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Review Paper

Fixed Point Results Using Rational Contraction in Modular Metric Spaces

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Abstract
The purpose of this paper is to establish some fixed point results for mappings, satisfying rational contractive inequalities on a closed ball in a modular metric space. Additionally, an example is also given to describe our main result in closed ball.

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I. Introduction

Banach in 1922 [1] proved his famous contraction mapping principle in a complete metric space. Since then this result of Banach has been characterized in different spaces such as those in multiplicative metric space, b-metric, G-metrics. On the other hand, in 2006, Chistyakov [4] introduced the notion of the metric modular on an arbitrary set and the corresponding modular space, which is more general than a metric space, and, based on this, he further studied Lipschitz continuity and a class of superposition operators on modular metric space (see also [8,5]). His purpose was to define the notion of a modular on an arbitrary set, develop the theory of metric spaces generated by modulars, called modular metric spaces in [6], [8],[9].This is a generalization of the classical modular spaces like Orlicz spaces (see [14]).

For more details on modular metric fixed point theory, the reader may consult the books [14, 7].

Let Ω be a nonempty set. Throughout this paper, for a function
$\partial : (0,\infty) \times \Omega \times \Omega \to [0,\infty)$, we write
$\partial_\kappa(\zeta,\eta) = \partial(\kappa,\zeta,\eta)$ for all $\kappa > 0$ and $\zeta,\eta \in \Omega$.

Definition 1.1. [6] Let Ω be a nonempty set. A function $\partial: (0,\infty) \times \Omega \times \Omega \to [0,\infty)$ is said to be a metric modular on Ω if it satisfies, for all $\zeta,\eta,w \in \Omega$, the following conditions:

1) $\partial_\kappa(\zeta,\eta) = 0$ for all $\kappa > 0$ if and only if $\zeta = \eta$,
2) $\partial_\kappa(\zeta,\eta) = \partial_\kappa(\eta,\zeta)$ for all $\lambda > 0$,
3) $\partial_{\kappa+\mu}(\zeta,\eta) \le \partial_\kappa(\zeta,w) + \partial_\mu(w,\eta)$ for all $\kappa,\mu > 0$.

If instead of (1) we have only the condition
1') $\partial_\kappa(\zeta,\zeta) = 0$ for all $\zeta \in \psi$, $\kappa > 0$ then ∂ is said to be a pseudo modular (metric) on Ω.

An important property of the (metric) pseudo modular on set Ω is that the mapping $\kappa \mapsto \partial_\kappa(\zeta,\eta)$ is non increasing for all $\zeta,\eta \in \Omega$.

Definition 1.2. [6] Let ∂ is a pseudo modular on Ω. Fixed $\zeta_0 \in \Omega$. The set
$\Omega_\partial = \Omega_\partial(\zeta_0) = \{ \zeta \in \Omega : \partial_\kappa(\zeta,\zeta_0) \to 0 \text{ as } \kappa \to \infty\}$ is said to be a modular metric space (around ζ_0).

Definition 1.3. [16] Let Ω_∂ be a modular metric space.

1) The sequence $\{\zeta_\eta\}$ in Ω_∂ is said to be ∂-convergent to $\zeta \in \Omega_\partial$ if and only if there exists a number $\kappa > 0$, possibly depending on (ζ_η) and ζ, such that $\lim_{n\to\infty} \partial_\kappa(\zeta_\eta,\zeta) = 0$.

2) The sequence $\{\zeta_\eta\}$ in Ω_∂ is said to be ∂-Cauchy if there exists $\kappa > 0$, possibly depending on the sequence, such that $\partial_\kappa(\zeta_m,\zeta_\eta) \to 0$ as $m,\eta \to \infty$

3) A subset C of Ω_∂ is said to be ∂ -complete if any ∂-Cauchy sequence in C is a convergent sequence and its limit is in C.

Definition 1.4. [3] Let ∂ be a metric modular on Ω and Ω_∂ be a modular metric space induced by ∂. If Ω_∂ is a ∂-complete modular metric space and $\mathcal{T}: \Omega_\partial \to \Omega_\partial$ be an arbitrary mapping $\mathcal{T}$ is called a contraction if for each $\zeta, \eta \in \Omega_\partial$ and for all $\kappa > 0$ there exists $0 \le \sigma < 1$ such that

$$\partial_\kappa(\mathcal{T}\zeta, \mathcal{T}\eta) \le \sigma\, \partial_\kappa(\zeta, \eta)$$

In 1975, Das and Gupta [10] derived some fixed point results for rational type contraction as follow:

Let β be a continuous self mapping on a complete metric space (Ω, d). If β is a rational type contraction, there exist α, β ∈ [0,1), where α + β < 1 such that

$$d(\beta\zeta, \beta\eta) \le \alpha\, d(\zeta, \eta) + \beta \frac{d(\eta, \beta\eta)[1 + d(\zeta, \beta\zeta)\}}{1 + d(\zeta, \eta)}$$

for all $\zeta, \eta \in \Omega$, then β has a unique fixed point in Ω.

On the other hand, in 1977 Jaggi [13] gave the fixed point results for rational type contraction as follows:

Let β be a continuous self mapping on a complete metric space (Ω, d). If β is a rational type contraction, there exist α, β ∈ [0, 1), where α + β < 1 such that

$$d(\beta\zeta, \beta\eta) \le \alpha\, d(\zeta, \eta) + \beta \frac{d(\zeta, \beta\zeta) d(\eta, \beta\eta)}{d(\zeta, \eta)}$$

for all $\zeta, \eta \in \Omega$, $\zeta \ne \eta$, then β has a unique fixed point in Ω.

Similarly, various results exist for self maps and pair of self mappings satisfying rational contraction in different spaces [11,2].

II. Main Results

In this section, we will prove the fixed point result for single map using rational type contraction in Modular Metric Space and then extend this for four maps in Modular Metric Space.

Theorem 2.1 Let γ be self - mapping of a complete modular metric space $(\Omega_\partial, \partial)$ such that

$$\partial_\kappa(\gamma\zeta, \gamma\eta) \le k \max\left\{\partial_\kappa(\zeta, \eta), \partial_\kappa(\zeta, \gamma\zeta), \partial_\kappa(\eta, \gamma\eta), \frac{\partial_\kappa(\zeta, \gamma\zeta)\partial_\kappa(\eta, \gamma\eta)}{1 + \partial_\kappa(\zeta, \eta)}, \frac{\partial_\kappa(\zeta, \gamma\zeta)\partial_\kappa(\eta, \gamma\eta)}{1 + \partial_\kappa(\gamma\zeta, \gamma\eta)}\right\} \quad (2.1)$$

for all $\zeta, \eta \in \overline{C(\zeta_0, \sigma)}$ and

$\partial_\kappa(c_0, \gamma c_0) \le \sigma(1 - k)$, where $k \in [0,1)$. (2.2)

Then γ has a unique fixed point.

Proof. Suppose $c_0 \in \Omega_\partial$ and $c_1 \in \Omega_\partial$ such that $c_1 = \gamma(c_0)$, $c_2 = \gamma(c_1)$,…, $c_{n+1} = \gamma(c_n)$. Now, by inequality (2.2), we have

$$\partial_\kappa(c_0, c_1) = \partial_\kappa(c_0, \gamma c_0) \le \sigma(1 - k) \le \sigma$$

This implies that $c_1 \in \overline{C(c_0, \sigma)}$. Suppose that $c_2, c_3, \ldots, c_q \in \overline{C(c_0, \sigma)}$.

Now, if $q = 2p + 1, p = 1,2,3, \ldots, p = \frac{q-1}{2}$

$$\begin{aligned}
\partial_\kappa(c_{2p+1}, c_{2p+2}) &= \partial_\kappa(\gamma c_{2p}, \gamma c_{2p+1}) \\
&\le k \max\left\{\begin{array}{c} \partial_\kappa(c_{2p}, c_{2p+1}), \partial_\kappa(c_{2p}, \gamma c_{2p}), \partial_\kappa(c_{2p+1}, \gamma c_{2p+1}), \\ \frac{\partial_\kappa(c_{2p}, \gamma c_{2p})\partial_\kappa(c_{2p+1}, \gamma c_{2p+1})}{1 + \partial_\kappa(c_{2p}, c_{2p+1})}, \frac{\partial_\kappa(c_{2p}, \gamma c_{2p})\partial_\kappa(c_{2p+1}, \gamma c_{2p+1})}{1 + \partial_\kappa(\gamma c_{2p}, \gamma c_{2p+1})} \end{array}\right\} \\
&\le k \max\left\{\begin{array}{c} \partial_\kappa(c_{2p}, c_{2p+1}), \partial_\kappa(c_{2p}, c_{2p+1}), \partial_\kappa(c_{2p+1}, c_{2p+2}), \\ \frac{\partial_\kappa(c_{2p}, c_{2p+1})\partial_\kappa(c_{2p+1}, c_{2p+2})}{1 + \partial_\kappa(c_{2p}, c_{2p+1})}, \frac{\partial_\kappa(c_{2p}, c_{2p+1})\partial_\kappa(c_{2p+1}, c_{2p+2})}{1 + \partial_\kappa(c_{2p+1}, c_{2p+2})} \end{array}\right\} \\
&\le k \max\{\partial_\kappa(c_{2p}, c_{2p+1}), \partial_\kappa(c_{2p+1}, c_{2p+2})\}
\end{aligned}$$

So, we have

$\partial_\kappa(c_{2p+1}, c_{2p+2}) \le k\partial_\kappa(c_{2p}, c_{2p+1})$ (2.3)

If $q = 2p + 1, p = 1,2,3, \ldots, p = \frac{q-1}{2}$

$$\begin{aligned}
\partial_\kappa(c_{2p}, c_{2p+1}) &= \partial_\kappa(\gamma c_{2p-1}, \gamma c_{2p}) \\
&\le k \max\left\{\begin{array}{c} \partial_\kappa(c_{2p-1}, c_{2p}), \partial_\kappa(c_{2p-1}, \gamma c_{2p-1}), \partial_\kappa(c_{2p}, \gamma c_{2p}), \\ \frac{\partial_\kappa(c_{2p-1}, \gamma c_{2p-1})\partial_\kappa(c_{2p}, \gamma c_{2p})}{1 + \partial_\kappa(c_{2p-1}, c_{2p})}, \\ \frac{\partial_\kappa(c_{2p-1}, \gamma c_{2p-1})\partial_\kappa(c_{2p}, \gamma c_{2p})}{1 + \partial_\kappa(\gamma c_{2p-1}, \gamma c_{2p})} \end{array}\right\}
\end{aligned}$$

$$\leq k \max\left\{\begin{array}{c}\partial_\kappa(c_{2p-1},c_{2p}),\partial_\kappa(c_{2p-1},c_{2p,}),\partial_\kappa(c_{2p},c_{2p+1}),\\ \dfrac{\partial_\kappa(c_{2p-1},c_{2p})\partial_\kappa(c_{2p},c_{2p+1})}{1+\partial_\kappa(c_{2p-1},c_{2p})},\\ \dfrac{\partial_\kappa(c_{2p-1},c_{2p})\partial_\kappa(c_{2p},c_{2p+1})}{1+\partial_\kappa(c_{2p},c_{2p+1})}\end{array}\right\}$$

$$\leq k \max\{\partial_\kappa(c_{2p-1},c_{2p}),\partial_\kappa(c_{2p},c_{2p+1})\}$$

So, we have

$$\partial_\kappa(c_{2p},c_{2p+1}) \leq k\partial_\kappa(c_{2p-1},c_{2p}) \qquad (2.4)$$

From (2.3), (2.4) and by induction, we have

$$\partial_\kappa(c_q,c_{q+1}) \leq k^q\partial_\kappa(c_0,c_1) \qquad (2.5)$$

$$\partial_\kappa(c_0,c_{q+1}) \leq \partial_{\frac{\kappa}{q+1}}(c_0,c_1) + \partial_{\frac{\kappa}{q+1}}(c_1,c_2) + \partial_{\frac{\kappa}{q+1}}(c_2,c_3) +\ldots+ \partial_{\frac{\kappa}{q+1}}(c_q,c_{q+1})$$

$$\partial_\kappa(c_0,c_{q+1}) \leq \partial_{\frac{\kappa}{q+1}}(c_0,c_1) + k\partial_{\frac{\kappa}{q+1}}(c_0,c_1) + k^2\partial_{\frac{\kappa}{q+1}}(c_0,c_1) +\ldots+ k^q\partial_{\frac{\kappa}{q+1}}(c_0,c_1)$$

$$\partial_\kappa(c_0,c_{q+1}) \leq (1+k+k^2+\cdots+k^q)\,\partial_{\frac{\kappa}{q+1}}(c_0,c_1)$$

$$\partial_\kappa(c_0,c_{q+1}) \leq \left(\frac{1}{1-k}\right)\partial_{\frac{\kappa}{q+1}}(c_0,c_1)$$

Since $c_1 \in \overline{C(c_0,\sigma)}$

$$\partial_\kappa(c_0,c_{q+1}) \leq \left(\frac{1-k}{1}\right).\left(\frac{1}{1-k}\right)\sigma \leq \sigma.$$

This implies that $c_{q+1} \in \overline{C(c_0,\sigma)}$. By mathematical induction $c_n \in \overline{C(c_0,\sigma)}$.

$$\partial_\kappa(c_n,c_{n+1}) \leq k^n\partial_\kappa(c_0,c_1) \qquad (2.6)$$

Next, we claim that $\{c_n\}$ is a Cauchy sequence for $m > n$.

$$\partial_\kappa(c_n,c_m) \leq \partial_{\frac{\kappa}{m-n}}(c_n,c_{n+1}) + \partial_{\frac{\kappa}{m-n}}(c_{n+1},c_{n+2}) + \partial_{\frac{\kappa}{m-n}}(c_{n+2},c_{n+3}) +\ldots+ \partial_{\frac{\kappa}{m-n}}(c_{m-1},c_m)$$

$$\leq k^n\partial_{\frac{\kappa}{m-n}}(c_0,c_1) + k^{n+1}\partial_{\frac{\kappa}{m-n}}(c_0,c_1) + k^{n+2}\partial_{\frac{\kappa}{m-n}}(c_0,c_1) + \ldots + k^{m-1}\partial_{\frac{\kappa}{m-n}}(c_0,c_1)$$

$$< (k^n + k^{n+1} +\ldots)\partial_{\frac{\kappa}{m-n}}(c_0,c_1) = \frac{k^n}{(1-k)}\partial_{\frac{\kappa}{m-n}}(c_0,c_1).$$

Applying $\lim_{n\to\infty}$, we get $\partial_\kappa(c_n,c_m) \leq 0$. We get that $\{c_n\}$ is a Modular Cauchy sequence in Ω_∂. and because Ω_∂ is complete so $c_n \to c^* \in \Omega_\partial$.

Now,

$$\partial_{2\kappa}(c^*,\gamma c^*) \leq \partial_\kappa(c^*,c_{n+1}) + \partial_\kappa(\gamma c_n,\gamma c^*)$$

$$\partial_{2\kappa}(c^*,\gamma c^*) \leq \partial_\kappa(c^*,c_{n+1}) + k\max\left\{\begin{array}{c}\partial_\kappa(c_n,c^*),\partial_\kappa(c_n,c_{n+1}),\partial_\kappa(c^*,\gamma c^*),\\ \dfrac{\partial_\kappa(c_n,c_{n+1})\partial_\kappa(c^*,\gamma c^*)}{1+\partial_\kappa(c_n,c^*)},\dfrac{\partial_\kappa(c_n,c_{n+1})\partial_\kappa(c^*,\gamma c^*)}{1+\partial_\kappa(\gamma c_n,\gamma c^*)}\end{array}\right\}$$

Taking $\lim_{n\to\infty}$, we get

$$\partial_{2\kappa}(c^*,\gamma c^*) \leq 0 + k\max\{0,0,\ \partial_{2\kappa}(c^*,\gamma c^*),0,0\}$$

$$\partial_{2\kappa}(c^*,\gamma c^*) \leq k\,\partial_{2\kappa}(c^*,\gamma c^*)$$

This implies that

$$\partial_{2\kappa}(c^*,\gamma c^*) \leq 0. \qquad (2.7)$$

So $c^*=\gamma c^*$. Hence c^* is a fixed point of γ. Let η be another fixed point of γ such that $\gamma\eta = \eta$.

$$\partial_\kappa(c^*,\eta) = \partial_\kappa(\gamma c^*,\gamma\eta)$$

$$\leq k\max\left\{\begin{array}{c}\partial_\kappa(c^*,\eta),\partial_\kappa(c^*,\gamma c^*),\partial_\kappa(\eta,\gamma\eta),\\ \dfrac{\partial_\kappa(c^*,\gamma c^*)\partial_\kappa(\eta,\gamma\eta)}{1+\partial_\kappa(c^*,\eta)},\dfrac{\partial_\kappa(c^*,\gamma c^*)\partial_\kappa(\eta,\gamma\eta)}{1+\partial_\kappa(\gamma c^*,\gamma\eta)}\end{array}\right\}$$

$$\leq k\max\{\partial_\kappa(c^*,\eta),0,0,0,0\}$$

$(1-k)\ \partial_\kappa(c^*,\eta) \leq 0,$

So $c^*=\eta$. Hence c^* is a unique fixed point of γ.

Corollary 2.2 Let $(\Omega_\partial,\partial)$ be a complete modular metric space such that

$$\partial_\kappa(\gamma\zeta,\gamma\eta) \leq k\,\partial_\kappa(\zeta,\eta),$$

for all $\zeta,\eta \in \overline{C(\zeta_0,\sigma)}$ and

$$\partial_\kappa(c_0,\gamma c_0) \leq \sigma(1-k),$$

where $k \in [0,1)$. Then γ has a unique common fixed point.

Theorem 2.3 Let α,β,γ and δ be self- mappings of a complete modular metric space $(\Omega_\partial,\partial)$ and (γ,α) and (δ,β) are weakly commutative with $\alpha\Omega_\partial \subset \delta\Omega_\partial$, $\beta\Omega_\partial \subset \gamma\Omega_\partial$ and α,β,γ and δ are continuous. Let $c_0 \in \Omega_\partial$ and $\alpha c_0 = \delta c_1 = \eta_0$. If there exists $k \in (0,\frac{1}{2})$ such that

$\partial_\kappa(\alpha\zeta, \beta\eta) \leq k(M(\zeta, \eta))$, for any $\zeta, \eta \in \overline{C(\eta_0, \sigma)}$ (2.8)
holds, where

$$M(\zeta, \eta) = \max \left\{ \begin{array}{c} \partial_\kappa(\gamma\zeta, \delta\eta), \partial_\kappa(\gamma\zeta, \alpha\zeta), \partial_\kappa(\delta\eta, \beta\eta), \\ \frac{\partial_\kappa(\gamma\zeta, \alpha\zeta)\partial_\kappa(\delta\eta, \beta\eta)}{1+\partial_\kappa(\gamma\zeta, \delta\eta)}, \frac{\partial_\kappa(\gamma\zeta, \alpha\zeta)\partial_\kappa(\delta\eta, \beta\eta)}{1+\partial_\kappa(\alpha\zeta, \beta\eta)} \end{array} \right\} \quad (2.9)$$

Then there exists a unique common fixed point of α, β, γ and δ in $\overline{C(\eta_0, \sigma)}$ provided that
$\partial_\kappa(\eta_0, \beta c_1) \leq (1-h)\sigma$, where $h = \frac{k}{1-k}$ and $h < 1$. (2.10)

Proof. Let c_0 be a given point in Ω_∂. Since $\alpha\Omega_\partial \subset \delta\Omega_\partial$, we can choose a point c_1 in Ω_∂ such that $\alpha c_0 = \delta c_1 = \eta_0$. Similarly, there exists a point c_2 in Ω_∂ such that $\beta c_1 = \gamma c_2 = \eta_1$. Indeed, it follows from the assumption that $\beta\Omega_\partial \subset \gamma\Omega_\partial$. Thus we can construct sequences $\{c_n\}$ and $\{\eta_n\}$ in Ω_∂ such that
$\eta_{2n} = \alpha c_{2n} = \delta c_{2n+1}$, $\eta_{2n+1} = \beta c_{2n+1} = \gamma c_{2n+2}$, $n = 0,1,2, ...$
Now, we show that $\{\eta_n\}$ is a sequence in $\overline{C(\eta_0, \sigma)}$.
By (2.10), $\partial_\kappa(\eta_0, \eta_1) = \partial_\kappa(\eta_0, \beta c_1) \leq (1-h)\sigma < \sigma$. Hence $\eta_1 \in \overline{C(\eta_0, \sigma)}$.
Assume $\eta_2, \eta_3, \ldots, \eta_q \in \overline{C(\eta_0, \sigma)}$ for some $q \in \mathbb{N}$.
Then if $q = 2k$, it follows from conditions (2.8) and (2.9), that

$$\begin{aligned} \partial_\kappa(\eta_{2k}, \eta_{2k+1}) &= \partial_\kappa(\alpha c_{2k}, \beta c_{2k+1}) \leq k(M(c_{2k}, c_{2k+1})) \\ &\leq k \max \left\{ \begin{array}{c} \partial_\kappa(\gamma c_{2k}, \delta c_{2k+1}), \partial_\kappa(\gamma c_{2k}, \alpha c_{2k}), \partial_\kappa(\delta c_{2k+1}, \beta c_{2k+1}), \\ \frac{\partial_\kappa(\gamma c_{2k}, \alpha c_{2k})\partial_\kappa(\delta c_{2k+1}, \beta c_{2k+1})}{1+\partial_\kappa(\gamma c_{2k}, \delta c_{2k+1})}, \frac{\partial_\kappa(\gamma c_{2k}, \alpha c_{2k})\partial_\kappa(\delta c_{2k+1}, \beta c_{2k+1})}{1+\partial_\kappa(\alpha c_{2k}, \beta c_{2k+1})} \end{array} \right\} \\ &\leq k \max \left\{ \begin{array}{c} \partial_\kappa(\eta_{2k-1}, \eta_{2k}), \partial_\kappa(\eta_{2k-1}, \eta_{2k}), \partial_\kappa(\eta_{2k}, \eta_{2k+1}), \\ \frac{\partial_\kappa(\eta_{2k-1}, \eta_{2k})\partial_\kappa(\eta_{2k}, \eta_{2k+1})}{1+\partial_\kappa(\eta_{2k-1}, \eta_{2k})}, \frac{\partial_\kappa(\eta_{2k-1}, \eta_{2k})\partial_\kappa(\eta_{2k}, \eta_{2k+1})}{1+\partial_\kappa(\eta_{2k}, \eta_{2k+1})} \end{array} \right\} \\ &\leq k \max \{\partial_\kappa(\eta_{2k-1}, \eta_{2k}), \partial_\kappa(\eta_{2k}, \eta_{2k+1})\} \end{aligned}$$

That is, $\partial_\kappa(\eta_{2k}, \eta_{2k+1}) \leq k\partial_\kappa(\eta_{2k-1}, \eta_{2k})$ (2.11)
Thus Similarly, if $q = 2k+1$, then
$\partial_\kappa(\eta_{2k+1}, \eta_{2k+2}) \leq k\partial_\kappa(\eta_{2k}, \eta_{2k+1})$ (2.12)
Hence from (2.11) and (2.12) , we have
$\partial_\kappa(\eta_k, \eta_{k+1}) \leq k\partial_\kappa(\eta_{k-1}, \eta_k)$ (2.13)
From (2.11), (2.12) and (2.13), we have
$\partial_\kappa(\eta_k, \eta_{k+1}) \leq k\partial_\kappa(\eta_{k-1}, \eta_k) \leq k^2\partial_\kappa(\eta_{k-2}, \eta_{k-1}) \leq \ldots \leq k^k\partial_\kappa(\eta_0, \eta_1) \quad \forall\, k \in \mathbb{N}$. (2.14)
Thus from (2.14), we have

$$\begin{aligned} \partial_\kappa(\eta_0, \eta_{p+1}) &\leq \partial_{\frac{\kappa}{p+1}}(\eta_0, \eta_1) + \partial_{\frac{\kappa}{p+1}}(\eta_1, \eta_2) + \partial_{\frac{\kappa}{p+1}}(\eta_2, \eta_3) + \ldots + \partial_{\frac{\kappa}{p+1}}(\eta_p, \eta_{p+1}) \\ &\leq \partial_{\frac{\kappa}{p+1}}(\eta_0, \eta_1) + k\partial_{\frac{\kappa}{p+1}}(\eta_0, \eta_1) + k^2\partial_{\frac{\kappa}{p+1}}(\eta_0, \eta_1) + \ldots + k^p\partial_{\frac{\kappa}{p+1}}(\eta_0, \eta_1) \\ &\leq (1 + k + k^2 + \ldots + k^p)\partial_{\frac{\kappa}{p+1}}(\eta_0, \eta_1) \\ &= \frac{(1-(k)^{p+1})}{1-k}\partial_{\frac{\kappa}{p+1}}(\eta_0, \eta_1). \end{aligned}$$

Since $\eta_1 \in \overline{C(\eta_0, \sigma)}$, we have

$$\begin{aligned} \partial_\kappa(\eta_0, \eta_{p+1}) &\leq (1-k)\frac{(1-(k)^{p+1})}{1-k}(\sigma) \\ &\leq (1-(k)^{p+1})\sigma \leq \sigma \end{aligned}$$

$\partial_\kappa(\eta_0, \eta_{p+1}) \leq \sigma$ for all $p \in \mathbb{N}$. (2.15)
Hence $\eta_{p+1} \in \overline{C(\eta_0, \sigma)}$. By induction on n, we conclude that $\{\eta_n\} \in \overline{C(\eta_0, \sigma)}$ for all $n \in \mathbb{N}$.
Now we claim that the sequence $\{\eta_n\}$ satisfies modular Cauchy criterion for convergence in ($\overline{C(\eta_0, \sigma)}$, ∂_κ). To show this let $m, n \in \mathbb{N}$ be such that $m > n$ and let $m = n + p$, then

$$\begin{aligned} \partial_\kappa(\eta_n, \eta_{n+p}) &\leq \partial_{\frac{\kappa}{p}}(\eta_n, \eta_{n+1}) + \partial_{\frac{\kappa}{p}}(\eta_{n+1}, \eta_{n+2}) + \ldots + \partial_{\frac{\kappa}{p}}(\eta_{n+p-1}, \eta_{n+p}) \\ &\leq k^n\partial_{\frac{\kappa}{p}}(\eta_0, \eta_1) + k^{n+1}\partial_{\frac{\kappa}{p}}(\eta_0, \eta_1) + \ldots + k^{n+p-1}\partial_{\frac{\kappa}{p}}(\eta_0, \eta_1) \\ &\leq (k^n + k^{n+1} + \ldots + k^{n+p-1})\partial_{\frac{\kappa}{p}}(\eta_0, \eta_1) \\ &= (1-k).\frac{k^n}{1-k}\sigma = k^n\sigma \end{aligned}$$

Letting $n \to \infty$, we have $\partial_\kappa(\eta_m, \eta_n) \to 0$, as $m, n \to \infty$. Hence the sequence $\{\eta_n\}$ is a modular Cauchy sequence.
As (Ω_∂, ∂_κ) is complete, so ($\overline{C(\eta_0, \sigma)}, \partial_\kappa$) is complete. Hence $\{\eta_n\}$ has a limit, say ζ in $\overline{C(\eta_0, \sigma)}$. The fact that $\{\alpha c_{2n}\} = \{\delta c_{2n+1}\} = \{\eta_{2n}\}$ and $\{\beta c_{2n+1}\} = \{\gamma c_{2n+2}\} = \{\eta_{2n+1}\}$ are subsequences of $\{\eta_n\}$ makes $\lim_{n\to\infty} \alpha c_{2n} = \lim_{n\to\infty} \delta c_{2n+1} = \lim_{n\to\infty} \beta c_{2n+1} = \lim_{n\to\infty} \gamma c_{2n+2} = \zeta$.

Suppose γ is continuous, then

$$\lim_{n\to\infty} \gamma(\alpha c_{2n}) = \gamma(\lim_{n\to\infty} \alpha c_{2n}) = \gamma(\lim_{n\to\infty} \gamma c_{2n+2}) = \gamma\zeta \tag{2.16}$$

By weak commutativity of a pair (γ, α), we have

$$\partial_\kappa(\gamma(\alpha c_{2n}), \alpha(\gamma c_{2n})) \leq \partial_\kappa(\gamma c_{2n}, \alpha c_{2n}) \tag{2.17}$$

Taking limit as $n \to \infty$, on both sides of (2.17) and by (2.16), we get

$$\partial_\kappa\left(\gamma(\zeta), \lim_{n\to\infty} \alpha(\gamma c_{2n})\right) \leq \partial_\kappa(\zeta, \zeta) \tag{2.18}$$

which further implies that $\lim_{n\to\infty} \alpha(\gamma c_{2n}) = \gamma(\zeta)$.

Now, by conditions (2.8) and (2.9), we have

$$\partial_\kappa(\alpha(\gamma c_{2n}), \beta c_{2n+1}) \leq k \max\left\{\begin{matrix} \partial_\kappa(\gamma\gamma c_{2n}, \delta c_{2n+1}), \partial_\kappa(\gamma\gamma c_{2n}, \alpha\gamma c_{2n}), \\ \partial_\kappa(\delta c_{2n+1}, \beta c_{2n+1}), \\ \frac{\partial_\kappa(\gamma\gamma c_{2n}, \alpha\gamma c_{2n})\partial_\kappa(\delta c_{2n+1}, \beta c_{2n+1})}{1+\partial_\kappa(\gamma\gamma c_{2n}, \delta c_{2n+1})}, \\ \frac{\partial_\kappa(\gamma\gamma c_{2n}, \alpha\gamma c_{2n})\partial_\kappa(\delta c_{2n+1}, \beta c_{2n+1})}{1+\partial_\kappa(\alpha\gamma c_{2n}, \beta c_{2n+1})} \end{matrix}\right\} \tag{2.19}$$

Taking limit as $n \to \infty$, on both sides of (2.19), we obtain

$$\partial_\kappa(\gamma\zeta, \zeta) \leq k \max\left\{\begin{matrix} \partial_\kappa(\gamma\zeta, \zeta), \partial_\kappa(\gamma\zeta, \gamma\zeta), \partial_\kappa(\zeta, \zeta), \\ \frac{\partial_\kappa(\gamma\zeta, \gamma\zeta)\partial_\kappa(\zeta, \zeta)}{1+\partial_\kappa(\gamma\zeta, \zeta)}, \frac{\partial_\kappa(\gamma\zeta, \gamma\zeta)\partial_\kappa(\zeta, \zeta)}{1+\partial_\kappa(\gamma\zeta, \zeta)} \end{matrix}\right\} \tag{2.20}$$

that is $\partial_\kappa(\gamma\zeta, \zeta) \leq k\, \partial_\kappa(\gamma\zeta, \zeta)$. Hence $\partial_\kappa(\gamma\zeta, \zeta) = 0$ and ζ is a fixed point of γ in $\overline{C(\eta_0, \sigma)}$. In similar way, by conditions (2.8) and (2.9), we have

$$\partial_\kappa(\alpha(\zeta), \beta c_{2n+1}) \leq k \max\left\{\begin{matrix} \partial_\kappa(\gamma\zeta, \delta c_{2n+1}), \partial_\kappa(\gamma\zeta, \alpha\zeta), \partial_\kappa(\delta c_{2n+1}, \beta c_{2n+1}), \\ \frac{\partial_\kappa(\gamma\zeta, \alpha\zeta)\partial_\kappa(\delta c_{2n+1}, \beta c_{2n+1})}{1+\partial_\kappa(\gamma\zeta, \delta c_{2n+1})}, \frac{\partial_\kappa(\gamma\zeta, \alpha\zeta)\partial_\kappa(\delta c_{2n+1}, \beta c_{2n+1})}{1+\partial_\kappa(\alpha\zeta, \beta c_{2n+1})} \end{matrix}\right\} \tag{2.21}$$

Taking limit as $n \to \infty$, on both sides of (2.21), we obtain

$$\partial_\kappa(\alpha(\zeta), \zeta) \leq k \max\left\{\begin{matrix} \partial_\kappa(\gamma\zeta, \zeta), \partial_\kappa(\gamma\zeta, \alpha\zeta), \partial_\kappa(\zeta, \zeta), \\ \frac{\partial_\kappa(\gamma\zeta, \alpha\zeta)\partial_\kappa(\zeta, \zeta)}{1+\partial_\kappa(\gamma\zeta, \zeta)}, \frac{\partial_\kappa(\gamma\zeta, \alpha\zeta)\partial_\kappa(\zeta, \zeta)}{1+\partial_\kappa(\alpha\zeta, \zeta)} \end{matrix}\right\}$$

$$\partial_\kappa(\alpha\zeta, \zeta) \leq 0.$$

Hence $\partial_\kappa(\alpha\zeta, \zeta) = 0$ and ζ is a fixed point of α in $\overline{C(\eta_0, \sigma)}$.

Because of the fact that $\zeta = \alpha(\zeta) \in \alpha\overline{C(\eta_0, \sigma)} \subseteq \delta\overline{C(\eta_0, \sigma)}$.

Let ζ^* in $\overline{C(\eta_0, \sigma)}$ be such that $\zeta = \delta(\zeta^*)$.

$$\partial_\kappa(\zeta, \beta\zeta^*) = \partial_\kappa(\alpha(\zeta), \beta\zeta^*) \leq k \max\left\{\begin{matrix} \partial_\kappa(\gamma\zeta, \delta\zeta^*), \partial_\kappa(\gamma\zeta, \alpha\zeta), \partial_\kappa(\delta\zeta^*, \beta\zeta^*), \\ \frac{\partial_\kappa(\gamma\zeta, \alpha\zeta)\partial_\kappa(\delta\zeta^*, \beta\zeta^*)}{1+\partial_\kappa(\gamma\zeta, \delta\zeta^*)}, \frac{\partial_\kappa(\gamma\zeta, \alpha\zeta)\partial_\kappa(\delta\zeta^*, \beta\zeta^*)}{1+\partial_\kappa(\alpha\zeta, \beta\zeta^*)} \end{matrix}\right\}$$

$$\partial_\kappa(\zeta, \beta\zeta^*) \leq 0.$$

This implies that $\beta\zeta^* = \zeta$.

Since the pair (δ, β) is weakly commutative from our assumptions, thus

$$\partial_\kappa(\delta\zeta, \beta\zeta) = \partial_\kappa(\delta\beta(\zeta^*), \beta\delta\zeta^*) \leq \partial_\kappa(\delta\zeta^*, \beta\zeta^*) = \partial_\kappa(\zeta, \zeta) = 0.$$

Hence $\delta\zeta = \beta\zeta$. By (2.8) and (2.9), we obtain

$$\partial_\kappa(\zeta, \beta\zeta) = \partial_\kappa(\alpha(\zeta), \beta\zeta) \leq k \max\left\{\begin{matrix} \partial_\kappa(\gamma\zeta, \delta\zeta), \partial_\kappa(\gamma\zeta, \alpha\zeta), \partial_\kappa(\delta\zeta, \beta\zeta), \\ \frac{\partial_\kappa(\gamma\zeta, \alpha\zeta)\partial_\kappa(\delta\zeta, \beta\zeta)}{1+\partial_\kappa(\gamma\zeta, \delta\zeta)}, \frac{\partial_\kappa(\gamma\zeta, \alpha\zeta)\partial_\kappa(\delta\zeta, \beta\zeta)}{1+\partial_\kappa(\alpha\zeta, \beta\zeta)} \end{matrix}\right\}$$

which implies $\zeta = \beta(\zeta)$. Hence ζ is a common fixed point of γ, δ, α and β in $\overline{C(\eta_0, \sigma)}$.

If δ is continuous, then following arguments similar to those given above, we obtain that $\zeta = \alpha(\zeta) = \gamma(\zeta) = \beta(\zeta) = \delta(\zeta)$.

Now suppose that α is continuous.

Thus

$$\lim_{n\to\infty} \alpha(\gamma c_{2n}) = \alpha(\lim_{n\to\infty} \gamma c_{2n}) = \alpha(\zeta) \tag{2.22}$$

As the pair (γ, α) is weakly commuting, we have

$$\partial_\kappa(\gamma(\alpha c_{2n}), \alpha(\gamma c_{2n})) \leq \partial_\kappa(\gamma c_{2n}, \alpha c_{2n}) \tag{2.23}$$

Taking limit as $n \to \infty$, on both sides of (2.23) , we obtain

$\partial_\kappa(\gamma(\alpha c_{2n}), \alpha\zeta) \leq \partial_\kappa(\zeta, \zeta) = 0$ and $\lim_{n\to\infty} \gamma(\alpha c_{2n}) = \alpha(\zeta)$.

By contractive condition (2.8) , we get

$\partial_\kappa(\alpha(\alpha c_{2n}), \beta c_{2n+1})$

$$\leq k \max\left\{\begin{array}{c}\partial_\kappa(\gamma\alpha c_{2n}, \delta c_{2n+1}), \partial_\kappa(\gamma\alpha c_{2n}, \alpha\alpha c_{2n}),\\ \partial_\kappa(\delta c_{2n+1}, \beta c_{2n+1}),\\ \frac{\partial_\kappa(\gamma\alpha c_{2n},\alpha\alpha c_{2n})\partial_\kappa(\delta c_{2n+1},\beta c_{2n+1})}{1+\partial_\kappa(\gamma\alpha c_{2n},\delta c_{2n+1})},\\ \frac{\partial_\kappa(\gamma\alpha c_{2n},\alpha\alpha c_{2n})\partial_\kappa(\delta c_{2n+1},\beta c_{2n+1})}{1+\partial_\kappa(\alpha\alpha c_{2n},\beta c_{2n+1})}\end{array}\right\} \quad (2.24)$$

Taking limit as $n \to \infty$, on both sides of (2.24), implies that

$\partial_\kappa(\alpha\zeta, \zeta) \leq k\, \partial_\kappa(\alpha\zeta, \zeta)$

Hence $\partial_\kappa(\alpha\zeta, \zeta) = 0$, and ζ is a fixed point of α in $\overline{C(\eta_0, \sigma)}$. Since $= \alpha(\zeta) \in \alpha(\overline{C(\eta_0, \sigma)}) \subseteq \delta\overline{C(\eta_0, \sigma)}$. Let ζ^* in $\overline{C(\eta_0, \sigma)}$ be such that $\zeta = \delta(\zeta^*)$. It follows from condition (2.8), that

$$\partial_\kappa(\alpha(\alpha c_{2n}), \beta\zeta^*) \leq k \max\left\{\begin{array}{c}\partial_\kappa(\gamma\alpha c_{2n}, \delta\zeta^*), \partial_\kappa(\gamma\alpha c_{2n}, \alpha\alpha c_{2n}), \partial_\kappa(\delta\zeta^*, \beta\zeta^*),\\ \frac{\partial_\kappa(\gamma\alpha c_{2n},\alpha\alpha c_{2n})\partial_\kappa(\delta\zeta^*,\beta\zeta^*)}{1+\partial_\kappa(\gamma\alpha c_{2n},\delta\zeta^*)}, \frac{\partial_\kappa(\gamma\alpha c_{2n},\alpha\alpha c_{2n})\partial_\kappa(\delta\zeta^*,\beta\zeta^*)}{1+\partial_\kappa(\alpha\alpha c_{2n},\beta\zeta^*)}\end{array}\right\} \quad (2.25)$$

Taking limit as $n \to \infty$, on both sides of (2.25), implies that

$\partial_\kappa(\zeta, \beta\zeta^*) \leq k\, \partial_\kappa(\zeta, \beta\zeta^*)$ (2.26)

Thus $\beta\zeta^* = \zeta$. Since the pair (β, δ) is weakly commutative from our hypothesis, then

$\partial_\kappa(\beta\zeta, \delta\zeta) = \partial_\kappa(\beta\delta\zeta^*, \delta\beta\zeta^*) \leq \partial_\kappa(\beta\zeta^*, \delta\zeta^*) = \partial_\kappa(\zeta, \zeta) = 0,$ (2.27)

which implies that $\delta\zeta = \beta\zeta$. From (2.8), we have

$$\partial_\kappa(\alpha c_{2n}, \beta\zeta) \leq k \max\left\{\begin{array}{c}\partial_\kappa(\gamma c_{2n}, \delta\zeta), \partial_\kappa(\gamma c_{2n}, \alpha c_{2n}), \partial_\kappa(\delta\zeta, \beta\zeta),\\ \frac{\partial_\kappa(\gamma c_{2n},\alpha c_{2n})\partial_\kappa(\delta\zeta,\beta\zeta)}{1+\partial_\kappa(\gamma c_{2n},\delta\zeta)}, \frac{\partial_\kappa(\gamma c_{2n},\alpha c_{2n})\partial_\kappa(\delta\zeta,\beta\zeta)}{1+\partial_\kappa(\alpha c_{2n},\beta\zeta)}\end{array}\right\} \quad (2.28)$$

Taking limit as $n \to \infty$, on both sides of (2.28), implies that

$\partial_\kappa(\zeta, \beta\zeta) \leq k\, \partial_\kappa(\zeta, \beta\zeta)$ and $\zeta = \beta(\zeta)$.

However, $\zeta = \beta(\zeta) \in \beta(\overline{C(\eta_0, \sigma)}) \subseteq \gamma(\overline{C(\eta_0, \sigma)})$, let $\eta \in (\overline{C(\eta_0, \sigma)}$) be such that $\zeta = \gamma(\eta)$. It follows from condition (2.8) , that

$\partial_\kappa(\alpha\eta, \zeta) = \partial_\kappa(\alpha\eta, \beta\zeta)$

$$\leq k \max\left\{\begin{array}{c}\partial_\kappa(\gamma\eta, \delta\zeta), \partial_\kappa(\gamma\eta, \alpha\eta), \partial_\kappa(\delta\zeta, \beta\zeta),\\ \frac{\partial_\kappa(\gamma\eta,\alpha\eta)\partial_\kappa(\delta\zeta,\beta\zeta)}{1+\partial_\kappa(\gamma\eta,\delta\zeta)}, \frac{\partial_\kappa(\gamma\eta,\alpha\eta)\partial_\kappa(\delta\zeta,\beta\zeta)}{1+\partial_\kappa(\alpha\eta,\beta\zeta)}\end{array}\right\}$$

which implies that $\partial_\kappa(\alpha\eta, \zeta) \leq k\, \partial_\kappa(\alpha\eta, \zeta)$.

Hence $\alpha\eta = \zeta$. Since α and γ are weakly commutative, so

$\partial_\kappa(\gamma\zeta, \alpha\zeta) = \partial_\kappa(\gamma\alpha\eta, \alpha\gamma\eta) \leq \partial_\kappa(\gamma\eta, \alpha\eta) = \partial_\kappa(\zeta, \zeta) = 0.$ (2.29)

This gives $\gamma(\zeta) = \alpha(\zeta)$. Applying condition (2.8), we obtain

$\partial_\kappa(\alpha\zeta, \zeta) = \partial_\kappa(\alpha\zeta, \beta\zeta)$

$$\leq k \max\left\{\begin{array}{c}\partial_\kappa(\gamma\zeta, \delta\zeta), \partial_\kappa(\gamma\zeta, \alpha\zeta), \partial_\kappa(\delta\zeta, \beta\zeta),\\ \frac{\partial_\kappa(\gamma\zeta,\alpha\zeta)\partial_\kappa(\delta\zeta,\beta\zeta)}{1+\partial_\kappa(\gamma\zeta,\delta\zeta)}, \frac{\partial_\kappa(\gamma\zeta,\alpha\zeta)\partial_\kappa(\delta\zeta,\beta\zeta)}{1+\partial_\kappa(\alpha\zeta,\beta\zeta)}\end{array}\right\}$$

which implies that $\zeta = \alpha(\zeta)$. Hence ζ is a common fixed point of γ, α, β and δ in $\overline{C(\eta_0, \sigma)}$.

If β is continuous, then by using arguments similar to those given above, we can easily obtain a common fixed point of γ, α, β and δ in $\overline{C(\eta_0, \sigma)}$.

We proceed to show the uniqueness of the common fixed point of the mappings γ, α, β and δ. So let $\eta \in \overline{C(\eta_0, \sigma)}$ be another common fixed point of γ, α, β and δ. By (2.8), we have

$\partial_\kappa(\zeta, \eta) = \partial_\kappa(\alpha\zeta, \beta\eta)$

$$\leq k \max\left\{\begin{array}{c}\partial_\kappa(\gamma\zeta, \delta\eta), \partial_\kappa(\gamma\zeta, \alpha\zeta), \partial_\kappa(\delta\eta, \beta\eta),\\ \frac{\partial_\kappa(\gamma\zeta,\alpha\zeta)\partial_\kappa(\delta\eta,\beta\eta)}{1+\partial_\kappa(\gamma\zeta,\delta\eta)}, \frac{\partial_\kappa(\gamma\zeta,\alpha\zeta)\partial_\kappa(\delta\eta,\beta\eta)}{1+\partial_\kappa(\alpha\zeta,\beta\eta)}\end{array}\right\}$$

that is, $\partial_\kappa(\zeta, \eta) \leq k\, \partial_\kappa(\zeta, \eta)$.

Hence $\zeta = \eta$ and this implies that the common fixed point of γ, α, β and δ is unique.

Example 2.4 Let $\Omega_\partial = \mathbb{R}$ and $\partial_\kappa : \Omega_\partial \times \Omega_\partial \to [0, \infty)$ be a modular metric defined by $\partial_\kappa(\zeta, \eta) = \frac{1}{\kappa}(|\zeta| + |\eta|)$. Note that $(\Omega_\partial, \partial)$ is a complete modular metric space, define mappings $\gamma, \delta, \alpha, \beta : \Omega_\partial \to \Omega_\partial$ by

$\gamma(\zeta) = \begin{cases}\zeta & if\ \zeta \leq 2\\ 30\zeta & if\ \zeta > 2\end{cases}$, $\quad \delta(\zeta) = \begin{cases}3\zeta & if\ \zeta \leq 2\\ 20\zeta & if\ \zeta > 2\end{cases}$

$\alpha(\zeta) = \begin{cases}2\zeta & if\ \zeta \leq 2\\ 100\zeta & if\ \zeta > 2\end{cases}$, $\quad \beta(\zeta) = \begin{cases}\frac{1}{3}\zeta & if\ \zeta \leq 2\\ 4\zeta & if\ \zeta > 2\end{cases}$

Obviously, Maps are continuous and (γ, α) and (β, δ) are weak commutative with $\alpha(\Omega_\partial) \subset \delta(\Omega_\partial)$, and $\beta(\Omega_\partial) \subset \gamma(\Omega_\partial)$. First we construct a closed ball such that $\zeta_0 = \frac{1}{7}$, $\varepsilon = 16$ and $\kappa = 2$

$\overline{C(\zeta_0,\varepsilon)} = \left\{\eta \in \Omega_\partial:\ \partial_\kappa(\eta,\frac{1}{7}) \le 16\right\}$
$= \left\{\eta \in \Omega_\partial: \frac{1}{2}(|\eta| + \left|\frac{1}{7}\right|) \le 16\right\}$
$= \left\{\eta \in \Omega_\partial: (|\eta| + \left|\frac{1}{7}\right|) \le 32\right\}$
$= \left\{\eta \in \Omega_\partial: |\eta| \le \frac{223}{7}\right\} = \left[-\frac{223}{7}, \frac{223}{7}\right].$

Clearly, the mappings are weakly commutative. Now choose $\zeta_0 = \frac{1}{7}$, then there exist
$\zeta_1 \in \overline{C(\frac{1}{7},\varepsilon)}$, such that $\alpha(\frac{1}{7}) = \delta(\zeta_1) = \eta_0 = \frac{2}{7}$ and $\eta_0 = \delta(\zeta_1) = 3\zeta_1$.
Now, $\eta_0 = \frac{2}{7} = \delta(\zeta_1) = 3\zeta_1 \Rightarrow \zeta_1 = \frac{2}{21}$
Also, $\beta(\zeta_1) = \beta(\frac{2}{21}) = \frac{2}{63} = \eta_1$
Thus,
$\partial_\kappa(\eta_0, \beta(\zeta_1)) = 0.06349$
where $\varepsilon = 16$, $k = \frac{3}{13}$, $k < 1$, then
$(1-k)\,\varepsilon = \left(1 - \frac{3}{13}\right) 16 \le 16$
So $\partial_\kappa(\eta_0, \beta(\zeta_0)) \le (1-k)\,\varepsilon$,
holds.
Thus for $\zeta, \eta \in \overline{C(\zeta_0,\varepsilon)}$ all the conditions of Theorem 2.3 hold.
Hence $\zeta = 0$ is the unique common fixed point of γ, β, α and δ.

III. Conclusion

The purpose of this paper is to prove some common fixed point theorems in Modular Metric Space for single map and four maps using rational type contraction mapping.

Competing Interests
The authors have declared that no competing interests exist.

References

[1]. Banach, S. : Sur les operations dans les ensembles abstracts et leure application aux equations integrals. Fundam. Math. 1922; 3: 133–181.
[2]. V. K. Bhardwaj, V. Gupta, N. Mani, Common fixed point theorems without continuity and compatible property of maps, Bol. Soc. Paran. Mat. 2017; 35(3): 67–77.
[3]. M. Chirasak, S.Wutiphol, and K. Poom, (2011) Fixed Point Theorems for Contraction Mappings in Modular Metric Spaces. Fixed Point Theory and Applications, 2011; 93.
[4]. Chistyakov, V.V. Metric modular and their applications. Dokl. Akad. Nauk 2006; 406: 165–168.
[5]. Chistyakov, V.V. Modular metric spaces, II: Application to superposition operators. Nonlinear Anal. 2010; 72: 15–30.
[6]. W. Chistyakov, Metric Modular Spaces and Their Application. Doklady Mathematics. 2006; 73: 32-35.
[7]. V.V. Chistyakov, Metric Modular Spaces-Theory and Applications, Springer International Publishing Switzerland 2015.
[8]. W. Chistyakov, Modular Metric Spaces, I: Basic Concepts. Nonlinear Analysis: Theory, Methods & Applications. 2010; 72: 1-14.
[9]. W. Chistyakov, Modular Metric Spaces Generated by F-Modular. Folia Mathematica, 2008; 14: 3-25.
[10]. Dass, B.K.; Gupta, S. An extension of Banach contraction principle through rational expressions. Indian J. Pure Appl. Math. 1975; 6: 1455–1458.
[11]. V. Gupta, N. Mani Existence and uniqueness of fixed point for contractive mapping of integral type, International Journal of Computing Science and Mathematics. 2013;4 (1): 72–83.
[12]. Hosseinzadeh, H.; Parvaneh, V. Meir-Keeler type contractive mappings in modular and partial modular metric spaces. Asian-Eur. J. Math. 2020; 13: 1–18.
[13]. Jaggi, D.S. Some unique fixed point theorems. Indian J. Pure Appl. Math. 1977; 8: 223–230.
[14]. M.A. Khamsi, W.M. Kozlowski, Fixed Point Theory in Modular Function Spaces, DOI 10.1007/978-3-319-14051-3, Springer International Publishing Switzerland 2015.
[15]. P.P. Murthy and K. N. V. V. Vara Prasad, Weak Contraction Condition Involving Cubic Terms of $d(\zeta, \eta)$ under the Fixed Point Consideration, Journal of Mathematics. 2013; 5 pages,
[16]. A. Padcharoen, D. Gopal, P. Chaipunya, and P. Kumam, Fixed Point and Periodic Point Results for α-Type F-Contractions in Modular Metric Spaces. Fixed Point Theory and Applications. 2016; 39.