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Research Paper

A Study on Fixed point theorems in metric-like spaces

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Abstract *In this paper, we introduce a generalized weak contraction and prove a fixed point theorem for above mentioned contraction in metric like space. Some corollaries are also provided, which can be easily deduced from our result. An example is also provided in the support of our main result.*
Keywords *Weakly contractive mapping, fixed point, metric like spaces.*
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I. Introduction

Fixed point theory is a branch of non-linear analysis which deals with finding the solutions of many problems of natural and social science by using the concept of fixed points. The notion of metric space is first introduced by French mathematician Maurice Frechet [7] in 1906. After that a lot of generalizations is done in metric space to reduce or modify the metric axioms. In the theory of metric space, Banach's Contraction Principle [3] is one of the most important theorem and a powerful tool. A mapping $P : X \to X$, where (X, g) is a metric space, is called a contraction mapping if there exists $\sigma < 1$ such that for all $u, v \in X$, $\mathcal{S}(Px, Py) \leq \sigma(x, y)$. If the metric space (X, g) is complete, then P has a unique fixed point.

Banach established how to find the desired fixed point by offering a smart and plain technique. This theorem has many applications, including establishing the existence and uniqueness of solutions of certain ordinary or partial differential equations, and providing a different proof of the implicit function theorem. These methods leads to increasing the possibility of solving various problems in different research fields is increased by these plain technique. In many abstract spaces for distinct operators, this result has been generalized. Contraction mappings are continuous. Contraction mappings have been extended or generalized in several directions by various authors (see[4], [11-14], [16]). Not only contractive mappings but also the concept of metric space is also extended in many ways in the literature (see[2], [10], [5-6]).

In 1996, Jungck [9] introduced the concept of weak compatibility. Since then, many interesting fixed point theorems of compatible and weakly compatible maps under various contractive conditions have been obtained by a number of authors. The study of common fixed points of mappings in dislocated metric space satisfying certain contractive conditions has been at the center of research activity. Dislocated metric space plays very important role in topology, logical programming and in electronics engineering.

The purpose of this paper is to introduce a generalization of Banach contraction mapping principle which includes the generalization of weakly contractive mapping. To prove the main results, we require some pre-requisite from literature as follows:

Definition 1.1 [1] Let X be a non-empty set and $g : X \times X \to [0, \infty)$ be a function satisfying the following conditions:

1. $g(x, y) = 0 \Rightarrow x = y$;
2. $g(x, y) = g(y, x)$;
3. $g(x, z) \leq g(x, y) + g(y, z)$.

for all $x, y, z \in X$.

Then g is called metric-like (dislocated metric) and (X, g) is called metric-like (dislocated metric) space.

Definition 1.2 [1] Let (X, g) be a metric like space.

1. A sequence $\{x_n\}$ in X is a Cauchy sequence if $\lim_{n,m\to\infty} g(x_n, y_m)$ exists and is finite.
2. (X, g) is complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, that is, $\lim_{n\to\infty} g(x, x_n) = g(x, x) = \lim_{n,m\to\infty} g(x_n, x_m)$.
3. A mapping $P: (X, g) \to (X, g)$ is continuous if for any sequence x_n in X such that $g(x_n, x) \to g(x, x)$ as $n \to \infty$, we have $g(Px_n, Px) \to g(Px, Px)$ as $n \to \infty$.

Lemma 1.3 [15] Let (X, g) be a metric-like space. Let $\{x_n\}$ be a sequence in X such that $\{x_n\} \to x$ where $x \in X$ and $g(x, x) = 0$. Then for all $y \in X$, we have
$\lim_{n\to\infty} g(x_n, y) = g(x, y)$.

Definition 1.4 [15] Let $P : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that P is an α-admissible mapping if $\alpha(x, y) \geq 1$ implies $\alpha(Px, Py) \geq 1$, for all $x, y \in X$.

Definition 1.5 [8**]** Let (X, g) be a metric-like space. Then the following statements hold in metric-like space:

i. In a Cauchy seuence, every subsequence will also be a Cauchy sequence.
ii. If a Cauchy sequence has a convergent subsequence, then that Cauchy sequence will be convergent.
iii. Convergent sequence has a unique limits.
iv. A metric-like g is continuous, that is for $\{u_n\}$ converges to u and $\{v_n\}$ converges to v imply that $g(u_n, v_n) \to g(u, v)$ as $n \to \infty$.

II. Main results

In this section, we shall prove the fixed point theorems.

Theorem 2.1 Let (X, g) be a metric-like space and $P : X \to X$ be a self-mapping satisfying the inequality
$\xi(g(Pu, Pv)) \leq \xi(g(u, v)) - \delta(g(u, v)) - \omega(g(Pu, u))$, (2.1)
where $\xi, \delta, \omega : [0, \infty) \to [0, \infty)$, are all continuous and monotone non-decreasing functions with $\xi(q) = 0 = \delta(q) = \omega(q)$ if and only if $q = 0$.
Then P has a unique fixed point.

Proof. For any $u_0 \in X$, we construct the sequence $\{u_n\}$ by $u_n = Pu_{n-1}$, $n = 1, 2, 3 \ldots \ldots$
Substituting $u = u_{n-1}$ and $v = u_n$ in (2.1), we get
$\xi(g(Pu_{n-1}, Pu_n)) \leq \xi(g(u_{n-1}, u_n)) - \delta(g(u_{n-1}, u_n)) - \omega(g(Pu_{n-1}, u_n))$,
$\xi(g(u_n, u_{n+1})) \leq \xi(g(u_{n-1}, u_n)) - \delta(g(u_{n-1}, u_n)) - \omega(g(u_n, u_{n-1}))$, (2.2)
which implies that
$g(u_n, u_{n+1}) \leq g(u_{n-1}, u_n)$ (by using monotone property of ξ-function). (2.3)
It follows that the sequence $\{g(u_n, u_{n+1})\}$ is monotone decreasing and consequently there exists $t \geq 0$ such that
$g(u_n, u_{n+1}) \to t$ as $n \to \infty$. (2.4)
Letting $n \to \infty$ in (2.2), we can write
$\xi(t) \leq \xi(t) - \delta(t) - \omega(t)$ (2.5)
which is a contradiction unless $t = 0$.
Hence
$g(u_n, u_{n+1}) \to 0$ as $n \to \infty$. (2.6)
Now we prove that $\{u_n\}$ is a Cauchy sequence. If possible, let $\{u_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ and sequences $\{u_{m(k)}\}$ and $\{u_{n(k)}\}$ such that for all positive integer k, we have
$n(k) > m(k) > k$,
such that
$g(u_{m(k)}, u_{n(k)}) \geq \epsilon$. (2.7)
And $g(u_{m(k)}, u_{n(k)-1}) \leq \epsilon$. (2.8)
By triangle inequality, we get
$\epsilon \leq g(u_{m(k)}, u_{n(k)}) \leq g(u_{m(k)}, u_{n(k)-1}) + g(u_{n(k)-1}, u_{n(k)})$
$< \epsilon + g(u_{n(k)-1}, u_{n(k)})$, (2.9)
Taking $k \to \infty$ in the above inequality and using equation (2.6), we get
$\lim_{n\to\infty} g(u_{m(k)}, u_{n(k)}) = \epsilon$. (2.10)
Again,
$g(u_{n(k)}, u_{m(k)}) \leq g(u_{n(k)}, u_{n(k)-1}) + g(u_{n(k)-1}, u_{m(k)-1}) + g(u_{m(k)-1}, u_{m(k)})$,
$g(u_{n(k)-1}, u_{m(k)-1}) \leq g(u_{n(k)-1}, u_{n(k)}) + g(u_{n(k)}, u_{m(k)}) + g(u_{m(k)}, u_{m(k)-1})$,
(2.11)
Letting $k \to \infty$ in the above two inequality and using equation (2.6) and (2.10), we get
$\lim_{k\to\infty} g(u_{n(k)-1}, u_{m(k)-1}) = \epsilon$. (2.12)

On putting $u = u_{m(k)-1}$ and $v = u_{n(k)-1}$ in (2.1), we can write

$$\xi(g(Pu_{m(k)-1}, Pu_{n(k)-1})) \leq \xi\left(g(u_{m(k)-1}, u_{n(k)-1})\right) - \delta\left(g(u_{m(k)-1}, u_{n(k)-1})\right) - \omega(g(Pu_{m(k)-1}, u_{m(k)-1})),$$

$$\xi(g(u_{m(k)}, u_{n(k)})) \leq \xi\left(g(u_{m(k)-1}, u_{n(k)-1})\right) - \delta\left(g(u_{m(k)-1}, u_{n(k)-1})\right) - \omega(g(u_{m(k)}, u_{m(k)-1})),$$

Now using (2.7) in above inequality, we obtained

$$\xi(\epsilon) \leq \xi(g(u_{m(k)}, u_{n(k)})) \leq \xi\left(g(u_{m(k)-1}, u_{n(k)-1})\right) - \delta\left(g(u_{m(k)-1}, u_{n(k)-1})\right) - \omega(g(u_{m(k)}, u_{m(k)-1})), \quad (2.13)$$

Letting $k \to \infty$, and utilizing (2.6), (2.10) and (2.12), we get

$$\xi(\epsilon) \leq \xi(\epsilon) - \delta(\epsilon) \quad (2.14)$$

which is a contradiction if $\epsilon > 0$.

This shows that $\{u_n\}$ is a Cauchy sequence. Since X is a complete metric-like space, then there exists $h \in X$ such that

$$\lim_{n\to\infty} g(u_m, h) = g(h, h) = \lim_{n,m\to\infty} g(u_m, u_n) = 0. \quad (2.15)$$

Since P is continuous, from equation (2.15), we get

$$\lim_{n\to\infty} g(u_{m+1}, Ph) = \lim_{n\to\infty} g(Pu_m, Ph) = g(Ph, Ph). \quad (2.16)$$

Using Lemma 1.3 and equation (2.15), we obtain

$$\lim_{n\to\infty} g(u_{m+1}, Ph) = g(h, Ph). \quad (2.17)$$

Comparing equation (2.16) and (2.17), we get

$g(h, Ph) = g(Ph, Ph)$.

By using $u = u_{n-1}$ and $v = h$ in (2.1), we get

$$\xi(g(u_n, Ph)) \leq \xi\left(g(u_{m-1}, h)\right) - \delta\left(g(u_{m-1}, h)\right) - \omega(g(u_m, u_{m-1})),$$

as $n \to \infty$

$$\xi(g(h, Ph)) \leq \xi(0) - \delta(0) - \omega(0) = 0,$$

This implies that $g(h, Ph) = 0$

we get $g(h, Ph) = 0$, that is, $Ph = h$. (2.18)

To prove uniqueness of the fixed point, let us suppose that h_1 and h_2 are two fixed points of P.

Putting $u = h_1$ and $v = h_2$ in (2.1), we can write

$$\xi(g(Ph_1, Ph_2)) \leq \xi\left(g(h_1, h_2)\right) - \delta\left(g(h_1, h_2)\right) - \omega(g(Ph_1, h_1)),$$

$$\xi(g(h_1, h_2)) \leq \xi\left(g(h_1, h_2)\right) - \delta\left(g(h_1, h_2)\right) - \omega(g(h_1, h_1)), \quad (2.19)$$

or

$\delta\left(g(h_1, h_2)\right) \leq 0$,

or equivalently $g(h_1, h_2) = 0$,

That is $h_1 = h_2$.

This proves the uniqueness of the fixed point.

Example 2.2 Let $X = [0, \infty)$ and define $g(x, y) = \max\{x, y\}$.

Then, clearly (X, g) is a complete metric like space.

Define $Px = \frac{x}{3}$, and $\xi t = 4t$, $\delta t = t$, $\omega t = t$.

Now, without loss of generality assume that $x \geq y$, then

$$\begin{aligned}\xi(g(Px, Py)) = 4\left(\max\left\{\frac{x}{3}, \frac{y}{3}\right\}\right) &= \frac{4x}{3} \\ &\leq 3x \\ &\leq 4x - x \\ &= 4\max\{x, y\} - \max\{x, y\} - \max\{\frac{x}{3}, x\} \\ &\leq \xi(g(x, y)) - \delta(g(x, y)) - \omega(g(Px, x)).\end{aligned}$$

Hence, equation (1.2) holds.

So, all the conditions of Theorem 2.1 satisfied and P has a unique fixed point.

Clearly, 0 is the unique fixed point of P.

Hence, Theorem 2.1 is verified.

Corollary 2.3 Let (X, g) be a metric-like space and $P : X \to X$ be a self-mapping satisfying the inequality

$$\xi(g(Pu, Pv)) \leq \xi\left(g(u, v)\right) - g(u, v) - \omega(g(Pu, u)),$$

where $\xi, \omega : [0, \infty) \to [0, \infty)$, are all continuous and monotone non-decreasing functions with $\xi(q) = 0 = \omega(q)$ if and only if $q = 0$.

Then P has a unique fixed point.

Proof Taking $\delta t = t$ in Theorem 2.1, one can obtain the proof easily.
Corollary 2.4 Let (X, g) be a metric-like space and $P : X \to X$ be a self-mapping satisfying the inequality
$\xi(g(Pu, Pv)) \leq \xi(g(u, v)) - g(u, v) - g(Pu, u),$
where $\xi : [0, \infty) \to [0, \infty)$, is continuous and monotone non-decreasing function with
$\xi(q) = 0$ if and only if $q = 0$.
Then P has a unique fixed point.
Proof Taking $\omega t = t$ in Corollary 2.3, one can obtain the proof easily.

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