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Research Paper

Conditions of ellipticity in three-dimension: application by energy potentials

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Abstract: *In this paper we are interested in the conditions of ellipticity in order to nd some relations which are functions of the material parameters from energy po- tentials and the eigenvalues of the gradient tensor of the deformations. As a methodology, we have chosen dierent forms of isotropic energy functions in compressible or in incompressible with kinematics of deformations in order to use them in the conditions of ellipticity to otain families of relations dened through lemmas. A comparison allowed us to see that for an adjustment of material parameters between two dierent energy potentials, we have the same conditions of ellipticity.*
Keywords: *Conditions of ellipticity, kinematics of deformation, eigenvalues, material pa- rameter, isotropic, compressible, incompressible.*

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I. Introduction

To describe the anisotropic hyperelastic mechanical behavior of a mechanical structure, it is still useful to use deformation energy functions in form polynomial, exponential, power or logarithmic. These energy potentials have been established as part of a phenomenological approach that describes the macroscopic nature of the material. This approach is basically used to find the parameters of the relations of behavior. However, modeling phenomenological is not always able to establish a direct correlation between the mechanisms of deformation and the physical characters of the structure which are known. Certain mathematical criteria such as convexity, ellipticity and objectivity should normally be satisfied when using the strain energy functions. These must take into account the principle of material indifference [1]. Once these criteria are met, the results become relatively easy to develop. The energy function is then a function of the transformation gradient tensor F and the various parameters of the model. Elliptic equations govern stationary problems, of equilibrium, generally defined on a bounded spatial domain of border on which the unknown is subject to boundary conditions. In fluid and solids mechanics, special attention has been paid to ellipticity equations from the study of compressible or incompressible material systems [2]. In elastostatic for example, Knowles and Sternberg [3] have shown that the loss of ellipticity in nonlinear systems is due to large local deformations whereas in acoustic, this loss is related to the non-singularity of the acoustic tensor. In plasticity it has been established that when elasto-viscoplastic behavior tends towards behavior elastoplastic, the ellipticity criterion can be linked to the analysis of stability [4]. So, the absence of a stability criterion may lead to the approximation of the condition of ellipticity considered as a criterion of local stability to the local form of classical stability analysis by the Lyapunov's theorem. Furthermore, it has long been known that compressible isotropic non-linearly elastic materials admit a loss of ellipticity at certain deformations. Knowles and Sternberg, in the context of crack problems, have been the first to show the possible loss of ellipticity at sufficiently severe deformations. The authors obtained ellipticity conditions on the power-law exponent for generalized neo-Hookean strain energy functions. Horgan proved a unified derivation of necessary and sufficients conditions for ellipticity of the three dimensional displacement equations of equilibrium for the generalized Blatz-Ko material. Qiu and Pence demonstrated some of the possibilities for loss of ordinary ellipticity in the unidirectionally reinforced neo-Hookean material.

II. Formulation of the problem

Consider a continuous material body. the whole of the particles of this body occupies, every moment, an open and connected domain or connected by arc of the physical space. At every point of the continuous medium corresponds one and only one particle. The initial undeformed configuration occupied by the body is used to define the strain, the strain rate and the stress and also to formulate the equilibrium equation. The actual position of a material particle $\mathbf{x}$ at time t is given by the map $\mathbf{y} = \mathbf{y}(\mathbf{x}, t)$.

Let $\mathbf{F} = \frac{\partial \mathbf{y}}{\partial \mathbf{x}}$ be the deformation gradient tensor, where $\mathbf{x}$ is the position vector of a material particle in the undeformed configuration and $\mathbf{y}$ is the corresponding position vector in the deformed configuration.
The right Cauchy-Green material metric tensor $\mathbf{C} = \mathbf{F}^{\mathbf{T}}\mathbf{F}$ is used as alternative objective deformation descriptor.
Objective is used as synonym for material frame indifference. This concept means that the strain measurement should not be influenced by rigid body motions. As illustration, the deformation gradient $\mathbf{F}$ is not objective as it includes part of the rotation in its formulation or polar decomposition: $\mathbf{F} = \mathbf{RU}$, $\mathbf{R}$ rotational tensor, $\mathbf{U}$ stretch tensor.
The definition of the material metric tensor $\mathbf{C}$ makes it possible to eliminate the rotational dependence for the strain descriptor:

$$\mathbf{C} = \mathbf{F}^{\mathbf{T}}\mathbf{F} = \mathbf{U}^{\mathbf{T}}\mathbf{R}^{\mathbf{T}}\mathbf{R}\mathbf{U} = \mathbf{U}^{\mathbf{T}}\mathbf{U}, \mathit{with}\ \ \mathbf{R}^{\mathbf{T}}\mathbf{R} = \mathbf{1}. \tag{1}$$

We consider an elastic homogeneous body having a density defined on all positive defined matrices and a homogeneous elastic body having a density:

$$W : \mathbf{M}_+^{n\,2} \rightarrow \mathbf{R}.$$

We hypothesize that the body occupies an open set $\Omega \subset \mathbf{R}^n$ when it is not subjected to any deformation.

The total energy of strain $\Psi : \Omega \rightarrow \mathbf{R}^n$ is given by:

$$E(x) = \int_\Omega W(\mathbf{grad}(\Psi)(X))dX. \tag{2}$$

We will say that W is isotropic if, in addition to the definition of total energy, we have:

$$W(\mathbf{F}) = W(\mathbf{FQ}), \forall \mathbf{F} \in M_+^{n\,2}, \forall \mathbf{Q} \in So(n) \tag{3}$$

where $So(n)$ is the orthogonal group of order n.
Similarly, Truesdell and Noll [5] demonstrate that W is isotropic if and only if there exist a symmetric function Φ such that:

$$\Phi : \mathbf{R}_+^n \rightarrow \mathbf{R},$$

$$\mathbf{R}_+^n = \{c = (c_1, c_2, ..., c_n) \in \mathbf{R}^n, c_i > 0, \leq 1 \leq i \leq n\}$$

and $W(\mathbf{F}) = \Phi(\lambda_1, \lambda_2, ..., \lambda_n), \forall \mathbf{F} \in M_+^{n2}$

and where $\lambda_1, \lambda_2, ..., \lambda_n$ denote the eigenvalues of $\mathbf{F}$.

For its part, Ball [6] notes that $W \in C^r(M_+^{n\,2})$ if and only if $\Phi \in C^r(\mathbf{R}_+^{n\,2})$, $r = 1, 2$.

Many elastic materials studied in mechanics are isotropic. This isotropy in addition to the objectivity of W implies:

$$W(\mathbf{FR}) = W(\mathbf{QF}) = W(\mathbf{F}), \forall \mathbf{F} \in R_+^{n\,2}, \forall \mathbf{Q}, \mathbf{R} \in So(n) \tag{4}$$

3 Ellipticity

In the case of the theory of elasticity, for transverse isotropic materials, some ellipticity conditions associated with the energy function have been established [7,8]:

$$\begin{aligned} W_{11}+4W_{12}+4W_{22}+2W_{13}+4W_{23}+W_{33} &= E_{11}/4, \\ W_2+W_3 &= (E_{12}-E_{11})/4, \\ W_1+W_2 &= E_{22}/2\ , \end{aligned} \qquad (5)$$

where $W_i = \frac{\partial W}{\partial \nu_i}$ and $W_{ij} = \frac{\partial^2 W}{\partial \nu_i \partial \nu_j}$, $(i,j=1,2,3)$ are calculated in the reference configuration, the coefficients E_{kk} are the elastic constants in the linear elastic standard and correspond to a material symmetry.

In the absence of volume force, the equilibrium equations in the reference configuration, as a function of the Jean tensor, have the following simple form:

$$div(\mathbf{S}) = 0 \qquad (6)$$

and can be written in the form:

$$A_{kjli} x_{i,kl} = 0 \qquad (7)$$

As $\mathbf{F}$ and $grad(\mathbf{F})$ being regular, let us denote by $\mathbf{n}$ the normal vector to the surface of the elastic body in the reference configuration and by $\mathbf{m}$ the vector such that $\mathbf{m} = \mathbf{Fn}$ in the deformed configuration.
Using the theory of waves [7,9], we can define the tensor $\mathbf{Q}(\mathbf{m})$ such that:

$$Q_{ji} = \tilde{A}_{pjqi} m_p m_q \qquad (8)$$

where

$$\tilde{A}_{pjqi} = \frac{1}{J} F_{pk} F_{ql} A_{kjli}, J = det(\mathbf{F}). \qquad (9)$$

The condition of elipticity is such

$$\tilde{A}_{pjqi} n_p n_q m_j m_i \neq 0, \forall \mathbf{m} \neq \mathbf{0}, \mathbf{n} \neq \mathbf{0}. \qquad (10)$$

Considering the previous equations, this condition of ellipticity can still be written in the more general and explicit form:

$$\sum_{i,j,k,l=1}^{n} \frac{\partial^2 W}{\partial F_{ik} \partial F_{jl}} n_i n_j m_k m_l \geq 0, \notin n, m \in \mathbb{R}^n, \forall \boldsymbol{F} \in (\mathbb{R}^n)^2 \qquad (11)$$

also called Legendre-Hadamard ellipticity condition.

The function W is elliptical if it is elliptical in every point of its continuous domain. It is first of all a necessary condition. Its field of application is wide and varied. For $n=2$ or $n=3$, for a compressible or incompressible medium, the field of wave propagation in acoustics is very well explored. For example, the strong condition of implies a positive definite acoustic tensor and excludes wave speeds equal to zero, i.e a strict inequality of (11).

In mathematics, there is a close link between ellipticity, coercivity and convexity.

3.1 Proposition

Let $W = \Phi(\mathbf{x}), \mathbf{x} = (\lambda_1, \lambda_2, ..., \lambda_n)$ be a class function C^2. Then W is elliptical if and only if there exists $\alpha > 0$ such as

$$\left\langle \nabla^2\Phi(\mathbf{x})\mathbf{h}, \mathbf{h} \right\rangle \geqslant \alpha \|\mathbf{h}\|^2, \forall \mathbf{x}, \mathbf{h} \in \mathbf{R}^n$$

Proof

Suppose that $\Phi(\mathbf{x})$ is elliptical.
Let $\mathbf{h} \in \mathbf{R}^n$ be fixed and denote by:

$\psi : \mathbf{R}^n \to \mathbf{R}$ the function given by $\psi(\mathbf{x}) = \langle \nabla\Phi(\mathbf{x}), \mathbf{h} \rangle$.

We then have:

$$\left\langle \nabla^2\Phi(\mathbf{x})\mathbf{h}, \mathbf{h} \right\rangle = \langle \nabla\psi(\mathbf{x}), \mathbf{h} \rangle = \frac{\partial\psi(\mathbf{x})}{\partial\mathbf{h}} = \lim_{t\to 0} \frac{\langle \nabla\Phi(\mathbf{x}+t\mathbf{h}), \mathbf{h} \rangle - \langle \nabla\Phi(\mathbf{x}), \mathbf{h} \rangle}{t}.$$

By using the bilinearity of the scalar product and then the fact that Φ is elliptic, we obtain:

$$\left\langle \nabla^2\Phi(\mathbf{x})\mathbf{h}, \mathbf{h} \right\rangle = \lim_{t\to 0} \frac{\langle \nabla\Phi(\mathbf{x}+t\mathbf{h}) - \nabla\Phi(\mathbf{x}), t\mathbf{h} \rangle}{t^2} \geqslant \alpha \frac{\|t\mathbf{h}\|}{t^2} = \alpha \|\mathbf{h}\|^2.$$

Suppose now that the inequality of the proposition is true and show that $\Phi(\mathbf{x})$ is elliptic.

Let $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$arbitrarily be fixed and the application $\chi : \mathbf{R}^n \to \mathbf{R}$ given by:

$\chi(\mathbf{z}) = \langle \nabla\Phi(\mathbf{z}), \mathbf{x} - \mathbf{y} \rangle, \forall \mathbf{z} \in \mathbf{R}^n$, then

$\langle \nabla\Phi(\mathbf{x}) - \nabla\Phi(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle = \chi(\mathbf{x}) - \chi(\mathbf{y}) = \langle \nabla\chi(\mathbf{y} + \theta(\mathbf{x} - \mathbf{y})) - \langle \nabla\Phi(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle$, $\theta \in]0, 1[$.

On the other hand, we have

$$\nabla\chi(\mathbf{z}) = \nabla^2\Phi(\mathbf{z})(\mathbf{x} - \mathbf{y}).$$

$$\nabla\chi(\mathbf{z}) = \nabla^2\Phi(\mathbf{z})(\mathbf{x} - \mathbf{y}).$$

From this we deduce, given the inequality of the proposition:

$$\langle \nabla\Phi(\mathbf{x}) - \nabla\Phi(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle = \left\langle \nabla^2\Phi(\mathbf{y} + \theta(\mathbf{x} - \mathbf{y})))(\mathbf{x} - \mathbf{y})), \mathbf{x} - \mathbf{y} \right\rangle$$

$$\geq \alpha \|\mathbf{x} - \mathbf{y}\|^2$$

which completes the proof.
The following proposition, concerning the energy potential W, shows an equiv-

alence between its convexity and its ellipticity.

3.2 Proposition

Let $W = \Phi(\mathbf{x})$ be a class function $C^2(\mathbf{R}^n_+)$ and Φ symmetrical, $\mathbf{x} = (\lambda_1, \lambda_2, ..., \lambda_n)$
Then W is called. convex if and only if the two following conditions are satisfied:

1. $\frac{\lambda_i\phi_i - \lambda_j\phi_j}{\lambda_i - \lambda_j} \geqslant 0, \lambda_i \neq \lambda_j, 1 \leqslant i < j \leqslant n$
2. $M^\epsilon = (m^\epsilon_{ij})$, is a symmetric positive definite matrix, $1 \leqslant i, j \leqslant n$

where

$$M^\epsilon = (m^\epsilon_{ij}) = \begin{cases} \phi_{ii} & \text{if } i = j \text{ or if } i < j, \lambda_i = \lambda_j \\ \epsilon_i\epsilon_j\phi_{ij} + \frac{\Phi_i - \epsilon_i\epsilon_j\Phi_j}{\lambda_i - \epsilon_i\epsilon_j\lambda_j} & \text{if } i < j, \lambda_i \neq \lambda_j, \text{or} \epsilon_i\epsilon_j \neq 1 \end{cases}$$

and for any choice of $\epsilon_j \in \{-1, 1\}$. $\phi_p = \frac{\partial\phi}{\partial\lambda_p}, \phi_{pq} = \frac{\partial^2\phi}{\partial\lambda_p\partial\lambda_q}$, with $\phi_{ii} \geq 0$.

Proof:
For proof of this result, see Dacorogna [10].
In his proof of this result, the author shows that the condition translated into equality 1 of the proposition is equivalent to the condition of Legendre-Hadamard.

The previous set of conditions read as follows for $\lambda_i \neq \lambda_j$ (if $\lambda_i = \lambda_j$, these inequalities are still valid when properly interpreted, cf. below)

3.3 Applications

In applications on the results on ellipticity and convexity,we will consider, in a system of cylindrical coordinates, in a 3-dimensional Euclidean space, an elastic body subjected to deformations. We define the kinematics of deformation as follows:

$$r = r(R, \Theta, Z); \theta = (R, \Theta, Z); z = z(R, \Theta, Z) \tag{12}$$

where (R, Θ, Z) and (r, θ, z) are respectively, the reference and the deformed position of material particle in the cylindrical system.

The gradient tensor of the transformation, in its diagonal form can be written by:

$$\mathbf{F} = \begin{pmatrix} \lambda_r & 0 & 0 \\ 0 & \lambda_\theta & 0 \\ 0 & 0 & \lambda_z \end{pmatrix} \tag{13}$$

where $\lambda_r, \lambda_\theta$ and λ_z are the eigenvalues of $\boldsymbol{F}$.
It follows the right Cauchy-Green deformation tensor.

$$\mathbf{C} = \begin{pmatrix} \lambda_r^2 & 0 & 0 \\ 0 & \lambda_\theta^2 & 0 \\ 0 & 0 & \lambda_z^2 \end{pmatrix} \tag{14}$$

The first three isotropic invariants becomes:

$$I_1 = tr(\mathbf{C}) = \lambda_r^2 + \lambda_\theta^2 + \lambda_z^2; \tag{15}$$

$$I_3 = det(\mathbf{C}) = \lambda_r^2 \lambda_\theta^2 \lambda_z^2; \tag{16}$$

$$I_2 = I_3 \mathbf{C}^{-1} = \lambda_r^2 \lambda_\theta^2 + \lambda_\theta^2 \lambda_z^2 + \lambda_z^2 \lambda_r^2. \tag{17}$$

The energy function depend on the invariants, so they depend on the eigenvalues of $\boldsymbol{F}$.
For the rest of our work, we will stand:

$$G_i = \partial W/\partial\lambda_i, G_{ij} = \partial^2 W/\partial\lambda_i \partial\lambda_j, (i, j = r, \theta, z) \tag{18}$$

3.3.1 Case of a polynomial isotropic compressible energy function

Let's consider a polynomial energy function in [11,12] defined by:

$$W = \alpha_1 (I_1 - 3) + \alpha_2 (I_2 - 3) + \alpha_3 (I_3 - 1) \tag{19}$$

Lemma 1
When we consider a material point M with the coordonates (R, Θ, Z) in the undeformed configuration and (r, θ, z) in the deformed configuration whose movement is described by the following kinematic:

$$r = r(R), \theta = \Theta, z = Z. \tag{20}$$

With the energy function of the relation (19), we have

$$\begin{aligned} (\lambda_r + \lambda_\theta)\left(\alpha_1 + \alpha_2 \lambda_z^2\right) \geq 0 \\ (\lambda_r + \lambda_z)\left(\alpha_1 + \alpha_2 \lambda_\theta^2\right) \geq 0 \\ (\lambda_\theta + \lambda_z)\left(\alpha_1 + \alpha_2 \lambda_r^2\right) \geq 0 \end{aligned} \tag{21}$$

and M^ϵ is a copositive matrix difined positive.

Proof
From (15) to (18), W defined in (19) becames a function of $\lambda_r^2, \lambda_\theta^2, \lambda_z^2$ defined by:

$$W = \alpha_1 \left(\lambda_r^2 + \lambda_\theta^2 + \lambda_z^2 - 3\right) + \alpha_2 \left(\lambda_r^2 \lambda_\theta^2 + \lambda_\theta^2 \lambda_z^2 + \lambda_z^2 \lambda_r^2 - 3\right) + \alpha_3 \left(\lambda_r^2 \lambda_\theta^2 \lambda_z^2 - 1\right) \tag{22}$$

The relation (22) gives the first order derivatives of the enrgy function by:

$$\begin{aligned} G_r &= 2\lambda_r \left[\alpha_1 + \alpha_2 \left(\lambda_\theta^2 + \lambda_z^2\right) + \alpha_3 \lambda_\theta^2 \lambda_z^2\right] \\ G_\theta &= 2\lambda_\theta \left[\alpha_1 + \alpha_2 \left(\lambda_r^2 + \lambda_z^2\right) + \alpha_3 \lambda_r^2 \lambda_z^2\right] \\ G_z &= 2\lambda_z \left[\alpha_1 + \alpha_2 \left(\lambda_r^2 + \lambda_\theta^2\right) + \alpha_3 \lambda_r^2 \lambda_\theta^2\right] \end{aligned} \tag{23}$$

When we use the condition of the Proposition 3.2 defined by:

$$\frac{\lambda_i G_i - \lambda_j G_j}{\lambda_i - \lambda_j} \geqslant 0, \lambda_i \neq \lambda_j, r \leqslant i < j \leqslant z \tag{24}$$

The relations of system (23) in (24) allow us to find after calculation and reduction

$$\begin{aligned} \tfrac{\lambda_r G_r - \lambda_\theta G_\theta}{\lambda_r - \lambda_\theta} &= \left(\lambda_r + \lambda_\theta\right)\left(\alpha_1 + \alpha_2 \lambda_z^2\right) \geq 0 \\ \frac{\lambda_r G_r - \lambda_z G_z}{\lambda_r - \lambda_z} &= \left(\lambda_r + \lambda_z\right)\left(\alpha_1 + \alpha_2 \lambda_\theta^2\right) \geq 0 \\ \tfrac{\lambda_\theta G_\theta - \lambda_z G_z}{\lambda_\theta - \lambda_z} &= \left(\lambda_\theta + \lambda_z\right)\left(\alpha_1 + \alpha_2 \lambda_r^2\right) \geq 0 \end{aligned} \tag{25}$$

Now if we calculate the second derivatives of the energy function with respect to the eigenvalues, we obtain

$$\begin{aligned} G_{rr} &= 2\left[\alpha_1 + \alpha_2 \left(\lambda_\theta^2 + \lambda_z^2\right) + \alpha_3 \lambda_\theta^2 \lambda_z^2\right] \\ G_{\theta\theta} &= 2\left[\alpha_1 + \alpha_2 \left(\lambda_r^2 + \lambda_z^2\right) + \alpha_3 \lambda_r^2 \lambda_z^2\right] \\ G_{zz} &= 2\left[\alpha_1 + \alpha_2 \left(\lambda_r^2 + \lambda_\theta^2\right) + \alpha_3 \lambda_r^2 \lambda_\theta^2\right] \end{aligned} \tag{26}$$

and

$$\begin{aligned} G_{r\theta} = G_{\theta r} &= 4\lambda_r \lambda_\theta \left[\alpha_2 + \alpha_3 \lambda_z^2\right] \\ G_{rz} = G_{zr} &= 4\lambda_r \lambda_z \left[\alpha_2 + \alpha_3 \lambda_\theta^2\right] \\ G_{\theta z} = G_{z\theta} &= 4\lambda_\theta \lambda_z \left[\alpha_2 + \alpha_3 \lambda_r^2\right] \end{aligned} \tag{27}$$

The others components of the matrix M^ϵ are:

$$\begin{aligned} m_{r\theta} &= \varepsilon_1 \varepsilon_2 G_{r\theta} + \tfrac{G_r - \varepsilon_1 \varepsilon_2 G_\theta}{\lambda_r - \varepsilon_1 \varepsilon_2 \lambda_\theta} \\ m_{rz} &= \varepsilon_1 \varepsilon_2 G_{rz} + \frac{G_r - \varepsilon_1 \varepsilon_2 G_z}{\lambda_r - \varepsilon_1 \varepsilon_2 \lambda_z} \\ m_{\theta z} &= \varepsilon_1 \varepsilon_2 G_{\theta z} + \tfrac{G_\theta - \varepsilon_1 \varepsilon_2 G_z}{\lambda_\theta - \varepsilon_1 \varepsilon_2 \lambda_z} \end{aligned} \tag{28}$$

That give all the components of the matrix $\boldsymbol{M^\varepsilon}$ as

$$\mathbf{M}^\epsilon = \begin{pmatrix} G_{rr} & m_{r\theta} & m_{rz} \\ m_{\theta r} & G_{\theta\theta} & m_{\theta z} \\ m_{zr} & m_{z\theta} & G_{zz} \end{pmatrix} \tag{29}$$

And according to the proposition 3.2, the matrix M^{ϵ} is a copositive and defined positive.
Whith the system (25) we have the proof of the Lemma 1.

Remarque: In incompressible, we have $I_3 = 1$, that give us a particulary case of the lemma 1 given by the following lemma:

Lemma 2
Let's consider a material point M with the coordonates (R, Θ, Z) in the undeformed configuration and (r, θ, z) in the deformed configuration whose movement is described by the following kinematic:

$$r = r\left(R\right), \theta = \Theta, z = Z. \tag{30}$$

With $\lambda_r > 0$ and the energy function of the relation (19), we have

$$\begin{aligned} &\alpha_1 + \alpha_2 \geq 0 \\ &\lambda_r^2 \geq sup\left(-\frac{\alpha_1}{\alpha_2}; -\frac{\alpha_2}{\alpha_1}\right) \end{aligned} \tag{31}$$

and M^{ϵ} is a copositive matrix difined positive.

Proof
In incompressible, we obtain $I_3 = 1$ and the kinematic give $\lambda_1 = 1$, that allow us to have:

$$\lambda_r = \frac{1}{\lambda_\theta}, \tag{32}$$

Since then, the use of the lemma 1 with (32) yield

$$\begin{aligned} &(\alpha_1 + \alpha_2)\left(\lambda_r + \tfrac{1}{\lambda_r}\right) \geq 0, \\ &(1 + \lambda_r)\left(\alpha_1 + \frac{\alpha_2}{\lambda_r^2}\right) \geq 0, \\ &\left(1 + \tfrac{1}{\lambda^r}\right)\left(\alpha_1 + \alpha_2\lambda_r^2\right) \geq 0, \end{aligned} \tag{33}$$

The first inequality and the two last inequalites of the system (33) give respectively

3.3.2 Case of a exponential energy function

Let's consider the the exponential energy function in isotropic and incompressibility [13] defined by:

$$W = \alpha\left[exp\left(\beta\left(I_1 - 3\right)\right) - 1\right] \tag{34}$$

Lemma 3
When we consider a material point M with the coordonates (R, Θ, Z) in the undeformed configuration and (r, θ, z) in the deformed configuration whose movement is described by the following kinematic:

$$r = r\left(R\right), \theta = \Theta, z = Z. \tag{35}$$

With the energy function of difined in (34), we have

$$
\begin{aligned}
2\alpha\beta(\lambda_r+\lambda_\theta) &\geq 0\\
2\alpha\beta(\lambda_r+\lambda_z) &\geq 0 \qquad (36)\\
2\alpha\beta(\lambda_\theta+\lambda_z) &\geq 0
\end{aligned}
$$

and M^ϵ is a copositive matrix difined positive.

Proof

When we use the expression (15) in the energy defined in (34), we find:

$$W=\alpha\left[exp\left(\beta\left(\lambda_r^2+\lambda_\theta^2+\lambda_z^2-3\right)\right)-1\right] \qquad (37)$$

From (37), we can calculate the first order derivatives of the energy which become:

$$
\begin{aligned}
G_r &= 2\alpha\beta\lambda_r\left[exp\left(\beta\left(\lambda_r^2+\lambda_\theta^2+\lambda_z^2-3\right)\right)\right]\\
G_\theta &= 2\alpha\beta\lambda_\theta\left[exp\left(\beta\left(\lambda_r^2+\lambda_\theta^2+\lambda_z^2-3\right)\right)\right] \qquad (38)\\
G_z &= 2\alpha\beta\lambda_z\left[exp\left(\beta\left(\lambda_r^2+\lambda_\theta^2+\lambda_z^2-3\right)\right)\right]
\end{aligned}
$$

The propostion 3.2 and the relations of the system (38) yield

$$
\begin{aligned}
\frac{\lambda_rG_r-\lambda_\theta G_\theta}{\lambda_r-\lambda_\theta} &= 2\alpha\beta(\lambda_r+\lambda_\theta)\left[exp\left(\beta\left(\lambda_r^2+\lambda_\theta^2+\lambda_z^2-3\right)\right)\right]\geq 0\\
\frac{\lambda_rG_r-\lambda_zG_z}{\lambda_r-\lambda_z} &= 2\alpha\beta(\lambda_r+\lambda_z)\left[exp\left(\beta\left(\lambda_r^2+\lambda_\theta^2+\lambda_z^2-3\right)\right)\right]\geq 0 \qquad (39)\\
\frac{\lambda_\theta G_\theta-\lambda_zG_z}{\lambda_\theta-\lambda_z} &= 2\alpha\beta(\lambda_\theta+\lambda_z)\left[exp\left(\beta\left(\lambda_r^2+\lambda_\theta^2+\lambda_z^2-3\right)\right)\right]\geq 0
\end{aligned}
$$

and as the exponential function is always positive, we fanally have

$$2\alpha\beta(\lambda_r+\lambda_\theta)\geq 0$$

$$2\alpha\beta(\lambda_r+\lambda_z)\geq 0$$

$$2\alpha\beta(\lambda_\theta+\lambda_z)\geq 0$$

The components of M^ϵ which are not in the main diagonal are:

$$
\begin{aligned}
m_{r\theta} &= 2\alpha\beta\left[1+\varepsilon_1\varepsilon_2\alpha\beta\lambda_r\lambda_\theta\right]\left[exp\left(\beta\left(\lambda_r^2+\lambda_\theta^2+\lambda_z^2-3\right)\right)\right]\\
m_{rz} &= 2\alpha\beta\left[1+\varepsilon_1\varepsilon_2\alpha\beta\lambda_r\lambda_z\right]\left[exp\left(\beta\left(\lambda_r^2+\lambda_\theta^2+\lambda_z^2-3\right)\right)\right] \qquad (40)\\
m_{\theta z} &= 2\alpha\beta\left[1+\varepsilon_1\varepsilon_2\alpha\beta\lambda_\theta\lambda_z\right]\left[exp\left(\beta\left(\lambda_r^2+\lambda_\theta^2+\lambda_z^2-3\right)\right)\right]
\end{aligned}
$$

With the system (38) the calculation of the second order derivatives of the energy function gives the others components of M^ϵ:

$$
\begin{aligned}
G_{rr} &= 2\alpha\beta\left(1+2\beta\lambda_r^2\right)\left[exp\left(\beta\left(\lambda_r^2+\lambda_\theta^2+\lambda_z^2-3\right)\right)\right]\\
G_{\theta\theta} &= 2\alpha\beta\left(1+2\beta\lambda_\theta^2\right)\left[exp\left(\beta\left(\lambda_r^2+\lambda_\theta^2+\lambda_z^2-3\right)\right)\right] \qquad (41)\\
G_{zz} &= 2\alpha\beta\left(1+2\beta\lambda_z^2\right)\left[exp\left(\beta\left(\lambda_r^2+\lambda_\theta^2+\lambda_z^2-3\right)\right)\right]
\end{aligned}
$$

with M^ϵ a copositive matrix defined positive according the proprosition 3.2. That give the proof of Lemma 3.

We can also calculate the others second order derivatives which are:

$$
\begin{aligned}
G_{r\theta} = G_{\theta r} &= 4\alpha\beta^2\lambda_r\lambda_\theta\left[exp\left(\beta\left(\lambda_r^2+\lambda_\theta^2+\lambda_z^2-3\right)\right)\right]\\
G_{rz} = G_{zr} &= 4\alpha\beta^2\lambda_r\lambda_z\left[exp\left(\beta\left(\lambda_r^2+\lambda_\theta^2+\lambda_z^2-3\right)\right)\right] \qquad (42)\\
G_{\theta z} = G_{z\theta} &= 4\alpha\beta^2\lambda_\theta\lambda_z\left[exp\left(\beta\left(\lambda_r^2+\lambda_\theta^2+\lambda_z^2-3\right)\right)\right]
\end{aligned}
$$

If we also consider the hypotesis of the proposition 3.2 that said that $\phi_{ii}\geq 0$, we obtain:

$$\begin{aligned} &2\alpha\beta\left(1+2\beta\lambda_r^2\right)\geq 0\\ &2\alpha\beta\left(1+2\beta\lambda_\theta^2\right)\geq 0\\ &2\alpha\beta\left(1+2\beta\lambda_z^2\right)\geq 0 \end{aligned} \tag{43}$$

3.3.3 Case of a rational energy function

In this part, we focus our study in order to know what is happening in the case of an isotropic, compressible rational energy function.
As an example we use the energy function in [14], which is a particular case of Blatz-Ko model discussed by Beatty and defined by:

$$W=\frac{\mu}{2}\left(\frac{I_2}{I_3}+2\sqrt{I_3}-5\right) \tag{44}$$

Lemma 4
Let's consider a material point M with the coordonates (R,Θ,Z) in the undeformed configuration and (r,θ,z) in the deformed configuration whose movement is described by the following kinematic:

$$r=r\left(R\right),\theta=\Theta,z=Z. \tag{45}$$

With $\lambda_r\neq 0,\lambda_\theta\neq 0$ and the energy function of the relation (44), we have

$$\begin{aligned} &\mu\left(\tfrac{1}{\lambda_r\lambda_\theta^2}+\tfrac{1}{\lambda_r^2\lambda_\theta}\right)\geq 0\\ &\mu\left(\frac{1}{\lambda_r\lambda_z^2}+\frac{1}{\lambda_r^2\lambda_z}\right)\geq 0\\ &\mu\left(\tfrac{1}{\lambda_\theta\lambda_z^2}+\tfrac{1}{\lambda_\theta^2\lambda_z}\right)\geq 0 \end{aligned} \tag{46}$$

and M^ϵ is a copositive matrix difined positive.

Proof
The relations(16) and (17) in (44) give

$$W=\frac{\mu}{2}\left(\frac{1}{\lambda_r^2}+\frac{1}{\lambda_\theta^2}+\frac{1}{\lambda_z^2}+2\lambda_r^2+\lambda_\theta^2+\lambda_z^2-5\right) \tag{47}$$

that allow us to find the first order derivatives of the energy function given by:

$$\begin{aligned} &G_r=\mu\left(\lambda_\theta\lambda_z-\tfrac{2}{\lambda_r^3}\right)\\ &G_\theta=\mu\left(\lambda_r\lambda_z-\frac{2}{\lambda_\theta^3}\right)\\ &G_z=\mu\left(\lambda_r\lambda_\theta-\tfrac{2}{\lambda_z^3}\right) \end{aligned} \tag{48}$$

The use of expressions of the system (48) in the proposition 3.2 allow us to have

$$\frac{\lambda_r G_r-\lambda_\theta G_\theta}{\lambda_r-\lambda_\theta} = \mu\left(\frac{1}{\lambda_r\lambda_\theta^2}+\frac{1}{\lambda_r^2\lambda_\theta}\right)\geq 0$$

$$\frac{\lambda_r G_r-\lambda_z G_z}{\lambda_r-\lambda_z} = \mu\left(\frac{1}{\lambda_r\lambda_z^2}+\frac{1}{\lambda_r^2\lambda_z}\right)\geq 0 \quad (49)$$

$$\frac{\lambda_\theta G_\theta-\lambda_z G_z}{\lambda_\theta-\lambda_z} = \mu\left(\frac{1}{\lambda_\theta\lambda_z^2}+\frac{1}{\lambda_\theta^2\lambda_z}\right)\geq 0$$

which gives us the sought inequalites which are

$$\mu\left(\frac{1}{\lambda_r\lambda_\theta^2}+\frac{1}{\lambda_r^2\lambda_\theta}\right)\geq 0$$

$$\mu\left(\frac{1}{\lambda_r\lambda_z^2}+\frac{1}{\lambda_r^2\lambda_z}\right)\geq 0$$

$$\mu\left(\frac{1}{\lambda_\theta\lambda_z^2}+\frac{1}{\lambda_\theta^2\lambda_z}\right)\geq 0$$

The second order derivatives of energy function from the system (48) become:

$$G_{rr}=\frac{3\mu}{\lambda_r^4}, G_{\theta\theta}=\frac{3\mu}{\lambda_\theta^4}, G_{zz}=\frac{3\mu}{\lambda_z^4}$$

$$G_{r\theta}=G_{\theta r}=\mu\lambda_z, G_{rz}=G_{zr}=\mu\lambda_\theta, \quad (50)$$

$$G_{\theta z}=G_{z\theta}=\mu\lambda_r.$$

To calculate the other components of the symmetric positive matrix M^ϵ, we consider the two following conditions
First we assume that $\varepsilon_1\varepsilon_2=1$, which give us

$$m_{r\theta}=\mu\left(\frac{1}{\lambda_r^3\lambda_\theta}+\frac{1}{\lambda_r^2\lambda_\theta^2}+\frac{1}{\lambda_r\lambda_\theta^3}\right)$$

$$m_{rz}=\mu\left(\frac{1}{\lambda_r^3\lambda_z}+\frac{1}{\lambda_r^2\lambda_z^2}+\frac{1}{\lambda_r\lambda_z^3}\right) \quad (51)$$

$$m_{\theta z}=\mu\left(\frac{1}{\lambda_\theta^3\lambda_z}+\frac{1}{\lambda_\theta^2\lambda_z^2}+\frac{1}{\lambda_\theta\lambda_z^3}\right)$$

In the second condition, we assume that $\varepsilon_1\varepsilon_2=-1$, which give us

$$m_{r\theta}=\mu\left(\frac{1}{\lambda_r^3\lambda_\theta}-\frac{1}{\lambda_r^2\lambda_\theta^2}+\frac{1}{\lambda_r\lambda_\theta^3}\right)$$

$$m_{rz}=\mu\left(\frac{1}{\lambda_r^3\lambda_z}-\frac{1}{\lambda_r^2\lambda_z^2}+\frac{1}{\lambda_r\lambda_z^3}\right) \quad (52)$$

$$m_{\theta z}=\mu\left(\frac{1}{\lambda_\theta^3\lambda_z}-\frac{1}{\lambda_\theta^2\lambda_z^2}+\frac{1}{\lambda_\theta\lambda_z^3}\right)$$

The last three systems give us the components of the matrix M^ϵ according to the case where the product $\varepsilon_1\varepsilon_2$ is equal to 1 or equal to -1.
The proposition 3.2 shows that M^ϵ is copositive and defined positive, that gives the complete proof of the Lemma 4.

3.3.4 Blatz-ko energy function

We propose to study the energy function of Blatz-ko model studied by Diouf in [15] and defined by:

$$W=\frac{\mu}{2}\left[I_1-3+\beta\left(I_3^{1/\beta}-1\right)\right] \tag{53}$$

Lemma 5

Let's consider a material point M with the coordonates (R,Θ,Z) in the undeformed configuration and (r,θ,z) in the deformed configuration whose movement is described by the following kinematic:

$$r=r\left(R,t\right),\theta=\Theta,z=\lambda Z+h\left(R,t\right). \tag{54}$$

With $\lambda_r\neq 0,\lambda_\theta\neq 0$ and the energy function of the relation (18), we have

$$\begin{aligned}\mu\left(\lambda_r+\lambda_\theta\right)&\geq 0\\ \mu\left(\lambda_r+\lambda_z\right)&\geq 0\\ \mu\left(\lambda_\theta+\lambda_z\right)&\geq 0\end{aligned} \tag{55}$$

Proof

The expressions (15) and (16) in the energy function defined in (53) yield

$$W=\frac{\mu}{2}\left[\lambda_r^2+\lambda_\theta^2+\lambda_z^2-3+\beta\left(\lambda_r^{2/\beta}\lambda_\theta^{2/\beta}\lambda_z^{2/\beta}-1\right)\right] \tag{56}$$

From (56), we calculate first order derivatives of this energy function that are:

$$\begin{aligned}G_r&=\mu\left[\lambda_r+\left(\lambda_r^{2-\beta}\lambda_\theta^2\lambda_z^2\right)^{1/\beta}\right]\\ G_\theta&=\mu\left[\lambda_\theta+\left(\lambda_r^2\lambda_\theta^{2-\beta}\lambda_z^2\right)^{1/\beta}\right]\\ G_z&=\mu\left[\lambda_z+\left(\lambda_r^2\lambda_\theta^2\lambda_z^{2-\beta}\right)^{1/\beta}\right]\end{aligned} \tag{57}$$

And with the proposition 3.2 we obtain:

$$\begin{aligned}\frac{\lambda_rG_r-\lambda_\theta G_\theta}{\lambda_r-\lambda_\theta}&=\mu\left(\lambda_r+\lambda_\theta\right)\geq 0\\ \frac{\lambda_rG_r-\lambda_zG_z}{\lambda_r-\lambda_z}&=\mu\left(\lambda_r+\lambda_z\right)\geq 0\\ \frac{\lambda_\theta G_\theta-\lambda_zG_z}{\lambda_\theta-\lambda_z}&=\mu\left(\lambda_\theta+\lambda_z\right)\geq 0\end{aligned} \tag{58}$$

That finally give:

$$\mu(\lambda_r + \lambda_\theta) \geq 0$$

$$\mu(\lambda_r + \lambda_z) \geq 0$$

$$\mu(\lambda_\theta + \lambda_z) \geq 0$$

That prove the Lemma 5.
The calculation of the second order derivatives also gives:

$$\begin{aligned} G_{rr} &= \mu\left[1 + \tfrac{2-\beta}{\beta}\left(\lambda_r^{1-\beta}\lambda_\theta\lambda_z\right)^{2/\beta}\right] \\ G_{\theta\theta} &= \mu\left[1 + \frac{2-\beta}{\beta}\left(\lambda_r\lambda_\theta^{1-\beta}\lambda_z\right)^{2/\beta}\right] \\ G_{zz} &= \mu\left[1 + \tfrac{2-\beta}{\beta}\left(\lambda_r\lambda_\theta\lambda_z^{1-\beta}\right)^{2/\beta}\right] \end{aligned} \quad (59)$$

and

$$\begin{aligned} G_{r\theta} &= G_{\theta r} = \tfrac{2\mu}{\beta}\left(\lambda_r^{2-\beta}\lambda_\theta^{2-\beta}\lambda_z^2\right)^{2/\beta} \\ G_{rz} &= G_{zr} = \frac{2\mu}{\beta}\left(\lambda_r^{2-\beta}\lambda_\theta^2\lambda_z^{2-\beta}\right)^{2/\beta} \\ G_{\theta z} &= G_{z\theta} = \tfrac{2\mu}{\beta}\left(\lambda_r^2\lambda_\theta^{2-\beta}\lambda_z^{2-\beta}\right)^{2/\beta} \end{aligned} \quad (60)$$

3.3.5 Case of Diouf-Zidi model

Now we study a Diouf-Zidi energy function given in [16] with the conditions $p = 2$ and $a_3 = 0$ so it becomes:

$$W = \frac{\mu}{2}\left[(I_1 - 3) + a_1(I_2 - 3) + a_2\left(\sqrt{I_3} - 1\right)^2\right] \quad (61)$$

Lemma 6
Let's consider a material point M with the coordonates (R, Θ, Z) in the undeformed configuration and (r, θ, z) in the deformed configuration whose movement is described by the following kinematic:

$$r = r(R,t), \theta = \Theta, z = \lambda Z + h(R,t). \quad (62)$$

With the energy function of the relation (24), we have

$$\begin{aligned} \mu(\lambda_r + \lambda_\theta)\left(1 + a_1\lambda_z^2\right) &\geq 0 \\ \mu(\lambda_r + \lambda_z)\left(1 + a_1\lambda_\theta^2\right) &\geq 0 \\ \mu(\lambda_\theta + \lambda_z)\left(1 + a_1\lambda_r^2\right) &\geq 0 \end{aligned} \quad (63)$$

Proof
Expressions (15), (16) and (17) in (61) give

$$W = \frac{\mu}{2}\left[\lambda_r^2 + \lambda_\theta^2 + \lambda_z^2 - 3 + a_1\left(\lambda_r^2\lambda_\theta^2 + \lambda_r^2\lambda_z^2 + \lambda_\theta^2\lambda_z^2\right) + a_2\left(\lambda_r\lambda_\theta\lambda_z - 1\right)^2\right] \tag{64}$$

The expression (64) allows us to find the first order derivatives of the energy which are:

$$\begin{aligned} G_r &= \mu\left[\lambda_r + a_1\left(\lambda_r\lambda_\theta^2 + \lambda_r\lambda_z^2\right) + a_2\lambda_\theta\lambda_z\left(\lambda_r\lambda_\theta\lambda_z - 1\right)\right] \\ G_\theta &= \mu\left[\lambda_\theta + a_1\left(\lambda_r^2\lambda_\theta + \lambda_\theta\lambda_z^2\right) + a_2\lambda_r\lambda_z\left(\lambda_r\lambda_\theta\lambda_z - 1\right)\right] \\ G_z &= \mu\left[\lambda_z + a_1\left(\lambda_r^2\lambda_z + \lambda_\theta^2\lambda_z\right) + a_2\lambda_r\lambda_\theta\left(\lambda_r\lambda_\theta\lambda_z - 1\right)\right] \end{aligned} \tag{65}$$

Using the system (65) in proposition 3.2, we have:

$$\begin{aligned} \tfrac{\lambda_r G_r - \lambda_\theta G_\theta}{\lambda_r - \lambda_\theta} &= \mu\left(\lambda_r + \lambda_\theta\right)\left(1 + a_1\lambda_z^2\right) \geq 0 \\ \frac{\lambda_r G_r - \lambda_z G_z}{\lambda_r - \lambda_z} &= \mu\left(\lambda_r + \lambda_z\right)\left(1 + a_1\lambda_\theta^2\right) \geq 0 \\ \tfrac{\lambda_\theta G_\theta - \lambda_z G_z}{\lambda_\theta - \lambda_z} &= \mu\left(\lambda_\theta + \lambda_z\right)\left(1 + a_1\lambda_r^2\right) \geq 0 \end{aligned} \tag{66}$$

and so we obtain

$$\mu\left(\lambda_r + \lambda_\theta\right)\left(1 + a_1\lambda_z^2\right) \geq 0$$

$$\mu\left(\lambda_r + \lambda_z\right)\left(1 + a_1\lambda_\theta^2\right) \geq 0$$

$$\mu\left(\lambda_\theta + \lambda_z\right)\left(1 + a_1\lambda_r^2\right) \geq 0$$

that end the proof of lemma 6.
From (65), we can have the second order derivatives of the energy which are:

$$\begin{aligned} G_{rr} &= \mu\left[1 + a_1\left(\lambda_\theta^2 + \lambda_z^2\right) + a_2\lambda_\theta^2\lambda_z^2\right] \\ G_{\theta\theta} &= \mu\left[1 + a_1\left(\lambda_r^2 + \lambda_z^2\right) + a_2\lambda_r^2\lambda_z^2\right] \\ G_{zz} &= \mu\left[1 + a_1\left(\lambda_r^2 + \lambda_\theta^2\right) + a_2\lambda_r^2\lambda_\theta^2\right] \end{aligned} \tag{67}$$

and

$$\begin{aligned} G_{r\theta} = G_{\theta r} &= \mu\left[2a_1\lambda_r\lambda_\theta + a_2\lambda_z\left(2\lambda_r\lambda_\theta\lambda_z - 1\right)\right] \\ G_{rz} = G_{zr} &= \mu\left[2a_1\lambda_r\lambda_z + a_2\lambda_\theta\left(2\lambda_r\lambda_\theta\lambda_z - 1\right)\right] \\ G_{\theta z} = G_{z\theta} &= \mu\left[2a_1\lambda_\theta\lambda_z + a_2\lambda_r\left(2\lambda_r\lambda_\theta\lambda_z - 1\right)\right] \end{aligned} \tag{68}$$

3.3.6 Remarks

We note in application that for each type of strain energy function characterizing the behavior of a specific material, the ellipticity conditions allow to have sets of inequalities that are in function of the eigenvalues and the material parameters of this considered material. Thus we observe that the ellipticity conditions obtained for the polynomial form are equivalent to those obtained for the the Diouf-Zidi model when we have $\alpha_1 = 1$ and $\alpha_2 = a_1$.
At the same time the ellipticity conditions of the exponential model are also equivalent to those of the Blatz-ko model for $\mu = 2\alpha\beta$.
This study makes it possible to highlight a very important result which shows that for a specified choice of material parameters, the ellipticity conditions can be identical for two energy functions of different types and which reflect the behavior of two different materials.

4 Conclusion

In this research work, we have proposed a study of the ellipticity of deformation energy functions in order to find inequalities defined through lemmas. Energy functions like polynomial type, exponential type, rational type and two cases of energy functions: the Blatz-ko model and the Diouf-Zidi model which are power types were used for the realization of this work.

This study allowed us to show that although there are various types of potentials modeling various materials, it is possible to obtain for a specific choice of material parameters between two different types of potentials, the same conditions of ellipticity. This proves that the criterion of ellipticity is not a particular character to each material.

Nevertheless, it should be noted that certain forms of energy functions, for example the exponatial and the rational case of energy functions have their special ellipticity conditions.

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