Quest Journals
Journal of Research in Applied Mathematics
Volume 9 ~ Issue 2 (2023) pp: 81-84
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

Uniqueness Results for the Euler Characteristic and the Pick Formula

Jack Weinstein
(Department of Mathematics, University of Haifa,Haifa, Israel)

ABSTRACT:*Two uniqueness results are proven on the class of binary polygonal images: one for the Euler characteristic, the other for the Pick formula.*
KEYWORDS:*binary polygonal image, Euler characteristic, Pick formula.*

Received 12 Feb., 2023; Revised 22 Feb., 2023; Accepted 25 Feb., 2023 © The author(s) 2023. Published with open access at www.questjournals.org

I. BASICS

A*binary polygonal image*(shape) P is defined as a finite collection of nonintersecting simple polygons (without holes or with nonintersecting holes). Each polygon in the collection is called a component; a hole is a polygon placed inside a component. A component is considered a *foreground* component; a hole is a *background* component. Some foreground components may be placed inside holes. A binary polygonal image has a hierarchical structure. If we denote by C the number of components and by H the total number of holes, then we define the *Euler characteristic* of P as $\chi(P)=C-H$.

If there are no holes in the components, then $\chi(P)=C$; if there is only one component, then $\chi(P)=1-H$; and if the only component has no holes, then $\chi(P)=1$.

A *triangulation* of a polygonal image P is a decomposition of P into triangles, such that each border vertex of P is a vertex of one or more triangles, each border edge of P is an edge of one triangle, and every two triangles do not intersect, or they have a common vertex or a common edge and two common vertices.. Given a triangulation of P, the Euler characteristic can be redefined as $\chi(P)=v(P)-e(P)+f(P)$, where $v(P)$ is the number of triangulation vertices, $e(P)$ is the number of triangulation edges, and $f(P)$ is the number of triangulation faces (triangles).

II. THE INTEGRAL LATTICE

Consider the integral lattice in the plane (the set of all points with integer coordinates). Let P be a binary polygonal image, all of whose vertices belong to the lattice (we call it a lattice polygonal image).A *lattice decomposition (lattice triangulation)* of P is a triangulation of P such that all ofits vertices have integer coordinates. Each face of this triangulation is a *lattice triangle*(its vertices are lattice points, i.e. their coordinates are integers).

A lattice triangle is *primitive* if there are no lattice points in its interior and the only lattice points on its boundary are its three vertices. As shown in [1], the area of a primitive lattice triangle is always $1/2$.

In the sequel we will also use the notations: $v(P)=v_i(P)+v_b(P)$, where $v_i(P)$ is the number of triangulation vertices interior to P and $v_b(P)$ is the number of triangulation vertices on its boundary; $e(P)=e_i(P)+e_b(P)$, where $e_i(P)$ is the number of triangulation edges interior to P and $e_b(P)$ is the number of triangulation edges on its boundary.

The Pick formula was defined in [2] and redefined in [3] as $A(P)=\frac{1}{2}f(P)=v_i(P)+\frac{1}{2}v_b(P)-\chi(P)$ ($A(P)$ is the area of P). As shown in [3], each lattice polygonal image admits a lattice decomposition (subdivision) into primitive triangles (not in a unique way).

III. FROM EULER TO PICK

The Pick formula can be derived from the Euler formula in a simple way; for a proof, see [4]. We give here a slightly modified variant of the proof in [5].

Consider a lattice polygonal image P with a given lattice decomposition into primitive triangles. Since $v_b(P)=e_b(P)$, we have $\chi(P)=v(P)-e(P)+f(P)=v_i(P)-e_i(P)+f(P)$. Each face contributes three edges to the count; each interior edge appears exactly twice and each boundary edge appears only once. Therefore, $3f(P)=2e_i(P)+e_b(P)=2e_i(P)+v_b(P)$ and $e_i(P)=\frac{3}{2}f(P)-\frac{1}{2}v_b(P)$.

This implies $\chi(P)=v_i(P)-\frac{3}{2}f(P)+\frac{1}{2}v_b(P)+f(P)=v_i(P)+\frac{1}{2}v_b(P)-\frac{1}{2}f(P)$ and, therefore, $A(P)=\frac{1}{2}f(P)=v_i(P)+\frac{1}{2}v_b(P)-\chi(P)$ (the Pick formula).

IV. MEDIAL SUBDIVISIONS

Let P be a lattice polygonal imagewith a given lattice decomposition into primitive triangles. We define the medial subdivision mP of P in the following way. Each edge is divided by its middle point into two edges. In each triangle, the three middle points of its edges are connected by three new edges; the triangle is thus divided into four equal sized smaller triangles. We also refine the integral lattice such that it contains all points with coordinates $\left(\frac{a}{2},\frac{b}{2}\right)$ (a and b integers). We then expand the plane by $(x,y)\to(2x,2y)$; the refined lattice turns into the original integral lattice. Each middle point of an edge becomes a lattice vertex; each new triangle becomes a primitive lattice triangle. The numbers of the various elements of mP are related to those of P by the following formulas:

$$\begin{aligned}
v_i(mP)&=v_i(P)+e_i(P)\\
v_b(mP)&=2v_b(P)\\
e_i(mP)&=2e_i(P)+3f(P)\\
e_b(mP)&=2e_b(P)\\
f(mP)&=4f(P)
\end{aligned}$$

The Euler characteristic is invariant under the medial subdivision operation:

$$\begin{aligned}
\chi(mP)&=v(mP)-e(mP)+f(mP)=v_i(mP)-e_i(mP)+f(mP)\\
&=v_i(P)+e_i(P)-2e_i(P)-3f(P)+4f(P)\\
&=v_i(P)-e_i(P)+f(P)=v(P)-e(P)+f(P)=\chi(P)
\end{aligned}$$

The Pick formula can be rewritten in the following way:

$$\chi(P)=v_i(P)+\frac{1}{2}v_b(P)-\frac{1}{2}f(P)$$

It is obviously an invariant under medial subdivision. Alternatively,

$$v_i(mP)+\frac{1}{2}v_b(mP)-\frac{1}{2}f(mP)=v_i(P)+e_i(P)+v_b(P)-2\cdot f(P)=$$
$$v_i(P)+\frac{3}{2}f(P)-\frac{1}{2}v_b(P)+v_b(P)-2\cdot f(P)=v_i(P)+\frac{1}{2}v_b(P)-\frac{1}{2}f(P)$$

V. UNIQUENESS OF THE EULER CHARACTERISTIC

Consider the class $\mathcal{LP}$ of all lattice polygonal images P, each with a given lattice decomposition. Our first uniqueness result is

Theorem 1. If $\lambda:\mathcal{LP}\to R$ is a function defined by

$$\lambda(P)=\alpha\cdot v_i(P)+\beta\cdot e_i(P)+\gamma\cdot f(P),\ \ \alpha,\beta,\gamma\in R,$$

and satisfying $\lambda(mP)=\lambda(P)$ for each $P\in\mathcal{LP}$.

Then $\lambda=\kappa\cdot\chi:\mathcal{LP}\to R$, where $\kappa\in R$ is a constant and the function is defined by

$$(\kappa\cdot\chi)(P)=\kappa\cdot\left(v_i(P)-e_i(P)+f(P)\right).$$

Proof: If $P\in\mathcal{LP}$, then

$$\begin{aligned}\lambda(mP)&=\alpha\cdot v_i(mP)+\beta\cdot e_i(mP)+\gamma\cdot f(mP)\\&=\alpha\cdot\left(v_i(P)+e_i(P)\right)+\beta\cdot\left(2e_i(P)+3f(P)\right)+4\gamma\cdot f(P)\\&=\alpha\cdot v_i(P)+(\alpha+2\beta)\cdot e_i(P)+(3\beta+4\gamma)\cdot f(P)\end{aligned}$$

The invariance condition $\lambda(mP)=\lambda(P)$ implies

$$(\alpha+\beta)\cdot e_i(P)+3(\beta+\gamma)\cdot f(P)=0$$

For the image P_1 that contains one triangle with vertices $(-1,0)$, $(0,0)$, $(0,1)$: $e_i(P_1)=0$, $f(P_1)=1$, then $\beta+\gamma=0$.For the image P_2 that contains the triangle in P_1 and one more triangle with vertices $(0,0)$, $(0,1)$, $(1,1)$: $e_i(P_2)=1$, $f(P_2)=2$, then $\alpha+\beta=0$. Therefore, $\alpha=-\beta=\gamma$ and we derive the result

$$\lambda(P)=\gamma\cdot\left(v_i(P)-e_i(P)+f(P)\right)$$

VI. UNIQUENESS OF THE PICK FORMULA

Our second uniqueness result is

Theorem 2. If $\lambda:\mathcal{LP}\to R$ is a function defined by

$$\lambda(P)=\alpha\cdot v_i(P)+\beta\cdot v_b(P)+\gamma\cdot f(P),\ \ \alpha,\beta,\gamma\in R,$$

and satisfying $\lambda(mP)=\lambda(P)$ for each $P\in\mathcal{LP}$.

Then $\lambda=\kappa\cdot\chi:\mathcal{LP}\to R$, where $\kappa\in R$ is a constant and the function is defined by

$$(\kappa\cdot\chi)(P)=\kappa\cdot\left(v_i(P)+\frac{1}{2}v_b(P)-\frac{1}{2}f(P)\right).$$

Proof: If $P\in\mathcal{LP}$, then

$$\lambda(mP) = \alpha \cdot v_i(mP) + \beta \cdot v_b(mP) + \gamma \cdot f(mP)$$
$$= \alpha \cdot (v_i(P) + e_i(P)) + 2\beta \cdot v_b(P) + 4\gamma \cdot f(P)$$

The invariance condition $\lambda(mP) = \lambda(P)$ implies

$$\alpha \cdot e_i(P) + \beta \cdot v_b(P) + 3\gamma \cdot f(P) = 0$$

Since $e_i(P) = \frac{3}{2} f(P) - \frac{1}{2} v_b(P)$, we get

$$\alpha \cdot \left(\frac{3}{2} f(P) - \frac{1}{2} v_b(P) \right) + \beta \cdot v_b(P) + 3\gamma \cdot f(P) = 0$$

therefore,

$$\left(\beta - \frac{\alpha}{2} \right) \cdot v_b(P) + 3 \cdot \left(\frac{\alpha}{2} + \gamma \right) \cdot f(P) = 0$$

For the image P_1 that contains one triangle with vertices $(-1,0)$, $(0,0)$, $(0,1)$: $v_b(P_1) = 3$, $f(P_1) = 1$, then $3\beta + 3\gamma = 0$.For the image P_2 that contains the triangle in P_1 and one more triangle with vertices $(0,0)$, $(0,1)$, $(1,1)$: $v_b(P_2) = 4$, $f(P_2) = 2$, then $\alpha + 4\beta + 6\gamma = 0$. Therefore, $2\alpha = \beta = -2\gamma$ and we derive the result

$$\lambda(P) = (-2\gamma) \cdot \left(v_i(P) + \frac{1}{2} v_b(P) - \frac{1}{2} f(P) \right)$$

REFERENCES

[1]. R. W. Gaskell, M. S. Klamkin, P. Watson, *Triangulations and Pick's theorem*, Math. Magazine 49 (1): 35-37 (1976).

[2]. G. A. Pick, *GeometrischeszurZahlenlehre*, Sitzenber. Lotos (Prague) 19 (1899), pp. 311-319.

[3]. Ross Honsberger, *Ingenuity in Mathematics*, Random House / Springer, New Mathematical Library, 1970; reprinted: Math. Assoc. ofAmerica, New York, 1998; Essay 5: *The FareySeries*, pp. 24-37.

[4]. W. W. Funkenbusch, From Euler's formula to Pick's formula using an edge theorem, Amer. Math, Monthly 81: 647-648 (1974).

[5]. M. Aigner & G. M. Ziegler, *Proofs from THE BOOK*, 6th ed., Springer, 2018; Chapter 13: *Three applications of Euler's formula*, pp. 89-94.