall t \in \mathbb{R}_+$.

($\mathcal{H}_4$) The function $\rho_1: \mathbb{R}_+ \to \mathbb{R}_+$ defined by the formula $\rho_2(t) = \int_0^t (t - s)g(s, x(s))ds$ bounded on $\mathbb{R}_+$ and $lim_{t\to\infty}\rho(t) = 0$ that is vanishing at infinity.

Remark (4.1): Note that the hypothesis **($\mathcal{H}_1$)−($\mathcal{H}_4$)** hold then there exist a constant function $\mathcal{K}_2 > 0$ such that $\mathcal{K}_1 = sup_{t\geq 0}\rho_2(t)$

Theorem (4.2): Assume that the hypothesis **($\mathcal{H}_1$)−($\mathcal{H}_4$)** holds, further if $\|\zeta\|\left(\left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \mathcal{K}_2\right) + \|\eta\| < 1$ then SNQDE (1.1) has locally attractive solution on the Banach space $\mathbb{X} = AC(\mathbb{R}_+, \mathbb{R})$.

Proof: Define a non- empty, convex, closed and bounded subset S of Banach space $\mathbb{X} = AC(\mathbb{R}_+, \mathbb{R})$ as $S = \{x \in \mathbb{X}: \|x\| \leq r\}$, where r satisfies the inequality $\|\zeta\|\left(\left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \mathcal{K}_2\right) + \|\eta\| < r$.

Now we define the operators $\mathbb{A}: \mathbb{X} \to \mathbb{X}$ and $\mathbb{B}: S \to \mathbb{X}$ and $\mathbb{C}: \mathbb{X} \to \mathbb{X}$ by,

$$\mathbb{A}x(t) = f(t, x(t),), t \in \mathbb{R}_+ \tag{4.2}$$

$$\mathbb{B}x(t) = \frac{x_0 - h(0,x_0)}{f(0,x_0)} + \int_0^t (t - s)g(s, x(s))ds, t \in \mathbb{R}_+ \tag{4.3}$$

$$\mathbb{C}x(t) = h(t, x(t)), t \in \mathbb{R}_+ \tag{4.4}$$

The SNQDE is equivalent to the operator equation

$$x(t) = \mathbb{A}x(t)\mathbb{B}x(t) + \mathbb{C}x(t), \forall t \in \mathbb{R}_+ \tag{4.5}$$

Now, we will prove that, the operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ satisfy all the axioms of theorem (2.1).

Step I: To show that $\mathbb{A}$ and $\mathbb{C}$ are Lipschitzian on $\mathbb{X}$.

For that, let $x, y \in \mathbb{X}$, then by hypothesis **($\mathcal{H}_1$)**, for $t \in \mathbb{R}_+$ we have,

$$|\mathbb{A}x(t) - \mathbb{A}y(t)| = |f(t, x(t)) - f(t, y(t))|$$
$$\leq \zeta(t)|x(t) - y(t)|$$

After taking supremum over t, we get

$$\|\mathbb{A}x - \mathbb{A}y\| \leq \|\zeta\|\|x - y\| \text{ for all } x, y \in \mathbb{R}.$$

Therefore, the operator $\mathbb{A}$ is lipschitzian with lipschitz constant $\|\zeta\|$.

Now to show $\mathbb{C}$ is Lipschitzian on $\mathbb{X}$ for any $x, y \in \mathbb{X}$we have

$$|\mathbb{C}x(t) - \mathbb{C}y(t)| = \big|h\big(t, x(t)\big) - h\big(t, y(t)\big)\big|$$
$$\leq \eta(t)|x(t) - y(t)|$$

Taking supremum over t, we obtain

$$\|\mathbb{C}x - \mathbb{C}y\| \leq \|\eta\|\|x - y\|, \text{ for all } x, y \in \mathbb{R}$$

Therefore, the operator $\mathbb{C}$ is lipschitzian with lipschitz constant $\|\eta\|$.

Step II: To show that $\mathbb{B}$ is completely continuous on $\mathbb{X}$.

This can be achieved by showing that $\mathbb{B}$ is continuous, uniformly bounded and equicontinuous.

Let $\{x_n\}$ be a sequence in S such that $\{x_n\} \to x$.Then by dominated convergence theorem,

$$\lim_{n\to\infty} \mathbb{B}x_n(t) = \lim_{n\to\infty} \left\{\frac{x_0 - h(0,x_0)}{f(0,x_0)} + \int_0^t (t-s) g\big(s, x_n(s)\big)\, ds\right\}$$

$$= \frac{x_0 - h(0,x_0)}{f(0,x_0)} + \int_0^t (t-s) g\big(s, x(s)\big)\, ds = \mathbb{B}x(t), , \forall t \in \mathbb{R}_+$$

This shows that $\mathbb{B}x_n$ converges to $\mathbb{B}x$ pointwise on S.

Next to show sequence $\{\mathbb{B}x_n\}$ is a uniformly convergent in S.

Let $t_1, t_2 \in \mathbb{R}_+$be arbitrary with $t_1 < t_2$ then

$$|\mathbb{B}x_n(t_2) - \mathbb{B}x_n(t_1)|$$
$$= \left|\frac{x_0 - h(0,x_0)}{f(0,x_0)} + \int_0^{t_2} (t_2 - s) g\big(s, x_n(s)\big)\, ds - \left(\frac{x_0 - h(0,x_0)}{f(0,x_0)} + \int_0^{t_1} g(t_1 - s)\big(s, x_n(s)\big)\, ds\right)\right|$$

$$|\mathbb{B}x_n(t_2) - \mathbb{B}x_n(t_1)| = \left|\int_0^{t_2} (t-s) g\big(s, x_n(s)\big)\, ds - \int_0^{t_1} (t-s) g\big(s, x_n(s)\big)\, ds\right|$$

$$|\mathbb{B}x_n(t_2) - \mathbb{B}x_n(t_1)| = \left|\int_0^{t_2} (t_2-s) g\big(s, x_n(s)\big)\, ds - \int_0^{t_1} (t_1-s) g\big(s, x_n(s)\big)\, ds\right|$$

$$|\mathbb{B}x_n(t_2) - \mathbb{B}x_n(t_1)| \leq \left|\int_0^{t_2} (t_2-s)\|p\|\, ds - \int_0^{t_1} (t_1-s)\|p\|\, ds\right|$$

$$|\mathbb{B}x_n(t_2) - \mathbb{B}x_n(t_1)| \leq \|p\| \left|\int_0^{t_2} (t_2-s)\, ds - \|p\| \int_0^{t_1} (t_1-s)\, ds\right|$$

$$|\mathbb{B}x_n(t_2) - \mathbb{B}x_n(t_1)| \leq \|p\| \left(\left|\int_0^{t_2} (t_2-s)\, ds - \int_0^{t_1} (t_1-s)\, ds\right|\right)$$

$$|\mathbb{B}x_n(t_2) - \mathbb{B}x_n(t_1)| \leq \|p\| \left(\frac{{t_2}^2}{2} - \frac{{t_1}^2}{2}\right) \to 0 \text{ as } t_1 \to t_2, \forall n \in \mathcal{N}$$

This shows that the sequence converges uniformly, and uniform convergence imply continuity by using this property of uniform convergence we can conclude that $\mathbb{B}$ is continuous on S.

Step III: To show $\mathbb{B}$ is compact operator on S.

For proving this, it is enough to show that $\mathbb{B}$ is uniformly bounded and equicontinuous in S.

First, we will show that $\mathbb{B}$ is uniformly bounded. Let $x \in S$ be arbitrary then

$$|\mathbb{B}x(t)| = \left|\frac{x_0 - h(0,x_0)}{f(0,x_0)} + \int_0^t (t-s)g\big(s,x(s)\big)\,ds\right|$$

$$\leq \left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \int_0^t \left|(t-s)g\big(s,x(s)\big)\right|ds$$

$$\leq \left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \rho_2(t)$$

Taking supremum over t, we obtain

$$\|\mathbb{B}x\| \leq \left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \sup_{t\geq 0}\rho_2(t) = \left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \mathcal{K}_2, \forall\, t \in \mathbb{R}_+$$

Hence $\mathbb{B}$ is uniformly bounded subset of S.

Now to show $\mathbb{B}$ is equicontinuous on S.

Let $t_1, t_2 \in \mathbb{R}_+$ then,

$$|\mathbb{B}x(t_1) - \mathbb{B}x(t_2)| = \left|\frac{x_0 - h(0,x_0)}{f(0,x_0)} + \int_0^{t_1} (t_1-s)g\big(s,x(s)\big)\,ds - \left(\frac{x_0 - h(0,x_0)}{f(0,x_0)} + \int_0^{t_2} (t_2-s)g\big(s,x(s)\big)\,ds\right)\right|$$

$$|\mathbb{B}x(t_1) - \mathbb{B}x(t_2)| \leq \left|\|p\|\int_0^{t_1} (t_1-s)\,ds - \|p\|\int_0^{t_2} (t_2-s)\,ds\right|$$

$$\leq \left|\|p\|\left(\frac{{t_1}^2}{2} - \frac{{t_2}^2}{2}\right)\right| \quad \to 0 \text{ as } t_1 \to t_2$$

Implies that $\mathbb{B}(S)$ is equicontinuous.

Hence $\mathbb{B}$ is compact subset of S. Hence the conclusion, that $\mathbb{B}$ is completely continuous on S.

Step IV: To show $x = \mathbb{A}x(t)\mathbb{B}y(t) + \mathbb{C}x(t) \Longrightarrow x \in S, \forall y \in S$.

Let $x \in \mathbb{X}$, and $y \in S$ such that $x = \mathbb{A}x(t)\mathbb{B}y(t) + \mathbb{C}x(t)$

By assumptions (**$\mathcal{H}_1$-$\mathcal{H}_4$**)

$$|x(t)| = |\mathbb{A}x(t)\mathbb{B}y(t) + \mathbb{C}x(t)|$$

$$\leq |\mathbb{A}x(t)||\mathbb{B}y(t)| + |\mathbb{C}x(t)|$$

$$\leq \left|f\big(t,x(t)\big)\right|\left(\left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \int_0^t \left|(t-s)g\big(s,y(s)\big)\right|ds\right) + \left|h\big(t,x(t)\big)\right|$$

$$\leq \mathbb{F}\left(\left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \int_0^t \left|(t-s)g\big(s,y(s)\big)\right|ds\right) + \mathbb{H}$$

$$\leq \mathbb{F}\left(\left|\frac{x_0-h(0,x_0)}{f(0,x_0)}\right| + \rho_2(t)\right) + \mathbb{H}$$

Taking supremum over t on $\mathbb{R}_+$, we obtain $\|x\| \leq \|\xi\| \left(\left|\frac{x_0-h(0,x_0)}{f(0,x_0)}\right| + \mathcal{K}_2\right) + \|\eta\|$, $\forall x \in S$.

That is, we have, $\|x\| = \|\mathbb{A}x(t)\mathbb{B}y(t) + \mathbb{C}x(t)\| \leq r$, $\forall x \in S$.

Hence assumption (c) of theorem (2.2.2) is proved.

Step V: Finally, we show that $\zeta\mathbb{M} + \eta < 1$ that is condition (d) of theorem (2.2.2) holds.

Since $\mathbb{M} = \|\mathbb{B}(\mathbb{S})\| = sup_{x\in\mathbb{S}}\{sup_{t\in\mathbb{R}_+}|\mathbb{B}x(t)|\}$

$$= sup_{x\in\mathbb{S}}\left\{sup_{t\in\mathbb{R}_+}\left|\frac{x_0 - h(0,x_0)}{f(0,x_0)} + \int_0^t (t-s)g\big(s,x(s)\big)ds\right|\right\}$$

$$\leq sup_{x\in\mathbb{S}}\left\{sup_{t\in\mathbb{R}_+}\left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \left|\int_0^t (t-s)g\big(s,x(s)\big)ds\right|\right\}$$

$$\leq sup_{x\in\mathbb{S}}\left\{\left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \rho_2(t)\right\}$$

$$\leq \left|\frac{x_0 - h(0,x_0)}{f(0,x_0)}\right| + \rho_2(t)$$

and therefore $\zeta\mathbb{M} + \eta = \left(\|\zeta\| \left(\left|\frac{x_0-h(0,x_0)}{f(0,x_0)}\right| + \mathcal{K}_2\right) + \|\eta\|\right) < 1,$

Thus, all the conditions of theorem (2.2.2) are satisfied and hence the operator equation $x = \mathbb{A}x\mathbb{B}x + \mathbb{C}x$ has a solution in $\mathbb{S}$.

Step VI: Now to show the solution is locally attractive on $\mathbb{R}_+$. Then we have

$$|x(t) - y(t)| = \left|\begin{array}{c}\left\{[f(t,x(t))]\left(\frac{x_0 - h(0,x_0)}{f(0,x_0)} + \int_0^t (t-s)g\big(s,x(s)\big)\,ds\right) + h\big(t,x(t)\big)\right\} - \\ \left\{[f(t,y(t))]\left(\frac{x_0 - h(0,x_0)}{f(0,x_0)} + \int_0^t (t-s)g\big(s,y(s)\big)ds\right) + h\big(t,y(t)\big)\right\}\end{array}\right|$$

$$\leq \left|f\big(t,x(t)\big)\right| \left(\frac{x_0-h(0,x_0)}{f(0,x_0)} + \int_0^t (t-s)g\big(s,x(s)\big)\,ds\right) + \left|h\big(t,x(t)\big)\right|$$

$$+\left|f\big(t,y(t)\big)\right| \left(\frac{x_0-h(0,x_0)}{f(0,x_0)} + \int_0^t (t-s)g\big(s,y(s)\big)\,ds\right) + \left|h\big(t,y(t)\big)\right|$$

$$\leq 2\mathbb{F}\left(\frac{x_0-h(0,x_0)}{f(0,x_0)} + \rho_2(t)\right) + 2\mathbb{H}, \forall t \in \mathbb{R}_+$$

Since $\lim_{t\to\infty}\rho_2(t) = 0$, $\lim_{t\to\infty}h\big(t,x(t)\big) = 0$

For $\epsilon > 0$, there is real number $\mathbb{T}' > 0, \mathbb{T}'' > 0$ such that $\rho_2(t) \leq \frac{\epsilon}{4F} - \frac{x_0-h(0,x_0)}{f(0,x_0)}$ $\forall$ $t \geq \mathbb{T}'$ and $\left|h\big(t,x(t)\big)\right| < \frac{\epsilon}{4}$, $\forall$ $t \geq \mathbb{T}''$ if we choose $\mathbb{T}^* = max\{\mathbb{T}', \mathbb{T}''\}$

Then from above inequality it follows that $|x(t) - y(t)| < \epsilon$ for all $t \geq \mathbb{T}^*$.

Hence SNQDE (1.1) has a locally attractive solution on $\mathbb{R}_+$.

This completes the proof.

Existence the extremal solutions for SNQDE (2.1.2)

Following definitions are useful in the forthcoming analysis for proving the existence of extremal solution.

Definition 2.5.8[15, 16, 18, 22, 23]: A function $a \in AC(\mathbb{R}_+, \mathbb{R})$ is called lower solution of (2.1.2) if

$$a(t) \leq h\big(t, a(t)\big) + f\big(t, a(t)\big)\left[\frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t g\big(s, a(s)\big)\, ds\right] \qquad (2.5.5)$$

$$t \in \mathbb{R}_+.$$

Definition 2.5.9[15, 16, 18,22,23,25]: A function $b \in AC(\mathbb{R}_+, \mathbb{R})$ is called upper solution of (2.1.2) if

$$b(t) \geq h\big(t, b(t)\big) + f\big(t, b(t)\big)\left[\frac{x_0 - h(0,x_0)}{f(0,x_0)} + \int_0^t (t-s) g\big(s, x(s)\big)\, ds\right] \qquad (2.5.6)$$

Definition 2.5.10a[[15, 16, 18,22,23,25]: A solution x_M of the SNQDE (2.1.2) is said to be maximal if x is any other solution of SNQDE (2.1.2) on $\mathbb{R}_+$ then we have $x(t) \leq x_M(t)$ for all $t \in \mathbb{R}_+$.

Definition 2.5.10 b [[15, 16, 18,22,23,25]: A solution x_m of the SNQDE (2.1.2) is said to be minimal if x is any other solution of SNQDE (2.1.2) on $\mathbb{R}_+$ then we have $x(t) \geq x_m(t)$ for all $t \in \mathbb{R}_+$.

We consider the following assumptions.

($\mathfrak{B}_5$) The function $h\big(t, x(t)\big)$, and function is monotone increasing in x almost every where for $t \in \mathbb{R}_+$.

($\mathfrak{B}_6$) The function $f\big(t, x(t)\big)$ be monotone increasing in x a. e. for $t \in \mathbb{R}_+$

($\mathfrak{B}_7$) The function $g\big(t, x(t)\big)$ is monotone increasing in x a.e. for $t \in \mathbb{R}_+$

($\mathfrak{B}_8$) The SNQDE (2.1.2) has a lower solution a and an upper solution b with $a \leq b$.

Remark (2.5.11): suppose that **$\mathfrak{B}_5$- $\mathfrak{B}_8$** is satisfied then ∃ a function $h_q(t) = \left|\int_0^t (t-s) g\big(s, a(s)\big) ds + \int_0^t (t-s) g\big(s, b(s)\big) ds\right|$ is Lesbegue measurable.

Theorem 2.5.12: Assume that the hypothesis $\boldsymbol{\mathcal{H}_5 - \mathcal{H}_8}$ and **($\mathfrak{B}_5$)- ($\mathfrak{B}_8$)** holds then SNQDE (2.1.2) has a minimal and a maximal solution.

Proof: Let $\mathbb{X} = AC(\mathbb{R}_+, \mathbb{R})$ and consider an ordered interval $[a, b]$ in X which is well defined in view of hypothesis (B_4). Let $\mathbb{A}, \mathbb{B}: [a, b]: \longrightarrow \mathcal{K}$ and $\mathbb{C}: [a, b] \longrightarrow \mathbb{X}$be three operators as defined in (2.4.3) and (2.4.4) and (2.4.5).

We will show that the operator $\mathbb{A}, \mathbb{B}$, and $\mathbb{C}$ are monotone increasing.

Let $x, y \in [a, b]$ be such that $x \leq y$,

$\mathbb{A}x = f\big(t, x(t)\big)$ then by (B_1)

$$f\big(t, x(t)\big) \leq f\big(t, y(t)\big).\, f\big(t, y(t)\big)$$

$$= \mathbb{A}y$$

$\mathbb{A}x \leq \mathbb{A}y$

$$\mathbb{B}x(t) = \frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t (t-s) g\big(s, x(s)\big) ds$$

$$\mathbb{B}x(t) \leq \frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t (t-s) g\big(s, y(s)\big) ds$$

$$\mathbb{B}x \leq \frac{x_0 - h(0, x_0)}{f(0, x_0)} + \int_0^t g\big(s, y(s)\big) ds$$

$$\mathbb{B}x \leq \mathbb{B}y$$

$$\mathbb{C}x(t) = h\big(t, x(t)\big)$$

$$h\big(t, x(t)\big) \leq h\big(t, y(t)\big)$$

$$h\big(t, y(t)\big) = \mathbb{C}y(t)$$

$$\mathbb{C}x(t) \leq \mathbb{C}y(t)$$

Hence the operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are strictly monotone increasing.

It can be shown, as in the proof of theorem (2.4.7) that operator $\mathbb{A}$ and $\mathbb{C}$ are Lipschitz.

It can be shown, using remark (2.5.11) as in the poof of theorem (2.4.7) that $\mathbb{B}$ is completely continuous. And $\alpha M + \beta < 1$,

Let $x \in [a, b]$ be any element then by (B_4) we have,

$$a \leq \mathbb{A}a.\mathbb{B}a + \mathbb{C}a \leq \mathbb{A}x.\mathbb{B}x + \mathbb{C}x \leq \mathbb{A}b.\mathbb{B}b + \mathbb{C}b \leq b$$

which shows that $\mathbb{A}x\mathbb{B}x + \mathbb{C}x \subset [a, b]$.

Thus, the operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ satisfy all the conditions of theorem (2.5.2) which yields that the operator inclusion $x \in \mathbb{A}x\mathbb{B}x + \mathbb{C}x$ and consequently the SNQDE (2.1.2) has maximal and minimal solution. This completes the proof.

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