Quest Journals
Journal of Research in Applied Mathematics
Volume 9 ~ Issue 6 (2023) pp: 01-13
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

Applications of Fractional Order Biological Population Model in Agriculture via Shehu Adomian Decomposition Method

Pradip R.Bhadane
Assistant Professor, Department of Mathematics,
Arts, Science and Commerce College Ozar (Mig) Nasik-424008(MS) India.

Abstract
We proposed a solid mix of Adomian Decomposition Method and Shehu transform called as Shehu Adomian Decomposition Method is intended to get an accurate solution for the Fractional order Biological Population Model with limit conditions. Which are used to determine maximum harvest for agriculturists to understand the dynamics of biological invasions, and for environmental conservation. The Fractional subsidiary is in Caputo sense and the nonlinear terms in Fractional order BPM can be taken care of by utilizing ADM. The techniques give an insightful arrangement of the Fractional order BPM in the form of a convergent series. The technique is described and illustrated with numerical example. A few plots are shown to illustrate the simplicity and reliability of the proposed method.
KEYWORDS: *Numerical Solution, Fractional Calculus(FC), Biological Population Model(BPM), Caputo Fractional derivatives, Adomian Decomposition method (ADM), Shehu transform(ST).*
2010 MSC: 26A33,92B05,49M27,44A05,44A20,.

*Received 01 June, 2023; Revised 08 June, 2023; Accepted 10 June, 2023 © The author(s) 2023. Published with open access at **www.questjournals.org***

I. Introduction

Fractional calculus is a piece of the investigation of mathematics and the Basic ideas of Fractional calculus are not new [1], [2]. The possibility of FC at first found by G. W. Leibniz in 1965 Later on, the hypothesis of FC grew quickly. The utilizations of FC have productive applications can be discovered these days in designing and science. Yet, some Fractional differential equations don't have accurate arrangements, so we require new strategies and Integral transforms [25-26]. Recently in 2019 [7,8,12], Chinees mathematician shehu Maitama developed a new transform called as Shehu transform which is a generalization of Laplace and Sumudu transform used for solving different differential equations. The ADM was presented by Adomian in 1991[9, 10, 11] and has been applied to a wide class of issues in many fields. The strategy gives the solution in a rapid convergent series with calculable terms. The basic thought of the technique is to expect an infinite solution of the structure $\theta = \sum_{n=0}^{\infty} \theta_n$, then, at that point, apply ST [3,4,5,6] to the differential equation. The nonlinear terms are then deteriorated as far as Adomian polynomials and an iterative algorithm is developed for the determination of the θ_n in a recursive way. The proficient computational devices are needed for logical and mathematical approximations of such models. The Shehu Adomian Decomposition Method is a blend of the ADM and ST. This technique was effectively utilized for tackling various issues.

In 2021 S.Nubpetchploy, utilized the mixture of ST and ADM for Solving Fractional Integro-Differential Equation [13]. Later on numerous analysts utilized this blend for addressing the linear and non linear Integral and Integro differential equations [14] and afterward it is likewise utilized for settling time fractional Schrdinger equation [15].Recently, in 2022, M.Liaqat and others, utilized the mixture of ST and ADM for settling Newell Whitehead Segel Equations [16]. For most recent couple of years numerous analysts have been giving their consideration on the presence of arrangement of the Fractional order BPM [17-24].

In this article, we consider the nonlinear time fractional order Biological Population Model in the structure

$$\frac{\partial^{\mu}\theta}{\partial t^{\mu}} = \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + f(\theta), \quad 0 < \mu \leq 1 \tag{1}$$

with the initial condition $\theta(\mathrm{x,y},0) = \theta_0(\mathrm{x,y},0)$.

Where θ signifies the population density and f addresses the population supply because of birth and death. This model considered concerning model in the number of inhabitants in creatures. The developments are made by and large either by mature creatures driven out by trespassers or by youthful creatures simply arriving at development moving out of their parental domain to build up their very own rearing area. In both cases, it is significantly more conceivable to assume that they will be coordinated towards an adjacent empty area. In this model, therefore, development will occur only "down" the population thickness inclination, and will be considerably faster at high population densities than at low ones. While trying to display the present circumstance, they thought about a walk through a rectangular grid, in which at each stage a creature may either remain at its current area or may move toward most minimal population thickness.

The point of this article is to concentrate on the utilization of SADM, to acquire estimated arrangement of time fractional order BPM with initial conditions of the structure by taking $\mathrm{f}(\theta) = h\theta^a\left(1 - r\theta^b\right)$, Where a , b, h are real constants, as

$$D_t^{\mu}\theta = \left(\theta^2\right)_{xx} + \left(\theta^2\right)_{yy} + h\theta^a\left(1 - r\theta^b\right); 0 \leq \mu < 1. \tag{2}$$

and the outcome is presented.

2 Fundamental Facts of the Fractional Calculus

Definition 1 *A new transform called the Shehu transform defined for function of exponential order, we Consider functions in the set A defined by*

$$A = \left\{v(t)\backslash\exists N, \lambda_1, \lambda_2 > 0, |v(t)| < Ne^{\frac{|t|}{\lambda_4}}, if \quad t \in (-1)' - X[0,\infty)\right\}$$

In the set A*, the constant* N *is finite number and* λ_1, λ_2 *are finite or infinite The Shehu transform denoted by the operator* S(.)& *defined by the integral equation.*

$$\mathrm{S}[v(t)] = V(s,u) = \int_0^{\infty} \exp\left(\frac{-st}{u}\right) v(t)dt$$

The inverse Shehu transform is given by

$$\mathrm{S}^{-1}[V(s,t)] = v(t),\ for\ t \geq 0$$

where s and u are the Shehu transform variables.

Shehu Transform for Simple Functions

We have $\mathrm{S}[v(t)] = V(s,u) = \int_0^{\infty} v(t)\left(e^{\frac{-st}{u}}\right)dt$

1. If v(t) = 1, then $\mathrm{S}[1] = V(s,u) = \int_0^{\infty}(1)\left(e^{\frac{-st}{u}}\right)dt = \frac{u}{s}$
2. If v(t)=t, then $\mathrm{S}[t] = V(s,u) = \int_0^{\infty}(t)\left(e^{\frac{-st}{u}}\right)dt = \frac{u^2}{s^2}$
   More generally $\mathrm{S}\left[t^n\right] = V(s,u) = \int_0^{\infty}\left(t^n\right)\left(e^{\frac{-st}{u}}\right)dt = n!\frac{u^{n+1}}{s^{n+1}}$
3. If v(t) = exp(αt), then $\mathrm{S}\left[e^{\alpha t}\right] = V(s,u) = \int_0^{\infty}\left(e^{\alpha t}\right)\left(e^{\frac{-st}{u}}\right)dt = \frac{u}{s-\alpha u}$

Theorem 1 Let V(s, u) is Shehu transform of v(t)& $\mathrm{S}[v(t)] = V(s,u)$, Then

1. $\mathrm{S}\left[\frac{d}{dt}(v(t))\right] = \mathrm{S}\left[v'(t)\right] = \int_0^{\infty}\left(v'(t)\right)\left(e^{\frac{-st}{u}}\right)dt$
   Integrating by parts, we find $\mathrm{S}\left[v'(t)\right] = V(s,u) - v(0)$
2. $\mathrm{S}\left[\frac{d^2}{dt^2}(v(t))\right] = \mathrm{S}\left[v''(t)\right] = \int_0^{\infty}\left(v''(t)\right)\left(e^{\frac{-st}{u}}\right)dt$ **then** $\mathrm{S}\left[v^*(t)\right] = \frac{s^2}{u^2}V(s,u) - \frac{s}{u}v(0) - v'(0)$
3. $\mathrm{S}\left[\frac{d^n}{dt^n}(v(t))\right] = \mathrm{S}\left[v^n(t)\right] = \int_0^{\infty}\left(v^n(t)\right)\left(e^{\frac{-st}{u}}\right)dt$, **then**

$$\mathrm{S}\left[v^n(t)\right] = \frac{s^n}{u^n}V(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-(k+1)}$$

Definition 2 *The Riemann-Liouville fractional integral operator of order $\mu \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as*

$$J^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t)dt, \mu > 0, x > 0,$$
$$J^0 f(x) = f(x).$$

Properties of the operator J^μ can be found, we mention only the following

$$f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0, \text{ and } \gamma > -1$$
$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$$
$$J^\alpha J^\beta f(x) = J^\alpha J^\beta f(x)$$
$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$$

The Riemann-Liouville derivative has certain disadvantage, when trying to model real world phenomenon with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_t^μ proposed by Caputo in his work.

Definition 3 *The fractional derivative of $f(x)$ in the Caputo sense is defined as*

$$D_t^\mu f(x) = J^{m-\mu} D^m f(x) = \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f^m(t)dt,$$

for $\mathrm{m}-1 < \mu \leq \mathrm{m}, \mathrm{m} \in \mathrm{N}, \mathrm{x} > 0$, For the Riemann-Liouville fractional integral and the Caputo fractional derivative, we have the following relation:

$$J_T^\mu D_t^\mu f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(o+)\frac{x^k}{k!}, x > 0,$$

Definition 4 *The Shehu transform of the Caputo fractional derivative is defined as*

$$\mathrm{S}\left[D_t^\mu f(t)\right] = \frac{s^\mu}{u^\mu}\left\{V(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{\mu-k-1}\left[f^K\left(0^+\right)\big|_{t=0}\right]\right\}, t > 0, n-1 < \mu \leq n, n \in N.$$

3 Shehu Adomian Decomposition Method

Consider a general nonlinear non-homogeneous partial fractional differential equation with starting state of the structure.

$$D_t^\mu \theta(x,y,t) + R\theta(x,y,t) + N\theta(x,y,t) = g(x,y,t), t \geq 0, n-1 < x < n, \quad (3)$$

$$\theta(x,y,0) = h(x,y), \theta_t(x,y,o) = f(x,y) \quad (4)$$

$0 < \mu \leq 1, \quad \theta(\mathrm{x},\mathrm{y},\mathrm{t}) \to \infty$
Where $D_t^\mu \theta(x,y,t)$ is the Caputo fractional derivative of the function $\theta(\mathrm{x},\mathrm{y},\mathrm{t})$, R is linear differential operator, N is the general nonlinear differential operator and $\mathrm{g}(\mathrm{x},\mathrm{t})$ is the source term.
Taking shehu transform on the two sides of equation (3), to get

$$\mathrm{S}\left[D_t^\mu \theta(x,y,t)\right] + \mathrm{S}[R\theta(x,y,t)] + \mathrm{S}[N\theta(x,y,t)] = \mathrm{S}[g(x,y,t)] \quad (5)$$

Utilizing the differentiation property of ST and introductory conditions (4), we have

$$\frac{s^\mu}{u^\mu}\theta(x,y,t) - \frac{s^{\mu-1}}{u^{\mu-1}}\theta(x,y,0) - \frac{s^{\mu-2}}{u^{\mu-2}}\theta_t(x,y,0) + \mathrm{S}[R\theta(x,y,t) + N\theta(x,y,t)] = \mathrm{S}[g(x,y,t)]$$

$$\theta(x,y,t) = \left(\frac{u}{s}\right)^1 h(x,y) + \left(\frac{u}{s}\right)^2 f(x,y) + \left(\frac{u}{s}\right)^\mu \mathrm{S}[g(x,y,t)] - \left(\frac{u}{s}\right)^\mu \mathrm{S}[R\theta(x,y,t) + N\theta(x,y,t)] \quad (6)$$

where $g(x,y,t)$ is emerging from the source term and the prescribed initial condition, By standard STADM characterizes the arrangement $\theta(x,y,t)$ known by the series

$$\theta(x,y,t) = \sum_{n=0}^{\infty} \theta_n(x,y,t) \quad (7)$$

The nonlinear operator is

$$N\theta(x,y,t) = \sum_{n=0}^{\infty} A_n(\theta) \quad (8)$$

where A_n are the Adomian polynomials and is given by $A_n(\theta_o, \theta_1, \ldots. \theta_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{n=0}^{\infty} p^i\theta_i\right)\right]_{p=0}, n = 0, 1, 2\ldots$ Subbing (7) and (8) in (6), we get

$$\sum_{n=0}^{\infty}\theta_n(x,y,t) = \left(\frac{u}{s}\right)^1 h(x,y) + \left(\frac{u}{s}\right)^2 f(x,y) + \left(\frac{u}{s}\right)^{\mu} \mathrm{S}[g(x,y,t)] - \left(\frac{u}{s}\right)^{\mu} \mathrm{S}[R\theta(x,y,t) + N\theta(x,y,t)]$$

This is the coupling of the ST and ADM, looking at the coefficients of like forces of the above condition are acquired.

$$\begin{aligned}
\theta_o(x,y,t) &= \left(\frac{u}{s}\right)^1 h(x,y) + \left(\frac{u}{s}\right)^2 f(x,y) + \left(\frac{u}{s}\right)^{\mu} \mathrm{S}[g(x,y,t)] = g(x,y,t)\\
\theta_1(x,y,t) &= \left(\frac{u}{s}\right)^{\mu} \mathrm{S}\left[R\theta_o(x,y,t) + A_o(\theta)\right]\\
\theta_2(x,y,t) &= \left(\frac{u}{s}\right)^{\mu} \mathrm{S}\left[R\theta_1(x,y,t) + A_1(\theta)\right]\\
&\vdots\\
\theta_{n+1}(x,y,t) &= \left(\frac{u}{s}\right)^{\mu} \mathrm{S}\left[R\theta_n(x,y,t) + A_n(\theta)\right]
\end{aligned}$$

Appling Inverse ST, condition becomes

$$\theta_o(x,y,t) = g(x,y,t)$$
$$\theta_{n+1}(x,y,t) = \mathrm{S}^{-1}\left(\left(\frac{u}{s}\right)^{\mu} \mathrm{S}\left[R\theta_n(x,y,t) + A_n(\theta)\right]\right), n \geq 1$$

at last we approximate the analytical w(x, y, t) as

$$\begin{aligned}
\theta(x,y,t) &= \sum_{n=0}^{\infty}\theta_n(x,y,t)\\
\theta(x,y,t) &= \theta_o(x,y,t) + \theta_1(x,y,t) + \theta_2(x,y,t) + \theta_3(x,y,t) + \ldots
\end{aligned}$$

4 APPLICATION

4.1 Example

Consider the following time fractional biological population model's equation as

$$\frac{\partial^{\mu}\theta}{\partial t^{\mu}} = \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} - \theta(x,y,t)\left(1 + \frac{8}{9}\theta(x,y,t)\right) 0 < \mu \leq 1 \tag{9}$$

dependent upon the condition

$$\theta(x,y,o) = e^{\frac{x+y}{3}} \tag{10}$$

Employing ST to the two sides of equation (9) dependent upon the underlying condition (10), we get

$$\mathrm{S}[\theta(x,y,t)] = \left(\frac{u}{s}\right) e^{\frac{x+y}{3}} + \left(\frac{u}{s}\right)^{\mu} \mathrm{S}\left[\theta_{xx} + \theta_{yy} - \theta\left(1 + \frac{8}{9}\theta\right)\right] \tag{11}$$

Utilizing the converse ST to the two sides of (11)

$$\theta(x,y,t) = e^{\frac{x+y}{3}} + \mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu} \mathrm{S}\left[\theta_{xx} + \theta_{yy} - \theta\left(1 + \frac{8}{9}\theta\right)\right]\right\} \tag{12}$$

Where $e^{\frac{x+y}{3}}$ address the term emerging from the source term.
Presently on Appling AD technique

$$\theta(x,y,t) = \sum_{n=0}^{\infty}\theta_n(x,y,t) \tag{13}$$

the non linear operator is due

$$N\theta(x,y,t) = \sum_{n=0}^{\infty} A_n(\theta) \tag{14}$$

Where A_n are the AD and is given by

$$A_n(\theta_0, \theta_1, \theta_2 \ldots\ldots\ldots \theta_n) = \frac{1}{n!}\left\{\sum_{n=0}^{\infty}\frac{\partial^n}{\partial p^n} N\left[p^i\theta_i(x,y,t)\right]\right\}, n = 0, 1, 2\ldots\ldots$$

Substituting (13) and (14) in equation (12), we obtain

$$\sum_{n=0}^{\infty}\theta_n(x,y,t)=e^{\frac{x+y}{3}}+\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}\left[\sum_{n=0}^{\infty}A_n(\theta)-\sum_{n=0}^{\infty}\theta_n\right]\right\} \tag{15}$$

Where

$$\begin{aligned}
&\theta_n=\theta_{nxx}^2+\theta_{nyy}^2-\frac{8}{9}\theta_n^2, n\in N\\
&A_0=\theta_{0xx}^2+\theta_{0yy}^2-\frac{8}{9}\theta_0^2\\
&A_1=2\left(\theta_0\theta_1\right)_{xx}+2\left(\theta_0\theta_1\right)_{yy}-\frac{16}{9}\theta_0\theta_1\\
&A_2=2\left(\theta_0\theta_2\right)_{xx}+\left(\theta_1^2\right)_{xx}+2\left(\theta_0\theta_2\right)_{yy}+\left(\theta_1^2\right)_{yy}-\frac{16}{9}\theta_0\theta_2-\frac{8}{9}\theta_1^2\\
&\vdots
\end{aligned}$$

This is mixture of the ST and Adomian polynomial strategy looking at the like forces of the above condition are gotten

$$\theta_0(x,y,t)=e^{\frac{x+y}{3}} \tag{16}$$

$$\begin{aligned}
\theta_1(x,y,t)&=\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}\left[A_0(\theta)-\theta_0\right]\right\}\\
&=\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}\left[\theta_{0xx}^2+\theta_{0yy}^2-\frac{8}{9}\theta_0^2-\theta_0\right]\right\}\\
&=\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{-\mu}\mathrm{S}\left[\left(e^{\frac{x+y}{3}}\right)^2_{xx}+\left(e^{\frac{x+y}{3}}\right)^2_{yy}-\frac{8}{9}\left(e^{\frac{x+y}{3}}\right)^2-e^{\frac{x+y}{3}}\right]\right\}\\
&=\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}\left[-e^{\frac{x+y}{3}}\right]\right\}\\
&=-e^{\frac{x+y}{3}}\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}[1]\right\}
\end{aligned}$$

$$\theta_1(x,y,t)=-e^{\frac{x+y}{3}}\frac{t^{\mu}}{\mu!} \tag{17}$$

$$\begin{aligned}
\theta_2(x,y,t)&=\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}\left[A_1(\theta)-\theta_1\right]\right\}\\
&=\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}\left[2\left(\theta_0\theta_1\right)_{xx}+2\left(\theta_0\theta_1\right)_{yy}-\frac{16}{9}\theta_0\theta_1-\theta_1\right]\right\}\\
&=\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}\left[-2\left(\left(e^{\frac{x+y}{3}}\right)^2\frac{t^{\mu}}{\mu!}\right)_{xx}-2\left(\left(e^{\frac{x+y}{3}}\right)^2\frac{t^{\mu}}{\mu!}\right)_{yy}+\frac{16}{9}\left(e^{\frac{x+y}{3}}\right)^2\frac{t^{\mu}}{\mu!}+e^{\frac{x+y}{3}}\frac{t^{\mu}}{\mu!}\right]\right\}\\
&=e^{\frac{x+y}{3}}\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}\left[\frac{t^{\mu}}{\mu!}\right]\right\}
\end{aligned}$$

$$\theta_2(x,y,t)=e^{\frac{x+y}{3}}\frac{t^{2\mu}}{(2\mu)!} \tag{18}$$

$$\theta_3(x,y,t)=\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}\left[A_2(\theta)-\theta_2\right]\right\}$$

$$\theta_3(x,y,t)=-e^{\frac{x+y}{3}}\frac{t^{3\mu}}{(3\mu)!} \tag{19}$$

Continuing thusly, the rest terms of $\theta_n(x,y,t)$ can be found.
The solution $\theta(x,y,t)$ of equation (9) by using (16), (17), (18)and(19), as

$$\begin{aligned}
&\theta(x,y,t)=\sum_{n=0}^{\infty}\theta_n(x,y,t)\\
&\theta(x,y,t)=\theta_o(x,y,t)+\theta_1(x,y,t)+\theta_2(x,y,t)+\theta_3(x,y,t)+\ldots\ldots\ldots\\
&\theta(x,y,t)=e^{\frac{x+y}{3}}-e^{\frac{x+y}{3}}\frac{t^{\mu}}{(\mu)!}+e^{\frac{x+y}{3}}\frac{t^{2\mu}}{(2\mu)!}-e^{\frac{x+y}{3}}\frac{t^{3\mu}}{(3\mu)!}+\ldots\ldots\ldots\\
&\theta(x,y,t)=e^{\frac{x+y}{3}}\left[1-\frac{t^{\mu}}{\mu!}+\frac{t^{2\mu}}{(2\mu)!}-\frac{t^{3\mu}}{(3\mu)!}+\ldots\ldots\ldots\right]\\
&\theta(x,y,t)=e^{\frac{x+y}{3}}E_{\mu}\left(-t^{\mu}\right)
\end{aligned}$$

where $E_\mu(t)$ is Mittag-Leffler function in one parameter. When $\mu = 1$, then

$$\theta(x, y, t) = e^{\frac{x+y}{3}} e^{-t} \tag{20}$$

Which is exact Solution of (9).

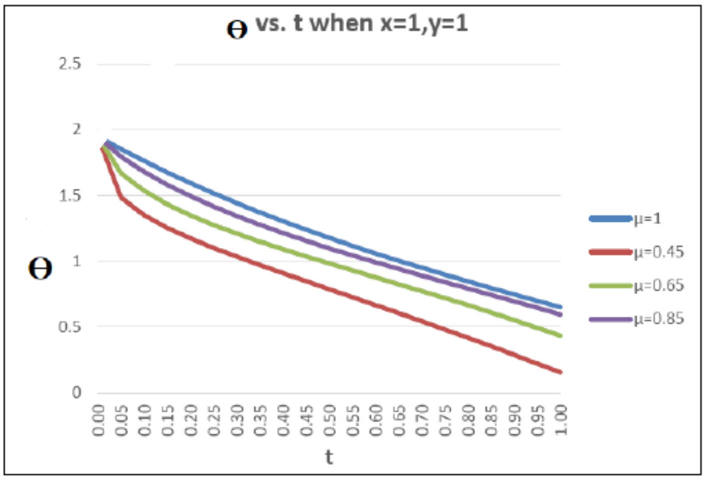

Figure 1: Comparison between w for different fractional orders.

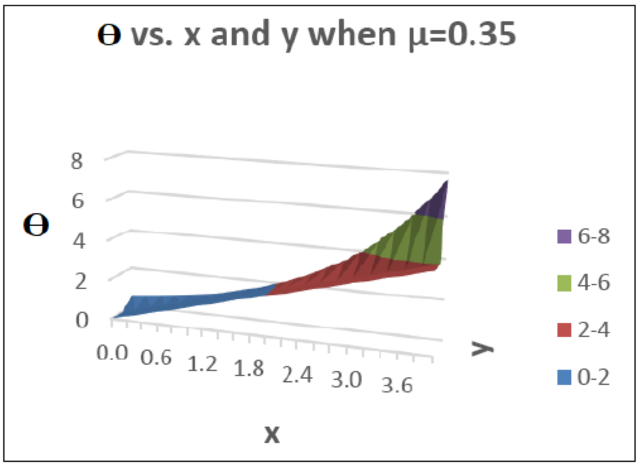

Figure 2: 3D behavior of an approximate solution for $\mu = 0.35$

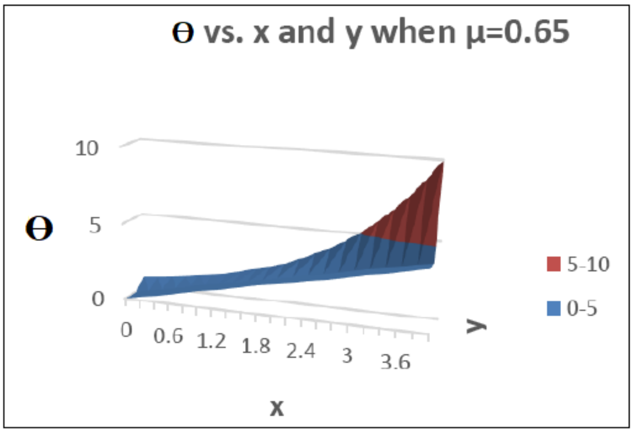

Figure 3: 3D behavior of an approximate solution for $\mu = 0.65$

4.2 Example

Consider the following time fractional biological population model's equation as

$$\frac{\partial^\mu \theta}{\partial t^\mu}=\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}+\frac{1}{4}\theta(x,y,t) \quad ,0<\mu\leq 1 \tag{21}$$

dependent upon the condition

$$\theta(x,y,0)=\sqrt{xy} \tag{22}$$

Employing ST to the two sides of equation (21) dependent upon the underlying condition (22), we get

$$\mathrm{S}[\theta(x,y,t)]=\left(\frac{u}{s}\right)\sqrt{xy}+\left(\frac{u}{s}\right)^\mu \mathrm{S}\left[\theta_{xx}+\theta_{yy}+\frac{1}{4}\theta\right] \tag{23}$$

Utilizing the converse ST to the two sides of (23)

$$\theta(x,y,t)=\sqrt{xy}-\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^\mu \mathrm{S}\left[\theta_{xx}+\theta_{yy}+\frac{1}{4}\theta\right]\right\} \tag{24}$$

Where $\sqrt{xy}$ address the term emerging from the source term.
Presently on Applying AD technique

$$\theta(x,y,t)=\sum_{n=0}^{\infty}\theta_n(x,y,t) \tag{25}$$

the non linear operator is due

$$N\theta(x,y,t)=\sum_{n=0}^{\infty}A_n(\theta) \tag{26}$$

Where A_n are the AD and is given by

$$A_n\left(\theta_0,\theta_1,\theta_2\ldots\ldots\ldots\theta_n\right)=\frac{1}{n!}\left\{\sum_{n=0}^{\infty}\frac{\partial^n}{\partial p^n}N\left[p^i\theta_i(x,y,t)\right]\right\},n=0,1,2\ldots\ldots$$

Substituting (25) and (26) in equation (24), we obtain Where

$$\begin{aligned}
&A_n=\theta_{nxx}^2+\theta_{nyy}^2, n\in N\\
&A_0=\theta_{0xx}^2+\theta_{0yy}^2\\
&A_1=2\left(\theta_0\theta_1\right)_{xx}+2\left(\theta_0\theta_1\right)_{yy}\\
&A_2=2\left(\theta_0\theta_2\right)_{xx}+\left(\theta_1^2\right)_{xx}+2\left(\theta_0\theta_2\right)_{yy}+\left(\theta_1^2\right)_{yy}\\
&\vdots
\end{aligned}$$

This is mixture of the ST and Adomian polynomial strategy looking at the like forces of the above condition are gotten

$$\theta_0(x,y,t)=\sqrt{xy} \tag{27}$$

$$\begin{aligned}
\theta_1(x,y,t)=&\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^\mu \mathrm{S}\left[A_0(\theta)+\frac{1}{4}\theta_0\right]\right\}\\
=&\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^\mu \mathrm{S}\left[\theta_{0xx}^2+\theta_{0yy}^2+\frac{1}{4}\theta_0\right]\right\}\\
=&\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^\mu \mathrm{S}\left[(\sqrt{xy})_{xx}^2+(\sqrt{xy})_{yy}^2+\frac{1}{4}\sqrt{xy}\right]\right\}\\
=&\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^\mu \mathrm{S}\left[\frac{1}{4}\sqrt{xy}\right]\right\}\\
=&\frac{1}{4}\sqrt{xy}\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^\mu \mathrm{S}[1]\right\}
\end{aligned}$$

$$\theta_1(x,y,t) = \frac{1}{4}\sqrt{xy}\frac{t^\mu}{\mu!} \tag{28}$$

$$\begin{aligned}
\theta_2(x,y,t) =& S^{-1}\left\{\left(\frac{u}{s}\right)^\mu S\left[A_1(\theta)+\frac{1}{4}\theta_1\right]\right\}\\
=& S^{-1}\left\{\left(\frac{u}{s}\right)^\mu S\left[2\left(\theta_0\theta_1\right)_{xx}+2\left(\theta_0\theta_1\right)_{yy}+\frac{1}{4}\theta_1\right]\right\}\\
=& S^{-1}\left\{\left(\frac{u}{s}\right)^\mu S\left[2\left(\frac{1}{4}(xy)\frac{t^\mu}{\mu!}\right)_{xx}+2\left(\frac{1}{4}(xy)\frac{t^\mu}{\mu!}\right)_{yy}+\frac{1}{16}\sqrt{xy}\frac{t^\mu}{\mu!}\right]\right\}\\
=& \frac{1}{16}\sqrt{xy}S^{-1}\left\{\left(\frac{u}{s}\right)^\mu S\left[\frac{t^\mu}{\mu!}\right]\right\}
\end{aligned}$$

$$\theta_2(x,y,t) = \frac{1}{16}\sqrt{xy}\frac{t^{2\mu}}{(2\mu)!} \tag{29}$$

$$\begin{aligned}
\theta_3(x,y,t) =& S^{-1}\left\{\left(\frac{u}{s}\right)^\mu S\left[A_2(\theta)+\frac{1}{4}\theta_2\right]\right\}\\
=& S^{-1}\left\{\left(\frac{u}{s}\right)^\mu S\left[2\left(\theta_0\theta_2\right)_{xx}+\left(\theta_1^2\right)_{xx}+2\left(\theta_0\theta_2\right)_{yy}+\left(\theta_1^2\right)_{yy}+\frac{1}{4}\theta_2\right]\right\}\\
=& S^{-1}\left[\left(\frac{u}{s}\right)^\mu S\left[2\left(\frac{1}{16}(xy)\frac{t^{2\mu}}{(2\mu)!}\right)_{xx}+\left(\frac{1}{16}(xy)\frac{t^{2\mu}}{(\mu!)^2}\right)_{xx}\right.\right.\\
&\left.\left.+2\left(\frac{1}{16}(xy)\frac{t^{2\mu}}{(2\mu)!}\right)_{yy}+\left(\frac{1}{16}(xy)\frac{t^{2\mu}}{(\mu!)^2}\right)_{yy}+\frac{1}{64}\sqrt{xy}\frac{t^{2\mu}}{(2\mu)!}\right]\right]
\end{aligned}$$

$$\theta_3(x,y,t) = \frac{1}{64}\sqrt{xy}\frac{t^{3\mu}}{(3\mu)!} \tag{30}$$

Continuing thusly, the rest terms of $\theta_n(x,y,t)$ can be found.
The solution $\theta(\mathrm{x,y,t})$ of equation (21) by using (27),(28),(29) and (30), as

$$\begin{aligned}
\theta(x,y,t) &= \lim_{p\to 1}\sum_{n=0}^{\infty}\theta_n(x,y,t)\\
\theta(x,y,t) &= \theta_o(x,y,t)+\theta_1(x,y,t)+\theta_2(x,y,t)+\theta_3(x,y,t)+\dots\dots\dots\\
\theta(x,y,t) &= \sqrt{xy}+\frac{1}{4}\sqrt{xy}\frac{t^\mu}{(\mu)!}+\frac{1}{16}\sqrt{xy}\frac{t^{2\mu}}{(2\mu)!}+\frac{1}{64}\sqrt{xy}\frac{t^{3\mu}}{(3\mu)!}+\dots\dots\dots\\
\theta(x,y,t) &= \sqrt{xy}\left[1+\frac{1}{4}\frac{t^\mu}{\mu!}+\left(\frac{1}{4}\right)^2\frac{t^{2\mu}}{(2\mu)!}+\left(\frac{1}{4}\right)^3\frac{t^{3\mu}}{(3\mu)!}+\dots\dots\dots\right]\\
\theta(x,y,t) &= \sqrt{xy}E_\mu\left(\frac{1}{4}t^\mu\right)
\end{aligned}$$

where $E_\mu(t)$ is Mittag-Leffler function in one parameter. when $\mu = 1$, then

$$\theta(x,y,t) = \sqrt{xy}e^{\frac{t}{4}} \tag{31}$$

Which is exact Solution of (21).

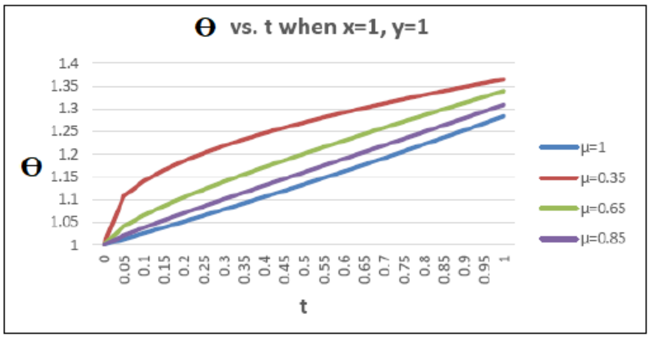

Figure 4: Comparison between w for different fractional orders.

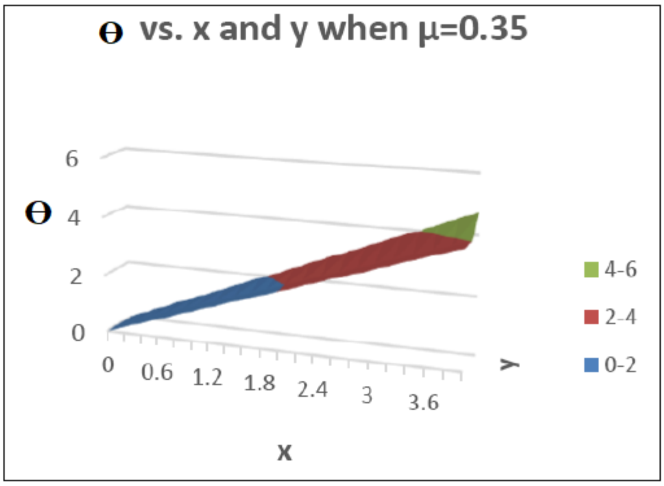

Figure 5: 3D behavior of an approximate solution for $\mu = 0.35$

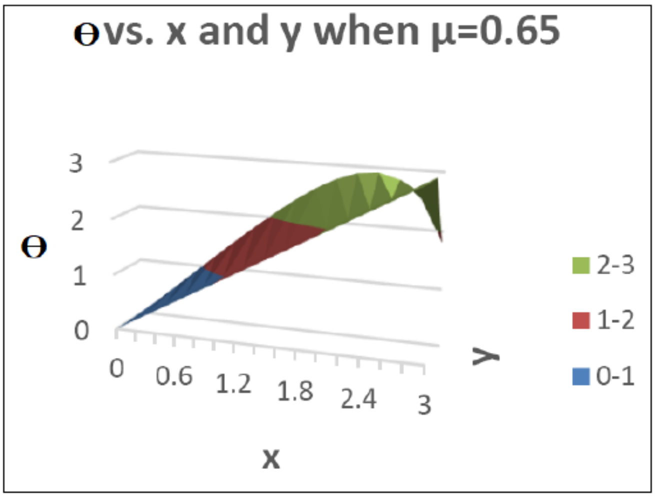

Figure 6: 3D behavior of an approximate solution for $\mu = 0.65$

4.3 Example

Consider the following time fractional biological population model's equation as

$$\frac{\partial^\mu \theta}{\partial t^\mu} = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \theta(x,y,t), 0 < \mu \leq 1 \tag{32}$$

dependent upon the condition

$$\theta(x,y,o) = \sqrt{\sin x \cdot \sinh y} \tag{33}$$

Employing ST to both sides of equation (32) subject to the initial condition (33), we get

$$\mathrm{S}[\theta(x,y,t)] = \left(\frac{u}{s}\right)\sqrt{\sin x \cdot \sinh y} + \left(\frac{u}{s}\right)^\mu \mathrm{S}\left[\theta_{xx} + \theta_{yy} + \theta\right] \tag{34}$$

Utilizing the converse ST to the two sides of (34)

$$\theta(x,y,t) = \sqrt{\sin x \cdot \sinh y} + \mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^\mu \mathrm{S}\left[\theta_{xx} + \theta_{yy} + \theta\right]\right\} \tag{35}$$

where $\sqrt{\sin x \cdot \sinh y}$ address the term emerging from the source term. Presently on Appling AD technique

$$\theta(x,y,t) = \lim_{p\to 1} \sum_{n=0}^{\infty} \theta_n(x,y,t) \tag{36}$$

The linear operator is due

$$N\theta(x,y,t)=\sum_{n=0}^{\infty}A_n(\theta) \tag{37}$$

Where A_n are the AD and is given by

$$A_n\left(\theta_0,\theta_1,_\theta 2\ldots\ldots\ldots.\theta_n\right)=\frac{1}{n!}\left\{\sum_{n=0}^{\infty}\frac{\partial^n}{\partial p^n}N\left[p^i\theta_i(x,y,t)\right]\right\},n=0,1,2\ldots\ldots$$

Substituting (36) and (37) in equation (35), we obtain

$$\sum_{n=0}^{\infty}\theta_n(x,y,t)=\sqrt{\sin x\cdot\sinh y}\cdot \mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}\left[\sum_{n=0}^{\infty}A_n(\theta)+\sum_{n=0}^{\infty}\theta_n\right]\right\} \tag{38}$$

Where

$$\begin{aligned}
&A_n=\theta_{nxx}^2+\theta_{nyy}^2, n\in N\\
&A_0=\theta_{0xx}^2+\theta_{0yy}^2-\frac{8}{9}\theta_0^2\\
&A_1=2\left(\theta_0\theta_1\right)_{xx}+2\left(\theta_0\theta_1\right)_{yy}\\
&A_2=2\left(\theta_0\theta_2\right)_{xx}+\left(\theta_1^2\right)_{xx}+2\left(\theta_0\theta_2\right)_{yy}+\left(\theta_1^2\right)_{yy}
\end{aligned}$$

This is mixture of the ST and Adomian polynomial strategy looking at the like forces of the above condition are gotten

$$\theta_0(x,y,t)=\sqrt{\sin x\cdot\sinh y} \tag{39}$$

$$\begin{aligned}
\theta_1(x,y,t)&=\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}\left[A_0(\theta)+\theta_0\right]\right\}\\
&=\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}\left[\theta_{0xx}^2+\theta_{0yy}^2+\theta_0\right]\right\}\\
&=\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}\left[(\sqrt{\sin x\cdot\sinh y})_{xx}^2+(\sqrt{\sin x\cdot\sinh y})_{yy}^2+\sqrt{\sin x\cdot\sinh y}\right]\right\}\\
&=\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}[\sqrt{\sin x\cdot\sinh y}]\right\}\\
&=\sqrt{\sin x\cdot\sinh y}\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}[1]\right\}
\end{aligned}$$

$$\theta_1(x,y,t)=\sqrt{\sin x\cdot\sinh y}\frac{t^{\mu}}{\mu!} \tag{40}$$

$$\begin{aligned}
\theta_2(x,y,t)&=\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}\left[A_1(\theta)+\theta_1\right]\right\}\\
&=\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}\left[2\left(\theta_0\theta_1\right)_{xx}+2\left(\theta_0\theta_1\right)_{yy}+\theta_1\right]\right\}\\
&=\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}\left[2\left(\sin x\cdot\sinh y\frac{t^{\mu}}{\mu!}\right)_{xx}+2\left(\sin x\cdot\sinh y\frac{t^{\mu}}{\mu!}\right)_{yy}+\sqrt{\sin x\cdot\sinh y}\frac{t^{\mu}}{\mu!}\right]\right\}\\
&=\sqrt{\sin x\sinh y}\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}\left[\frac{t^{\mu}}{\mu!}\right]\right\}
\end{aligned}$$

$$\theta_2(x,y,t)=\sqrt{\sin x\sinh y}\frac{t^{2\mu}}{(2\mu)!} \tag{41}$$

$$\begin{aligned}
\theta_3(x,y,t)&=\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}\left[A_2(\theta)+\theta_2\right]\right\}\\
&=\mathrm{S}^{-1}\left\{\left(\frac{u}{s}\right)^{\mu}\mathrm{S}\left[2\left(\theta_0\theta_2\right)_{xx}+\left(\theta_1^2\right)_{xx}+2\left(\theta_0\theta_2\right)_{yy}+\left(\theta_1^2\right)_{yy}+\theta_2\right]\right\}
\end{aligned}$$

$$\theta_3(x,y,t)=\sqrt{\sin x\sinh y}\frac{t^{3\mu}}{(3\mu)!} \tag{42}$$

Continuing thusly, the rest terms of$\theta_n(x,y,t)$ can be found.
The solution $\theta(x,y,t)$ of equation (32) by using (39),(40),(41) and (42), as

$$\theta(x,y,t) = \lim_{p\to 1}\sum_{n=0}^{\infty}\theta_n(x,y,t)$$
$$\theta(x,y,t) = \theta_o(x,y,t)+\theta_1(x,y,t)+\theta_2(x,y,t)+\theta_3(x,y,t)+\ldots\ldots\ldots$$

$$\begin{aligned}\theta(x,y,t) =&\sqrt{\sin x\sinh y}+\sqrt{\sin x\sinh y}\frac{t^{\mu}}{(\mu)!}+\sqrt{\sin x\sinh y}\frac{t^{2\mu}}{(2\mu)!}\\&+\sqrt{\sin x\sinh y}\frac{t^{3\mu}}{(3\mu)!}+\ldots\ldots\ldots\\\theta(x,y,t) =&\sqrt{\sin x\sinh y}\left[1+\frac{t^{\mu}}{\mu!}+\frac{t^{2\mu}}{(2\mu)!}+\frac{t^{3\mu}}{(3\mu)!}+\ldots\ldots\ldots\right]\\\theta(x,y,t) =&\sqrt{\sin x\sinh y}E_{\mu}\left(t^{\mu}\right)\end{aligned}$$

Where $E_{\mu}(t)$ is Mittag-Leffler function in one parameter. When $\mu=1$, then

$$\theta(x,y,t) = \sqrt{\sin x\cdot\sinh y}\cdot e^{t}$$

Which is exact Solution of (32).

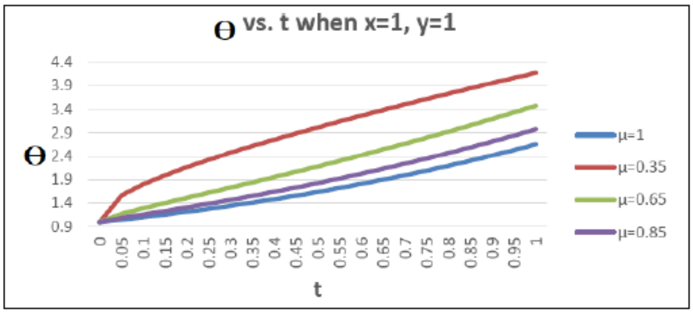

Figure 7: Comparison between w for different fractional orders.

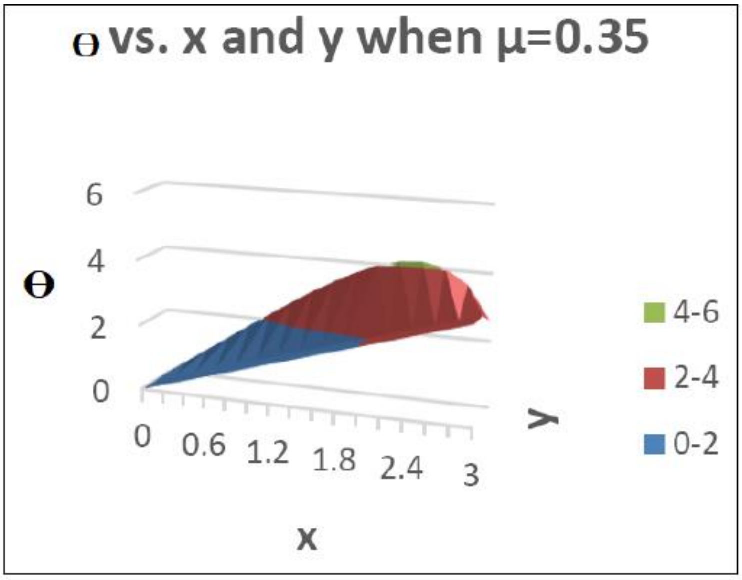

Figure 8: 3D behavior of an approximate solution for $\mu = 0.35$

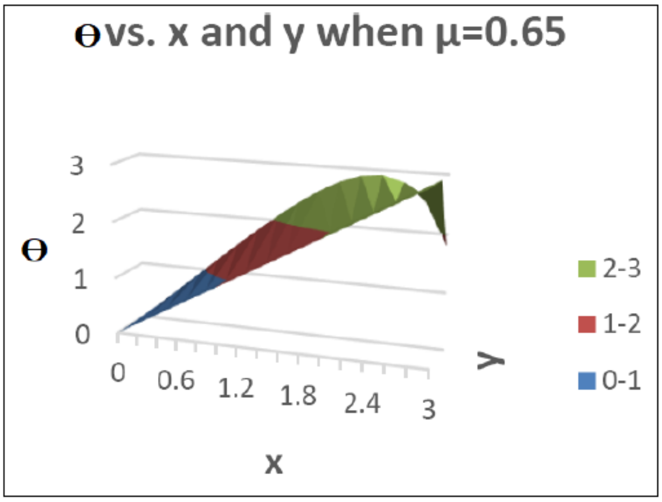

Figure 9: 3D behavior of an approximate solution for $\mu = 0.65$

5 Applications

1. Enviromental scientists use the models to describe how populations grow over time.
2. Population biologists frequently used mathematical models to investigate the behaviors of animal populations.
3. Population models are also used to understand the spread of parasites, viruses and disease.
4. Polpulation models are used to determined maximum harvest for agriculturists to understand the dynamics of biological invasions and for enviromental conservation.

6 Conclusion

This paper intends to show the applicability of the SADM to obtain an numerical solution in terms of convergent series for nonlinear time fractional order BPM. The evaluations shows that the proposed method (SADM) is extremely successful and suitable and is an influential and professional tool for fractional order BPM. Because the obtained results using proposed techniques are in an excellent agreement with the exact solution. We also conclude that the complete territory of population density is shown in different plotted graphs.

Acknowledgments

The author offers his grateful thanks to friends and the referee for their help and active guidance in the preparation of this revised paper.

References

[1] I. Podlubny, *Fractional Differential Equations Calculus*, Academic, Press, New York; 1999.

[2] K.B. Oldham, J. Spanier, *the Fractional Calculus, Academic Press*, New York; 1974.

[3] S. Aggarwal, Anjana Gupta and Swarg Sharma, A New Application of Shehu Transform for Handling Volterra Integral Equations of First Kind, In. J. of Res. in Advent Tech, **7**(4) (2019), 439-445.

[4] S. Aggarwal and Anjana Gupta, Shehu Transform for Solving Abel's Integral Equation, JETIR, **6**(5) (2019),101-110.

[5] R. Belgacem, D. Baleanu and Ahmed Bokhari, Shehu Transform and Applications to Caputo-Fractional Differential Equations, Int. J. of Analysis and Appl., **17**(6),(2019),917-927.

[6] Ahmed Bokharia , D.Baleanub & R. Belgacema, Application of Shehu transform to Atangana-Baleanu Derivatives, J. Math. Computer Sci., 20 (2020), 101–107.

[7] Shehu Maitama, W. Zhao, New Integral Transform: Shehu Transform A Generalization of Sumudu and Laplace Transform for Solving Differential Equations. School of Mathematics, Shandong University, Jinan, Shandong, P. R. Chaina. (2019),1-22.

[8] Shehu Maitama & W. Zhao, New Homotopy Analysis Transform Method for Solving Multidimensional Fractional Diffusion Equations, Arab J. of Basic and Appl. Sci., **27**(1) (2019), 27-44.

[9] G.Adomian, *A review of the decomposition method and some recent results for nonlinear equations*, Comput. Math. Appl.,V-**21**,I-5,pp: 101-127,1991.

[10] G.Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer, Dordrecht, 1994.

[11] D. Ziane, R. Belgacem, and Ahmed Bokhari,*A new modified Adomian decomposition method for nonlinear partial differential equations*, Open J. Math. Anal, Vol-**3**(2), pp:81-90,2019.

[12] L. Akinyemi, O. Iyiola,*Exact and approximate solutions of time-fractional models arising from physics via Shehu transform*, Mathematical Methods in Applied Science., ISSN:**1099-1476**,April 2020.

[13] S. Nubpetchploy, P. Chaisuwan, D. Poltem,*Shehu Adomian Decomposition Method for Solving Fractional Integro-Differential Equation*, Burapha Science Journal, Vol-**26**,1, pp:232-248, 2021.

[14] B. M. Yisa, M. A. Baruwa,*Shehu Transform Adomian Decomposition Method For The Solution of Linear and Nonlinear Integral and Integro-Differential Equations*, J. of the Nigerian Mathematical Society , Vol-**41**,2, p:105-128, 2022.

[15] M. Kapoor, N. Ali Shah, W. Weera, *Analytical solution of time fractional Schrodinger equations via Shehu Adomian Decomposition Method*, AIMS Mathematics, Vol-**7**,10, 19562–19596. DOI: 10.3934/math.20221074, 2022.

[16] M. Liaqat , Adnan Khan , Md. Ashraful Alam, *Approximate and Closed-Form Solutions of Newell-Whitehead-Segel Equations via Modified Conformable Shehu Transform Decomposition Method*, Maths Problems in Engg, Article ID-**6752455**, pp:1-14,2022.

[17] Yunqi Xu, D.Lu, *Study on approximate solution of fractional order biological population Model*, Archives of Current Res. Int., Vol-**15**, I-4 pp: 1-11, 2018.

[18] M.E. Gurtin, Richard C. *On the diffusion of biological populations Mathematical Biosciences*, Vol-**33**, I-1-2, pp-35-49, 1977.

[19] Yun - Guang Lu, *Holder Estimates of solutions of biological population equations, Applied Mathematics Letters*, Vol-**13**, pp: 123-126,2000.

[20] F.Shakeri, M.Dehghan, *Numerical solution of a biological population model using He's Variational Iteration Method*, Computer's and Mathematics with applications, Vol-**54**, pp: $1197-1209, 2007$.

[21] A.M.A. El- Sayed,S.Z. Rida, *Exact Solution of Fractional Order Biological Population Model*, Chinese physical society IOP, Publishing Ltd., Vol-**52**, pp: 992-996, 2009.

[22] R.N. Prajapati,R.Mohan, *Acoupling Technique for analytical solution of Time Fractional Biological Population Model*, J.of Engg. Computers and App. Sci., Vol-**2**, I-2, pp: 30-38, 2013.

[23] V.G.Gupta, Pramod Kumar, *Approximate solutions of fractional biological Population model by homotopy analysis Sumudu transform method*, Int. J.Sci.and Res, Vol-**5**,I-5,pp:908-916, 2016.

[24] Samia Bushanaq, Sajjad Ali, *Exact Solution to Non-linear biological population model with Fractional Order*, Thermal Science, Vol-**22**, I-1, pp:317-327,2018.

[25] P R Bhadane, K P Ghadle, *Approximate Analytical Solution of Fractional Gas Dynamic Equation By ETHPM*, J. of Basic and App. Res. Int.,Vol-**20**, I-1,pp: 29-36, 2017.

[26] P R Bhadane, K P Ghadle, *Approximate solution of Fractional Black-Schole's european option pricing equation by using ETHPM*, Nonlinear Functional Analysis and Applications Vol. **25**, I-2,pp:3331-334,2020.