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Research Paper

Fixed point theorems for weakly contractive mappings in S-metric space

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Abstract *In this paper, we shall introduce the new notion of generalized $\psi_{\int \varphi}$ weak contraction and (ψ, ξ) weak contraction. Some fixed point theorems in S-metric space are also proved using these contractions.*
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I. Introduction

Fixed point theory is an important tool in analysis. It plays an important role in solving the problems based on computer optimization theory, boundary value problems, engineering science as well as medical science. The first result in this direction is given by Banach [1] in 1922, it is known as Banach's contraction mapping theorem, which states as

Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is said to be a contractive mapping, if
$d(Tx, Ty) \leq a\, d(x, y)$, for all $x, y \in X$, where $0 < a < 1$.

If (X, d) is complete then the contractive mapping T has a unique fixed point, this is known as Banach contraction mapping principle. It is regarded as one of the most important theorems in functional analysis. This theorem establish the existence of solutions for nonlinear equations and integral equations. Since then, because of simplicity and usefulness, it has become a very popular tool in solving a variety of problems such as control theory, economic theory, nonlinear analysis and global analysis. Many mathematics problems requires one to find a distance between two or more objects which is not easy to measure precisely in general. There exist different approaches to obtaining the appropriate concept of a metric structure.

Over last few decades, a number of generalizations of metric spaces have thus appeared in several papers, such as G-metric spaces, Partial metric spaces, D^*- metric spaces and cone metric spaces. These generalizations were then used to extend the scope of the study of fixed point theory. For more discussions of such generalizations, we refer to [2, 3, 4, 9, 10, 12, 13,15, 17, 18, 22, 23, 24, 25]. Sedghi et al [20] have introduced the notion of an S-metric space and proved that, this notion is a generalization of a G-metric and a D^*-metric space. Also, they have proved properties of S-metric spaces and some fixed point theorems for a self-map on an S-metric space.

In metric fixed point theory, the concept of altering distance function has been used by many authors in a number of works on fixed points. An altering distance function is actually a control function which alters the distance between two points in a metric space. This concept was introduced by Khan et al. in 1984 in [16] in which they addressed new category of metric fixed point problems by use of such functions. Altering distance function have been generalized to functions of two variables [5] and three variables [6] and have been used in fixed point theory. It have also been extended to fixed point problems of multivalued and fuzzy mappings [7]. It has also been extended to probabilistic fixed point theory ([8] and [11]). In the present paper we make another use of such concept in proving fixed point results in S-metric space.

Definition 1.1 [20]
Let X be a non-empty set, an S-metric on X is a function $d : X^3 \to [0,+\infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

1. $d(x, y, z) \geq 0$,
2. $d(x, y, z) = 0$ if and only if $x = y = z$,
3. $d(x, y, z) \leq d(x, x, a) + d(y, y, a) + d(z, z, a)$,
   for all $x, y, z, a \in X$.

Definition 1.2 [19]
Let (X, d) be an S-metric space.

(i) A sequence $\{x_n\} \subset X$ converges to $x \in X$ if $d(x_n, x_n, x) \to 0$ as $n \to +\infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $d(x_n, x_n, x) < \epsilon$.

(ii) A sequence $\{x_n\} \subset X$ is a Cauchy sequence if $d(x_n, x_n, x_m) \to 0$ as $n, m \to +\infty$. That is for each $\epsilon > 0$, there exists $n_0 \in N$ such that for all $n, m \geq n_0$ we have
$d(x_n, x_n, x_m) < \epsilon$.

(iii) S-metric space $(X, \mathcal{S})$ is complete if every Cauchy sequence is a convergent sequence.

Lemma 1.3 [21] Let (X, d) be an S-metric space. If sequence $\{x_n\}$ converges to x, then x is unique.

Lemma 1.4 [21] Let (X, d) be a S-metric space. If sequence $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.
In 2011 Cai *et al.* gave the following definition of $\psi_{\int \varphi}$ -weakly contractive mappings and obtained the common fixed point theorems in metric space as follows:

Definition 1.5 [26] Let (X, d) be a complete metric space and $P, Q : X \to X$ two self-mappings such that for all $x, y \in X$,

$$\psi\left(\int_0^{d(Px,Qy)} \varphi(t)dt\right) \leq \psi\left(\int_0^{M(x,y)} \varphi(t)dt\right) - \phi\left(\int_0^{M(x,y)} \varphi(t)dt\right), \qquad (1.5)$$

Where $M(x, y) = \max\{d(x, y), d(Px, x), d(Qy, y), \frac{d(y,Px)+d(x,Qy)}{2}\}$
and

(i) $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable, non-negative and such that
$\int_0^{\varepsilon} \varphi(t)dt > 0$, for each $\varepsilon > 0$.

(ii) $\psi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function and $\varphi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous and non-decreasing function such that $\psi(t) = 0 = \varphi(t)$ if and only if $t = 0$.

Then there exists a unique point $u \in X$ such that $u = Pu = Qu$.

II. Fixed Point Theorem for $\psi_{\int \varphi}$ –Weakly Contractive Mapping

In this section, we shall prove some fixed point theorems for generalized $\psi_{\int \varphi}$-weak contraction in S-metric space

Theorem 2.1. Let $(X, \mathcal{S})$ be a S-metric space and $P, Q : X \to X$ two self-mappings such that for all $x, y \in X$,

$$\psi\left(\int_0^{d(Px,Px,Qy)} \varphi(t)dt\right) \leq \psi\left(\int_0^{M(x,x,y)} \varphi(t)dt\right) - \phi\left(\int_0^{M(x,x,y)} \varphi(t)dt\right) \qquad (2.1)$$

Where $M(x, x, y) = \max\{d(x, x, y), d(Px, Px, x), d(Qy, Qy, y), \frac{d(y,y,Px)+d(x,x,Qy)}{2}\}$
and

(i) $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable, non-negative and such that
$\int_0^{\varepsilon} \varphi(t)dt > 0$, for each $\varepsilon > 0$.

(ii) $\psi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function and $\varphi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous and non-decreasing function such that $\psi(t) = 0 = \varphi(t)$ if and only if $t = 0$.

Then there exists a unique point $u \in X$ such that $u = Pu = Qu$.

Proof.
For any $x_0 \in X$, we construct the sequence $\{x_n\}, n \geq 0$ by
$x_{2n+1} = Qx_{2n},\ x_{2n+1} = Px_{2n+1}, n = 0, 1, 2, \ldots$
We divide the proof into four steps:

Step 1.
We prove that $\lim_{n\to\infty} \mathcal{S}(x_{n+1}, x_{n+1}, x_n) = 0$.

Suppose that n is an odd number. Taking $x = x_n$, $y = x_{n+1}$ in (2.1), we have

$$\psi\left(\int_0^{\mathcal{S}(x_{n+1},x_{n+1},x_n)} \varphi(t)dt\right) \leq \psi\left(\int_0^{M(x_n,x_n,x_{n-1})} \varphi(t)dt\right) - \phi\left(\int_0^{M(x_n,x_n,x_{n-1})} \varphi(t)dt\right)$$

$$\leq \psi\left(\int_0^{M(x_n,x_n,x_{n-1})} \varphi(t)dt\right).$$

Since ψ is a monotone non-decreasing function, so

$$\int_0^{\mathcal{S}(x_{n+1},x_{n+1},x_n),} \varphi(t)dt \leq \int_0^{M(x_n,x_n,x_{n-1})} \varphi(t)dt. \tag{2.2}$$

and

$$M(x_n,x_n,x_{n-1}) = max\{\mathcal{S}(x_n,x_n,x_{n-1}), \mathcal{S}(x_{n+1},x_{n+1},x_n), \mathcal{S}(x_n,x_n,x_{n-1}),$$

$$\frac{\mathcal{S}(x_{n-1},x_{n-1},x_{n-1}) + \mathcal{S}(x_n,x_n,x_n)}{2}\}$$

$$\leq \max\{\mathcal{S}(x_n,x_n,x_{n-1}), \mathcal{S}(x_{n+1},x_{n+1},x_n), \frac{\mathcal{S}(x_{n-1},x_{n-1},x_n) + \mathcal{S}(x_n,x_n,x_{n+1})}{2}\}$$

If $\mathcal{S}(x_{n+1},x_{n+1},x_n) > \mathcal{S}(x_n,x_n,x_{n-1})$,

Then, we can write

$M(x_n,x_n,x_{n-1}) = \mathcal{S}(x_{n+1},x_{n+1},x_n)$.

It follows from (2.1) that

$$\psi\left(\int_0^{\mathcal{S}(x_{n+1},x_{n+1},x_n),} \varphi(t)dt\right) \leq \psi\left(\int_0^{\mathcal{S}(x_{n+1},x_{n+1},x_n)} \varphi(t)dt\right) - \phi\left(\int_0^{\mathcal{S}(x_{n+1},x_{n+1},x_n),} \varphi(t)dt\right)$$

$$< \psi\left(\int_0^{\mathcal{S}(x_{n+1},x_{n+1},x_n)} \varphi(t)dt\right),$$

a contradiction.

Hence, $M(x_n,x_n,x_{n-1}) = \mathcal{S}(x_n,x_n,x_{n-1})$.

Now by (2.2), we have

$$\int_0^{\mathcal{S}(x_{n+1},x_{n+1},x_n)} \varphi(t)dt \leq \int_0^{M(x_n,x_n,x_{n-1})} \varphi(t)dt = \int_0^{\mathcal{S}(x_n,x_n,x_{n-1})} \varphi(t)dt.$$

Set $y_n = \int_0^{\mathcal{S}(x_{n+1},x_{n+1},x_n)} \varphi(t)dt$,

then $0 \leq y_n \leq y_{n-1}$ for all $n \geq 1$.

Therefore, the sequence $\{y_n\}, n \geq 0$ is monotone decreasing and has lower bound. So, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \int_0^{\mathcal{S}(x_{n+1},x_{n+1},x_n),} \varphi(t)dt$$
$$= \lim_{n\to\infty} \int_0^{M(x_n,x_n,x_{n-1})} \varphi(t)dt$$
$$= r.$$

By the lower semi-continuity of ϕ, we have

$\phi(r) \leq \lim_{n\to\infty} \inf \phi\left(\int_0^{M(x_n,x_n,x_{n-1})} \varphi(t)dt\right)$.

We show that $r = 0$.

In fact, taking $lim\ sup$ on both side of the following inequality

$\psi\left(\int_0^{\mathcal{S}(x_{n+1},x_{n+1},x_n)} \varphi(t)dt\right) \leq \psi\left(\int_0^{M(x_n,x_n,x_{n-1})} \varphi(t)dt\right) - \phi\left(\int_0^{M(x_n,x_n,x_{n-1})} \varphi(t)dt\right)$,
from here, we get

$\psi(r) \leq \psi(r) - \phi(r)$

which implies that $\phi(r) \leq 0$. Thus $\phi(r) = 0$.

By the property of the function ϕ, we find

$\lim_{n\to\infty} \int_0^{\mathcal{S}(x_{n+1},x_{n+1},x_n),} \varphi(t)dt = 0.$

We know that $\lim_{n\to\infty} \mathcal{S}(x_{n+1},x_{n+1},x_n) = 0.$ (2.3)

Step 2.
We show that $\{x_{n,}\}$ is a Cauchy sequence.
By step 1, we know $\lim_{n\to\infty} \mathcal{S}(x_{n+1},x_{n+1},x_n) = 0,$ so we will only prove that the subsequence $\{x_{2n}\}$ of $\{x_n\}$ is a Cauchy sequence. Suppose that the subsequence is not a Cauchy sequence so there exists an $\varepsilon > 0$ and subsequences $\{x_{2m(k)}\}$ and $\{x_{2n(k)}\}$ of $\{x_{2n}\}$
such that $n(k)$ is the smallest integer for which
$n(k) > m(k) > k,$
$\mathcal{S}\left(x_{2n(k)},x_{2n(k)},x_{2m(k)}\right) \geq \varepsilon,$
and
$\mathcal{S}(x_{2n(k)-2},x_{2n(k)-2},x_{2m(k)}) < \varepsilon.$

Then, we have

$$\begin{aligned}
\varepsilon &\leq \mathcal{S}(x_{2n(k)}, x_{2n(k)},x_{2m(k)}) \\
&\leq \mathcal{S}(x_{2n(k)},x_{2n(k)},x_{2n(k)-1}) + \mathcal{S}(x_{2n(k)},x_{2n(k)},x_{2n(k)-1}) + \mathcal{S}(x_{2n(k)-1}, x_{2n(k)-1},x_{2n(k)-2}) + \\
&\quad \mathcal{S}(x_{2n(k)-1}, x_{2n(k)-1},x_{2n(k)-2}) + \mathcal{S}(x_{2n(k)-2}, x_{2n(k)-2}, x_{2m(k)}) \\
&= 2\mathcal{S}(x_{2n(k)},x_{2n(k)},x_{2n(k)-1}) + 2\,\mathcal{S}(x_{2n(k)-1}, x_{2n(k)-1}, x_{2n(k)-2}) + \\
&\quad \mathcal{S}(x_{2n(k)-1}, x_{2n(k)-1},x_{2n(k)-2}) + \mathcal{S}(x_{2n(k)-2}, x_{2n(k)-2}, x_{2m(k)}) \\
&< \varepsilon + 2\,\mathcal{S}(x_{2n(k)}, x_{2n(k)}, x_{2n(k)-1}) + 2\mathcal{S}\,(x_{2n(k)-1}, x_{2n(k)-1}, x_{2n(k)-2}).
\end{aligned}$$

Thus, we obtained

$$\begin{aligned}
0 < \delta &= \int_0^{\varepsilon} \varphi(t)dt \\
&\leq \int_0^{\mathcal{S}(x_{2n(k)}, x_{2n(k)},x_{2m(k)})} \varphi(t)dt \\
&\leq \int_0^{\varepsilon+2\mathcal{S}(x_{2n(k)}, x_{2n(k)},x_{2n(k)-1})+2\mathcal{S}(x_{2n(k)-1}, x_{2n(k)-1},x_{2n(k)-2})} \varphi(t)dt.
\end{aligned}$$

Taking $\lim_{k\to\infty}$ and using (2.3), we obtain

$$\lim_{n\to\infty} \int_0^{\mathcal{S}(x_{2n(k)}, x_{2n(k)},x_{2m(k)})} \varphi(t)dt = \delta. \quad (2.4)$$

By the triangle inequality, we have

$$\mathcal{S}(x_{2m(k)}, x_{2m(k)},x_{2n(k)}) \leq \mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) + \mathcal{S}\left(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}\right) + \mathcal{S}\,(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)})$$

$$= 2\mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) + \mathcal{S}(x_{2m(k)-1}, x_{2m(k)-1}x_{2n(k)})$$

$$\mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2n(k)}) \le 2\mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) + \mathcal{S}(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}).$$

$$\mathcal{S}(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}) \le \mathcal{S}(x_{2n(k)-1}, x_{2n(k)-1}, x_{2m(k)}) + \mathcal{S}(x_{2n(k)-1}, x_{2n(k)-1}, x_{2m(k)}) + \mathcal{S}(x_{2n(k)}, x_{2n(k)}x_{2m(k)})$$

$$= 2\mathcal{S}(x_{2n(k)-1}, x_{2n(k)-1}, x_{2m(k)}) + \mathcal{S}(x_{2n(k)}, x_{2n(k)}x_{2m(k)})$$

$$\le 2\mathcal{S}(x_{2n(k)-1}, x_{2n(k)-1}, x_{2m(k)}) + \mathcal{S}(x_{2m(k)}, x_{2m(k)}x_{2n(k)}).$$

Therefore, we can write

$$\int_0^{\mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2n(k)})} \varphi(t)dt \le \int_0^{2\mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) + \mathcal{S}(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)})} \varphi(t)dt$$

$$\int_0^{\mathcal{S}(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)})} \varphi(t)dt \le \int_0^{2\mathcal{S}(x_{2n(k)-1}, x_{2n(k)-1}, x_{2m(k)}) + \mathcal{S}(x_{2m(k)}, x_{2m(k)}x_{2n(k)})} \varphi(t)dt.$$

Taking $k \to \infty$ in the above two inequalities, and using (2.3), (2.4), we get

$$\lim_{n\to\infty} \int_0^{\mathcal{S}(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)})} \varphi(t)dt = \delta. \tag{2.5}$$

Again by triangle inequality, we have

$$\mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1}) \le \mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2n(k)}) + \mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2n(k)}) \mathcal{S}(x_{2n(k)+1}, x_{2n(k)+1}, x_{2n(k)})$$

$$= 2\mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2n(k)}) + \mathcal{S}(x_{2n(k)+1}, x_{2n(k)+1}, x_{2n(k)})$$

$$= 2\mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2n(k)}) + \mathcal{S}(x_{2n(k)}, x_{2n(k)}, x_{2n(k)+1}).$$

$$\mathcal{S}\left(x_{2m(k)}, x_{2m(k)}, x_{2n(k)}\right) \le \mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1}) + \mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1}) \mathcal{S}(x_{2n(k)}, x_{2n(k)}, x_{2n(k)+1})$$

$$= 2\mathcal{S}\left(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1}\right) + \mathcal{S}(x_{2n(k)}, x_{2n(k)}, x_{2n(k)+1})$$

$$\le 2\mathcal{S}\left(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1}\right) + \mathcal{S}\left(x_{2n(k)+1}, x_{2n(k)+1}, x_{2n(k)}\right).$$

and

$$\int_0^{\mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1})} \varphi(t)dt \le \int_0^{2\mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2n(k)}) + \mathcal{S}(x_{2n(k)}, x_{2n(k)}, x_{2n(k)+1})} \varphi(t)dt,$$

$$\int_0^{\mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2n(k)})} \varphi(t)dt \le \int_0^{2\mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1}) + \mathcal{S}(x_{2n(k)+1}, x_{2n(k)+1}, x_{2n(k)})} \varphi(t)dt.$$

Let $k \to \infty$ in the both sides of the above inequalities and notice (2.3), (2.4), we have

$$\lim_{n\to\infty} \int_0^{\mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1})} \varphi(t)dt = \delta. \tag{2.6}$$

Moreover, we have

$$\mathcal{S}(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)+1}) \le \mathcal{S}(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}) + \mathcal{S}(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}) + \mathcal{S}(x_{2n(k)+1}, x_{2n(k)+1}, x_{2n(k)})$$

$$= 2\mathcal{S}(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}) + \mathcal{S}(x_{2n(k)+1}, x_{2n(k)+1}, x_{2n(k)})$$

$$\le 2\mathcal{S}(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}) + \mathcal{S}(x_{2n(k)}, x_{2n(k)}, x_{2n(k)+1}).$$

$$\mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1}) \le \mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) + \mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) + \mathcal{S}(x_{2n(k)+1}, x_{2n(k)+1}, x_{2m(k)-1})$$

$$= 2\mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) + \mathcal{S}(x_{2n(k)+1}, x_{2n(k)+1}, x_{2m(k)-1})$$

$$\le 2\mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) + \mathcal{S}(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)+1}).$$

and $\int_0^{\mathcal{S}(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)+1})} \varphi(t)dt \le \int_0^{2\mathcal{S}(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}) + \mathcal{S}(x_{2n(k)}, x_{2n(k)}, x_{2n(k)+1})} \varphi(t)dt,$

$$\int_0^{\mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1})} \varphi(t)dt \le \int_0^{2\mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2n(k)-1}) + \mathcal{S}(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)+1})} \varphi(t)dt .$$

Again let $k \to \infty$ and by (2.3), (2.5) and (2.6), we get

$$\lim_{n\to\infty} \int_0^{\mathcal{S}(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)+1})} \varphi(t)dt = \delta. \qquad (2.7)$$

From the definition of $M(x, x, y)$, we get

$$M(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}) = max \begin{pmatrix} \mathcal{S}(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}), \mathcal{S}(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}), \mathcal{S}(x_{2n(k)+1}, x_{2n(k)+1}, x_{2n(k)}), \\ \dfrac{\mathcal{S}(x_{2n(k)}, x_{2n(k)}, x_{2m(k)}) + \mathcal{S}(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)+1})}{2} \end{pmatrix}.$$

By (2.4), (2.5) and (2.7), we get

$$\lim_{n\to\infty} \int_0^{M(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)})} \varphi(t)dt = \delta. \qquad (2.8)$$

Taking $x = x_{2m(k)-1},\ y = x_{2n(k)}$ in (2.2), we have

$$\psi\left(\int_0^{\mathcal{S}(x_{2m(k)}, x_{2n(k)}, x_{2n(k)+1})} \varphi(t)dt\right) \le \psi\left(\int_0^{M(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)})} \varphi(t)dt\right) - \phi\left(\int_0^{M(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)})} \varphi(t)dt\right).$$

Letting k $\to \infty$, and by virtue of (2.6), (2.8) and also by using the property of ψ, ϕ, we get

$\psi(\delta) \le \psi(\delta) - \phi(\delta),$

a contradiction as $\delta > 0$.

Then $\{x_n\}$ is a Cauchy sequence. Since X is a $\mathcal{S}$-metric space, there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$.

Step 3.

We prove that v is a common fixed point of P and Q.

We know that $M(v, v, x_{2n}) = \mathcal{S}(v, v, Pv)$.

Putting v and x and x_{2n} for y in (2.1), we have

$$\psi\left(\int_0^{\mathcal{S}(Pv, Pv, x_{2n+1})} \varphi(t)dt\right) \le \psi\left(\int_0^{M(v,v,x_{2n})} \varphi(t)dt\right) - \phi\left(\int_0^{M(v,v,x_{2n})} \varphi(t)dt\right).$$

Letting $n \to \infty$, we obtain

$$\psi\left(\int_0^{\mathcal{S}(Pv,Pv,v)} \varphi(t)dt\right) \le \psi\left(\int_0^{\mathcal{S}(Pv,Pv,v)} \varphi(t)dt\right) - \phi\left(\int_0^{\mathcal{S}(Pv,Pv,v)} \varphi(t)dt\right).$$

This is a contradiction unless $\int_0^{\mathcal{S}} \varphi(t)dt = 0$.

Notice the condition of φ, we know that $\mathcal{S}(Pv, Pv, v) = 0$,

That is, $Pv = v$.

Since $\mathcal{S}(v, v, Qv) = \mathcal{S}(Pv, Pv, Qv)$, then we have

$$\psi\left(\int_0^{\mathcal{S}(v,v,Qv)} \varphi(t)dt\right) = \psi\left(\int_0^{\mathcal{S}(Pv,Pv,Qv)} \varphi(t)dt\right)$$

$$\leq \psi\left(\int_0^{M(v,v,v)} \varphi(t)dt\right) - \phi\left(\int_0^{M(v,v,v)} \varphi(t)dt\right)$$

$$= \psi\left(\int_0^{\mathcal{S}(v,v,Qv)} \varphi(t)dt\right) - \phi\left(\int_0^{\mathcal{S}(v,v,Qv)} \varphi(t)dt\right).$$

Hence, $\mathcal{S}(v, v, Qv) = 0$,

That is $v = Pv = Qv$.

Step 4.

We prove that v is a unique fixed point of Q and P.

Suppose that there exists another $v' \epsilon X$ such that $v' = Pv' = Qv'$

Then, we can write

$$\psi\left(\int_0^{\mathcal{S}(v,v,v')} \varphi(t)dt\right) = \psi\left(\int_0^{\mathcal{S}(Pv,Pv,Qv')} \varphi(t)dt\right)$$

$$\leq \psi\left(\int_0^{M(v,v,v')} \varphi(t)dt\right) - \phi\left(\int_0^{M(v,v,v')} \varphi(t)dt\right)$$

$$= \psi\left(\int_0^{\mathcal{S}(v,v,v')} \varphi(t)dt\right) - \phi\left(\int_0^{\mathcal{S}(v,v,v')} \varphi(t)dt\right).$$

This is a contradiction unless $\int_0^{\mathcal{S}(v,v,v')} \varphi(t)dt = 0$, and notice the condition of ψ, we have $\mathcal{S}(v, v, v') = 0$.

Hence $v = v'$.

This completes the proof of Theorem.

Corollary 2.2. Let $(X, \mathcal{S})$ be a S-metric space and $P, Q : X \to X$ two self-mappings such that for all $x, y \in X$,

$$\int_0^{d(Px,Px,Qy)} \varphi(t)dt \leq \int_0^{M(x,x,y)} \varphi(t)dt - \phi\left(\int_0^{M(x,x,y)} \varphi(t)dt\right) \qquad (2.9)$$

Where $M(x, x, y) = \max\{d(x, x, y), d(Px, Px, x), d(Qy, Qy, y), \frac{d(y,y,Px)+d(x,x,Qy)}{2}\}$

and

$\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable, non-negative and such that

$\int_0^{\varepsilon} \varphi(t)dt > 0$, for each $\varepsilon > 0$.

Then there exists a unique point $u \in X$ such that $u = Pu = Qu$.

Proof. By taking $\psi(t) = t$ in Theorem 2.1, one can obtain the proof easily.

Corollary 2.3. Let $(X, \mathcal{S})$ be a S-metric space and $P, Q : X \to X$ two self-mappings such that for all $x, y \in X$,

$$\psi\left(\int_0^{d(Px,Px,Qy)} \varphi(t)dt\right) \leq \psi\left(\int_0^{d(x,x,y)} \varphi(t)dt\right) - \phi\left(\int_0^{d(x,x,y)} \varphi(t)dt\right) \qquad (2.10)$$

and

(i) $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable,

non-negative and such that
$\int_0^\varepsilon \varphi(t)dt > 0$, for each $\varepsilon > 0$.
(ii) $\psi : [0,\infty) \to [0,\infty)$ is a continuous and non-decreasing function and $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous and non-decreasing function such that $\psi(t) = 0 = \varphi(t)$ if and only if $t = 0$.
Then there exists a unique point $u \in X$ such that $u = Pu = Qu$.
Proof. By taking $M(t) = d(t)$ in Theorem 2.1, one can obtain the proof easily.

Definition 2.4 [14] Let (X,d) be a complete b-metric space with parameter $s \geq 1$, and $T : X \to X$ be a self-mapping satisfying the (ψ, ξ)-weakly contractive condition

$$\psi(sd(Tx,Ty)) \leq \psi\left(\frac{d(x,y)}{s^2}\right) - \xi(d(x,y)) \quad (2.11)$$

for all $x, y \in X$, where

$\psi \in \omega = \{\psi : [0,\infty) \to [0,\infty), \psi$ is an altering distance function$\}$,

and

$$\xi \in \Upsilon = \begin{Bmatrix} \xi : [0,\infty) \to [0,\infty), \xi \text{ is continuous }, \xi(t) = 0 \text{ if an only if } t = 0, \\ \xi(\lim_{n\to\infty} inf\ c_n) \leq \lim_{n\to\infty} inf\ \xi(c_n) \end{Bmatrix}$$

Then P has a unique fixed point.

III. Fixed point theorem for (ψ, ξ)-weakly contractive mapping

In this section, we generalize the (ψ, ξ)-weakly contractive mapping in S-metric space.
Theorem 3.1 Let $(X, \mathcal{S})$ be a S-metric space and $P : X \to X$ be a self-mapping satisfying the (ψ, ξ)-weakly contractive condition

$$\psi(\mathcal{S}(Px,Px,Py)) \leq \psi(\mathcal{S}(x,x,y)) - \xi(\mathcal{S}(x,x,y)), \quad \text{for all } x, y \in X, \quad (3.1)$$

$\psi \in \omega = \{\psi : [0,\infty) \to [0,\infty), \psi$ is an altering distance function$\}$,

and

$$\xi \in \Upsilon = \begin{Bmatrix} \xi : [0,\infty) \to [0,\infty), \xi \text{ is continuous }, \xi(t) = 0 \text{ if an only if } t = 0, \\ \xi(\lim_{n\to\infty} inf\ c_n) \leq \lim_{n\to\infty} inf\ \xi(c_n) \end{Bmatrix}$$

Then P has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Consider the iterated sequence $\{x_n\}$, where $x_{n+1} = Tx_n$

for $n = 0, 1, 2, 3, ...$We will prove that $\mathcal{S}(x_n, x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Using (3.1), we can write

$$\psi(\mathcal{S}(x_n, x_n, x_{n+1})) \leq \psi(\mathcal{S}(x_{n-1}, x_{n-1}, x_n) - \xi(\mathcal{S}(x_{n-1}, x_{n-1}, x_n)), \quad (3.2)$$

$n = 0, 1, 2, 3 ...$

Therefore,

$$\psi(\mathcal{S}(x_n, x_n, x_{n+1}) \leq \psi(\mathcal{S}(x_{n-1}, x_{n-1}, x_n), \quad n = 0, 1, 2, 3 ...$$

ψ is strictly increasing, we have

$\mathcal{S}(x_n, x_n, x_{n+1}) \leq (\mathcal{S}(x_{n-1},\ x_{n-1},\ x_n)$, $\qquad n = 0, 1, 2, 3$ ...

Therefore, $\{\mathcal{S}(x_n, x_n, x_{n+1})\}$ is a non-increasing sequence and hence it is convergent. Let $\mathcal{S}(x_n, x_n, x_{n+1}) \to g$, where $g \geq 0$.

Letting $n \to \infty$ in (3.2) and using the continuity of ξ and ψ, we obtain

$\psi(g) \leq \psi(g) - \xi(g)$.

This implies,

$g = 0$, that is

$$\mathcal{S}(x_n, x_n, x_{n+1}) \to 0 \qquad (3.3)$$

We claim that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$ and

$$\mathcal{S}(x_{m(k)}, x_{m(k)}, x_{n(k)}) \geq \varepsilon, \qquad (3.4)$$

and

$$\mathcal{S}(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) \leq \varepsilon. \qquad (3.5)$$

Using (3.4) and (3.5), we obtain

$\varepsilon \leq \mathcal{S}(x_{m(k)}, x_{m(k)}, x_{n(k)})$

$\leq [\mathcal{S}(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) + \mathcal{S}\big(x_{m(k)}, x_{m(k)}, x_{n(k)-1}\big) + \mathcal{S}(x_{n(k)-1}, x_{n(k)-1}, x_{n(k)})]$

$= 2\mathcal{S}(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) + \mathcal{S}(x_{n(k)-1}, x_{n(k)-1}, x_{n(k)})$

$\leq 2(\varepsilon) + \mathcal{S}\big(x_{n(k)-1}, x_{n(k)-1}, x_{n(k)}\big)$.

for all $k \geq 1$.

Therefore

$$\varepsilon \leq \lim_{k\to\infty} sup\, \mathcal{S}(x_{m(k)}, x_{m(k)}, x_{n(k)}) \leq 2\varepsilon, \qquad (3.6)$$

Moreover, for all $k \geq 1$, we have

$\varepsilon \leq \mathcal{S}(x_{m(k)}, x_{m(k)}, x_{n(k)})$

$\leq [\mathcal{S}(x_{m(k)}, x_{m(k)}, x_{m(k)+1}) + \mathcal{S}(x_{m(k)}, x_{m(k)}, x_{m(k)+1}) + \mathcal{S}(x_{n(k)}, x_{n(k)}, x_{m(k)+1})]$

$= 2\mathcal{S}(x_{m(k)}, x_{m(k)}, x_{m(k)+1}) + \mathcal{S}(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)})$

$\leq 2\mathcal{S}\big(x_{m(k)}, x_{m(k)}, x_{m(k)+1}\big) + [\mathcal{S}(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)+1}) + \mathcal{S}\big(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)+1}\big) + \mathcal{S}(x_{n(k)}, x_{n(k)}, x_{n(k)+1})]$

$= 2\mathcal{S}\big(x_{m(k)}, x_{m(k)}, x_{m(k)+1}\big) + [2\mathcal{S}(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)+1}) + \mathcal{S}(x_{n(k)+1}, x_{n(k)+1}, x_{n(k)})]$

$\leq 2\mathcal{S}\big(x_{m(k)}, x_{m(k)}, x_{m(k)+1}\big) + 2\mathcal{S}(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)+1}) + \mathcal{S}(x_{n(k)+1}, x_{n(k)+1}, x_{n(k)})]$.

Using (3.3), we obtain

$$\varepsilon \leq \lim_{k\to\infty} sup\ 2\, \mathcal{S}(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)+1}). \tag{3.7}$$

Also letting $k \to \infty$ and using (3.4) for all $k \geq 1$, we get

$$\varepsilon \leq \lim_{k\to\infty} inf\ \mathcal{S}(x_{m(k)}, x_{m(k)}, x_{n(k)}). \tag{3.8}$$

Using (3.1) and (3.7), we have

$$\psi(\varepsilon) \leq \psi\left(\lim_{k\to\infty} sup\ 2\, \mathcal{S}(x_{m(k)+1}, x_{m(k)+1}, x_{n(k)+1})\right) \tag{3.9}$$

$$= \psi\left(\lim_{k\to\infty} sup\ 2\, \mathcal{S}(Px_{m(k)}, P\, x_{m(k)}, Px_{n(k)})\right)$$

$$\leq \lim_{k\to\infty} sup\left(\psi\big(\mathcal{S}(x_{m(k)}, x_{m(k)}, x_{n(k)})\big)\right) - \xi\big(\mathcal{S}(x_{m(k)}, x_{m(k)}, x_{n(k)})\big)$$

$$\leq \lim_{k\to\infty} sup\left(\psi\left(\mathcal{S}\big(x_{m(k)}, x_{m(k)}, x_{n(k)}\big)\right)\right) - \lim_{k\to\infty} inf\ \xi\left(\mathcal{S}\big(x_{m(k)}, x_{m(k)}, x_{n(k)}\big)\right).$$

Using (3.6) and also using property of ξ,

Then, we obtained

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \xi\left(\lim_{k\to\infty} inf\ \mathcal{S}(x_{m(k)}, x_{m(k)}, x_{n(k)})\right).$$

Hence, we have

$$\xi\left(\lim_{k\to\infty} inf\ \mathcal{S}(x_{m(k)}, x_{m(k)}, x_{n(k)})\right) = 0.$$

Since by property of ξ, we get

$$\lim_{k\to\infty} inf\ \mathcal{S}\big(x_{m(k)}, x_{m(k)}, x_{n(k)}\big) = 0,$$

which contradict (3.8).

Hence $\{x_n\}$ is a Cauchy sequence. The completeness of X implies that there exists $x^* \in X$ such that $x_n \to x^*$.

Using (3.1), we have

$$\psi(\mathcal{S}(Px_n, Px_n, Px^*) \leq \psi\big(\mathcal{S}(x_n, x_n, x^*)\big) - \xi(\mathcal{S}(x_n, x_n, x^*))$$

$$\leq \psi\big(\mathcal{S}(x_n, x_n, x^*)\big), \quad n = 0,1,2,3 \ldots$$

Since ψ is strictly increasing, we have

$$\mathcal{S}(Px_n, Px_n, Px^*) \leq \big(\mathcal{S}(x_n, x_n, x^*)\big), \qquad n = 0,1,2,3 \ldots$$

when limit $n \to \infty$, we get

$$Px_n \to Px^*.$$

Now we have

$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} P x_n = Px^*,$ (3.10)

That is, x^* is a fixed point of P.

Now to see the uniqueness of the fixed point x^*,

Assume to the contrary that $Py^* = y^*$ and $x^* \neq y^*$.

From (3.2), we have

$$\psi(\mathcal{S}(Px^*, Px^*, Py^*)) \leq \psi\big(\mathcal{S}(x^*, x^*, y^*)\big) - \xi(\mathcal{S}(x^*, x^*, y^*)).$$

Then, we can write

$$\psi\big(\mathcal{S}(x^*, x^*, y^*)\big) \leq \psi\big(\mathcal{S}(x^*, x^*, y^*)\big) - \xi(\mathcal{S}(x^*, x^*, y^*)). \quad (3.11)$$

Hence $\xi(\mathcal{S}(x^*, x^*, y^*)) = 0$

which implies that $x^* = y^*$.

This completes the proof of theorem.

Corollory 3.2 Let $(X, \mathcal{S})$ be a S-metric space and $P : X \to X$ be a self-mapping satisfying the (ψ, ξ)-weakly contractive condition and taking $\alpha \in [0, 1)$ satisfying
$\mathcal{S}(Px, Px, Py)) \leq \alpha\mathcal{S}(x, x, y)$, for all $x, y \in X$. Then P has a unique fixed point.

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