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Research Paper

A Novel Method for Evaluating Explicit Integration of Certain Recurrent Form

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ABSTRACT: *A novel method is presented to evaluating explicit integration which has certain recurrent form. The proposed method re-writes the recurrent form in product operator with which closed-form solutions can be obtained.*
KEYWORDS: *Explicit integration, recurrent form, product operator, closed form solution*

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I. INTRODUCTION

Step-by-step time integration is widely used in structural dynamics problems, wave propagation problems, and even in AI recurrent neural network (RNN), which is a type of AI neural network that is designed to handle time-series data. Time integration methods can be categorized into two groups: explicit methods [1-4] and implicit methods [5]. An explicit method doesn't use an unknow derivative, i.e., the instantaneous rate of change of the status at the current time step to determine the current status, while an implicit method does. Consequently, it is not necessary for an explicit method to solve a system of linear equations at each time step, but small steps and thus a large number of steps may be needed for stability. In contrast, an implicit method needs to solve a system of linear equations once or more times at each time step, but large steps can be chosen. In general, each type of integration has its own advantages and disadvantages. Explicit algorithms are efficient for wave propagation problems, while implicit algorithms are effective for structural dynamics problems.

In this paper, a novel method is presented to evaluating explicit integration which has certain recurrent form. The proposed method re-writes the recurrent formula of integration to the form of product $\prod$ operator with which closed-form solutions can be obtained.

II. RECURRENT FORM

The explicit integration which has a recurrent formula in the form of

$$X_{k+1} = a_k + b_k X_k \qquad k = 0, 1, 2, \ldots \tag{1}$$

can be written in the form of product $\prod$ operator by expanding the recurrent formula at each time step.

$X_1 = a_0 + b_0 X_0$

$X_2 = a_1 + b_1 X_1 = a_1 + b_1(a_0 + b_0 X_0) = a_1 + b_1 a_0 + b_1 b_0 X_0$

$X_3 = a_2 + b_2 X_2 = a_2 + b_2(a_1 + b_1 a_0 + b_1 b_0 X_0) = a_2 + b_2 a_1 + b_2 b_1 a_0 + b_2 b_1 b_0 X_0$

...

$X_{k+1} = a_k + b_k X_k$

$$= a_k + b_k a_{k-1} + b_k b_{k-1} a_{k-2} + b_k b_{k-1} b_{k-2} a_{k-3} + \cdots + b_k b_{k-1} b_{k-2} \ldots b_1 a_0 + b_k b_{k-1} b_{k-2} \ldots b_1 b_0 X_0$$

$$= (b_k b_{k-1} b_{k-2} \ldots b_1)\left(\frac{a_k}{b_k b_{k-1} b_{k-2} \ldots b_1} + \frac{a_{k-1}}{b_{k-1} b_{k-2} \ldots b_1} + \frac{a_{k-2}}{b_{k-2} b_{k-3} \ldots b_1} + \frac{a_{k-3}}{b_{k-3} b_{k-4} \ldots b_1} + \cdots + \frac{a_1}{b_1} + a_0 + b_0 X_0\right)$$

Thus, the explicit integration can be written as

$$X_{k+1} = B_k \left(\sum_{i=0}^{k} \frac{a_i}{B_i} + b_0 X_0 \right) \quad (2)$$

, where

$$B_i = \prod_{r=1}^{i} b_r \qquad i = 0, \ldots, k \quad (3)$$

$B_i \neq 0$ under the condition of $b_r \neq 0 \; or \; \frac{1}{b_r} \neq 0, r = 1, 2, \ldots, k$

III. PRODUCT ∏ EQUATIONS

Here is some product ∏ equations which can be used to evaluate $\prod_{r=1}^{k} b_r$ and find closed-form solutions of the explicit integration.

$$\prod_{r=1}^{k} (\alpha r + \beta) = \alpha^k \frac{\Gamma(k+1+\frac{\beta}{\alpha})}{\Gamma(1+\frac{\beta}{\alpha})} \quad (4)$$

$$\prod_{r=1}^{k} \alpha = \alpha^k \quad (5)$$

$$\prod_{r=1}^{k} r = k! \quad (6)$$

$$\prod_{r=1}^{k} (r+1) = (k+1)! \quad (7)$$

$$\prod_{r=1}^{k} (r+2) = \frac{(k+2)!}{2!} \quad (8)$$

$$\prod_{r=1}^{k} (r+m) = \frac{(k+m)!}{m!} \quad (9)$$

$$\prod_{r=1}^{k} (r-m) = \frac{(k-m)!}{(-m)!} = \begin{cases} \frac{(-1)^k (m-1)!}{(m-1-k)!}, k \leq m \\ 0, k \leq m \end{cases} \quad (10)$$

$$\prod_{r=1}^{k} \left(r + \frac{1}{2}\right) = \frac{(2k+1)!}{2^{2k} k!} \quad (11)$$

$$\prod_{r=1}^{k} \left(r - \frac{1}{2}\right) = \frac{(2k)!}{2^{2k} k!} \quad (12)$$

$$\prod_{r=1}^{k} \left(r + \frac{m}{2}\right) = \frac{\left(\frac{m-1}{2}\right)!(m+2k)!}{2^{2k} m! \left(\frac{m-1}{2}+k\right)!}, m = odd \quad (13)$$

$$\prod_{r=1}^{k} \left(r - \frac{m}{2}\right) = \begin{cases} \frac{(-1)^{\frac{m-1}{2}} (m-1)!(2k-m+1)!}{2^{2k} \left(\frac{m-1}{2}\right)! \left(\frac{2k-m+1}{2}\right)!}, m < 2k+2 \\ \frac{(-1)^k (m-2)! \left(\frac{m-1-2k}{2}\right)!}{2^{2k-1} \left(\frac{m-3}{2}\right)! (m-2-2k)!}, m > 2k+2 \end{cases} \quad m = odd \quad (14)$$

IV. EXAMPLE

Consider the explicit integration which has a recurrent formula shown in formula (15)

$$\int \frac{dx}{\cos^m ax} = \frac{1}{(m-1)a}\frac{\sin ax}{\cos^{m-1} ax} + \frac{m-2}{m-1}\int \frac{dx}{\cos^{m-2} ax} \quad m = 2, \ldots \tag{15}$$

a) If m is an odd number, i.e., m = 2k+1, k = 1, 2, 3, ...

$$\int \frac{dx}{\cos^{2k+1} ax} = \frac{1}{2ka}\frac{\sin ax}{\cos^{2k} ax} + \frac{k-\frac{1}{2}}{k}\int \frac{dx}{\cos^{2k-1} ax} \tag{16}$$

To write the explicit integration in product form (2), let

$$a_k = \frac{1}{2ka}\frac{\sin ax}{\cos^{2k} ax}$$

$$b_k = \frac{k-\frac{1}{2}}{k}$$

$$X_0 = \int \frac{dx}{cosax} = \frac{1}{a} log \frac{1+\sin ax}{cosax}$$

$$B_k = \prod_{r=1}^{k} \frac{r-\frac{1}{2}}{r} = \frac{(2k)!}{2^{2k}k!k!}$$

Thus, the close-form of this explicit integration is

$$\int \frac{dx}{\cos^{2k+1} ax} = \frac{(2k)!}{2^{2k}k!k!}\frac{1}{a}[log\frac{1+\sin ax}{cosax} + sinax\sum_{i=1}^{k}\frac{2^{2i-1}i!(i-1)!}{(2i)!\cos^{2i} ax}] \tag{17}$$

b) If m is an even number, i.e., m = 2k+2, k = 0, 1, 2, ...

$$\int \frac{dx}{\cos^{2k+2} ax} = \frac{1}{(2k+1)a}\frac{\sin ax}{\cos^{2k+1} ax} + \frac{k}{k+\frac{1}{2}}\int \frac{dx}{\cos^{2(k-1)+2} ax} \tag{18}$$

, let

$$a_k = \frac{1}{(2k+1)a}\frac{\sin ax}{\cos^{2k+1} ax} = \frac{tanax}{ax}\frac{1}{(2k+1)\cos^{2k} ax}$$

$$b_k = \frac{k}{k+\frac{1}{2}}$$

Then we have

$$B_k = \prod_{r=1}^{k} \frac{r}{r+\frac{1}{2}} = \frac{2^{2k}k!k!}{(2k+1)!}$$

Thus, the close-form of this explicit integration is

$$\int \frac{dx}{\cos^{2k+2} ax} = \frac{2^{2k}k!k!}{(2k+1)!}\frac{tanax}{ax}\sum_{i=0}^{k}\frac{(2i)!}{2^{2i}i!i!\cos^{2i} ax} \tag{19}$$

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In summary,

$$\int \frac{dx}{\cos^m ax} = \begin{cases} \frac{(m-1)!}{2^{m-1}(\frac{m-1}{2})!(\frac{m-1}{2})!}\frac{1}{a}[log\frac{1+\sin ax}{cos ax} + sin ax\sum_{i=1}^{\frac{m-1}{2}}\frac{2^{2i-1}i!(i-1)!}{(2i)!\cos^{2i}ax}], \ m = 3, 5 \\ \frac{2^{m-2}(\frac{m}{2}-1)!(\frac{m}{2}-1)!}{(m+1)!}\frac{tan ax}{ax}\sum_{i=0}^{\frac{m}{2}-1}\frac{(2i)!}{2^{2i}i!i!\cos^{2i}ax}, \ m = 2, 4, 6 \end{cases} \quad (20)$$

V. CONCLUSION

A novel method is presented to evaluating explicit integration which has certain recurrent form. The recurrent formula is written in the form of product $\prod$ operator with which closed-form solutions can be obtained. Some useful product $\prod$ equations are listed. The given example illustrates that the method is an efficient way to find the closed-form solutions of this type of explicit integration.

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