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Review Paper

A study on Lower and Upper bounds of packing chromatic number of Cartesian product of Graphs and subdivision of Graphs

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Abstract: *The packing chromatic number $\chi_{pc}(H)$ of a graph H is the least integer m in such a way that there is a mapping $C : V(H) \to \{1,2,\dots,m\}$ such that the distance between any two nodes of colour l is greater than $l + 1$. In this paper, we compute the lower bound for the packing chromatic number and upper bound for the packing chromatic number of Cartesian product of any two graphs and the lower and upper bound for the packing chromatic number of subdivision of graphs.*
Keywords: *Packing chromatic number, Cartesian product of graphs, Subdivision graph.*

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I. INTRODUCTION

Let $H = (V(H), E(H))$ be a graph with n vertices and m edges, we are asked to colour its vertices such that any two adjacent vertices in the graph not sharing the same colour, now we are in a position to count the minimum requirement of colour to satisfying the above said technique. This originates a colouring problem. Having coloured the vertices we can classify them into non-identical sets such that one set containing of one colour and the other set containing of other colour and so on. This is partitioning problem. The colouring and partitioning can be done on vertices or edges of a graph.
A *proper colouring* is nothing but colouring all the vertices in which no adjacent vertices have the same colour.

Definition 1.1. A proper k –vertex colouring of a simple graph $H = (V, E)$ is defined as a vertex colouring from a set of k colours such that no two adjacent vertices share same colour.
$C : V(H) \to \{c_1, c_2, \dots, c_k\}$ Such that $\forall u, v \in E(H) / C(u) \neq C(v)$.

The intention of the packing chromatic number of a graph was established in [2] by Goddard et al. There will be more difficulty to identify the points in a mesh where two non-identical points are managing the unchanged frequency unless both the points are positioned not to very close. With this representation all points are located at nodes in the i packing, X_i are authorized to transmit at the same frequency only after distance i. This ideology has been developed a lot and used in many fields such as biological diversity, resource placements and so on.

Let $H = \left(V(H), E(H)\right)$ be a graph and let r be any non-negative integer. Then X be the subset of the vertex set of the graph H, is a r-packing provided the vertices of X are pairwise at a distance greater than the value of r. The uttermost number of a r-packing in a graph H is called the r-packing number of H indicated by $\rho_r(H)$, this non-negative integer r is nothing but the width of the packing X.
Definition 1.2 [7] Let H be any graph and we need to find the smallest integer l in such a way that the vertex set of the graph H, $V(H)$ can be partitioned into l packings as $X_1, X_2, \dots, X_l$ with pairwise unlike widths. As above mentioned split-up has packings of l inconsistent widths and the aim is to minimize l, the non-negative integer l meant as the packing chromatic number of H and marked it by $\chi_{pc}(H)$.

Definition1.3: **[9]** Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be any two simple graphs with $V_1 = \{v_1, v_2, \dots, v_i\}$ and $V_2 = \{v_1', v_2', \dots, v_j'\}$.Then the direct product of the graphs G and H denoted as $G \times H$ and defined as the Cartesian product of the vertex sets V_1 *and* V_2 in such a way that any two different vertices (v_1, v_1') and (v_2, v_2') of $G \times H$ are adjacent if $v_1 v_2 \in E_1(G)$ and $v_1' v_2' \in E_2(H)$.
We clearly know that associative and commutative properties holds for Cartesian product of two graphs, see **[8]**
The subgraph of $G_1 \times G_2$ induced by $\{a\} \times V(G_2)$ is isomorphic to G_2.It is called *G- fiber* G^b for a vertex b of G_2.
Definition1.4: **[10]** Let G_1 be any connected graph, The diameter, denoted as $\dim(G_1)$ and defined as the maximum distance between any two vertices of G_1, that is,
$\dim(G_1) = \{a \in V(G_1) / \, d(a, b) \leq m\}$.
Let H be any graph then the order of highest complete subgraph of H is denoted as $\omega(H)$ and the order of the highest independent set of H is denoted as $\gamma(H)$.
In this paper, Section 1, we have given the introduction about the graph colouring and preliminary definitions of colouring of the graph. In Section 2, we discuss about the lower bound for the packing chromatic number of Cartesian product of any two graphs, in Section 3 we scrutinize about the upper bound for the packing chromatic number of Cartesian product of any two graphs and in Section 4, we study about the lower and upper bound for packing chromatic number of subdivision of graph.

II. LOWER BOUND FOR PACKING CHROMATIC NUMBER OF CARTESIAN PRODUCT OF GRAPHS.

In this section we consider the lower bound for packing chromatic number of Cartesian product of graphs. We clearly know the result $\dim(G_1 \times G_2) = \dim(G_1) + \dim(G_2)$.We will use this result in the forthcoming proof.

Theorem 2.1: Let G_1 & G_2 be any two connected graphs with at least two vertices and order of the graphs G_1 & G_2 denoted as $O(G_1)$ and $O(G_2)$ respectively then

$$\chi_{pc}(G_1 \times G_2) \geq \left[\chi_{pc}(G_1) + 1\right] O(G_1) - dim(G_1 \times G_2)[O(G_2) - 1\,] - 1.$$

Proof. We are given that G_1 & G_2 are any two connected graphs with greater than or equal to two vertices. Order of the graph G_1 and order of the graph $G_2$are denoted as $O(G_1)$ and $O(G_2)$ respectively.

Let y be any integer such that $\chi_{pc}(G_1) = \dim(G_1) + y \;\; \dots \;\; (1)$

Claim: $\chi_{pc}(G_1 \times G_2) \geq \left[\chi_{pc}(G_1) + 1\right] O(G_1) - dim(G_1 \times G_2)[O(G_2) - 1\,] - 1.$

Let us divide this proof into two categories, one is holding the values of y which are less than or equal to zero and the other one is greater than equal to one, note that the values are belonging to integers.

Case (i): If $y \leq 0$

Clearly $(O(G_2) - 1) \leq (O(G_2) - 1)\, dim\big(O(G_2)\big)$

$$\begin{aligned}
\text{Therefore } \chi_{pc}(G_1 \times G_2) &\geq \chi_{pc}(G_1) \\
&\geq \dim(G_1) + y \\
&\geq \dim(G_1) + O(G_2)y \quad (\text{ because } y \leq 0\text{ }) \\
&\geq \dim(G_1) + O(G_2)y + (O(G_2) - 1) - (O(G_2) - 1)\dim(G_2) \text{ as } y \leq 0 \\
&\geq \dim(G_1) + O(G_2)y + O(G_2) - 1 - O(G_2)\dim(G_2) + \dim(G_2) \\
&\geq \dim(G_1) + \dim(G_2) + O(G_2)y - O(G_2)\dim(G_2) + O(G_2) - 1 \\
&\geq dim(G_1 \times G_2) + O(G_2)[y - \dim(G_2) + 1] - 1
\end{aligned}$$

This proves the claim for $y \leq 0$

Case (i): If $y \geq 1$

As $O(G_2) > \dim(G_2)$, assume that $y - \dim(G_2) + 1 < 0$

$$\begin{aligned}
\chi_{pc}(G_1 \times G_2) &\geq dim(G_1) + y \\
&\geq \dim(G_1) + 1 \\
&\geq \dim(G_1) + [\dim(G_2) - O(G_2)] + 1 \\
&\geq \dim(G_1) + \dim(G_2) - O(G_2) + 1
\end{aligned}$$

$$\geq dim(G_1 \times G_2) + O(G_2)[y - \dim(G_2) + 1] + 1$$
$$\geq dim(G_1 \times G_2) + O(G_2)[y - \dim(G_2) + 1] - 1$$

Beginning of this case, we have assumed that $y - \dim(G_2) + 1$ is negative, suppose that $y - \dim(G_2) + 1$ is zero that is , $y - \dim(G_2) + 1 = 0$ which implies that $\dim(G_2) = y + 1$ and therefore

$$\begin{aligned}\chi_{pc}(G_1 \times G_2) &\geq dim(G_1) + y \\ &\geq (dim(G_1) + y + 1) + 0 - 1 \\ &\geq (\dim(G_1) + \dim(G_2)) + (y - \dim(G_2) + 1) - 1 \\ &\geq dim(G_1 \times G_2) + O(G_2)[y - \dim(G_2) + 1] - 1\end{aligned}$$

We can conclude that $y - \dim(G_2) + 1 \geq 1$

Let us assume that at most $dim(G_1 \times G_2) + O(G_2)[y - \dim(G_2) + 1] - 2$ colours to colouring $G_1 \times G_{2.}$ then we can choose every colours from the following interval which is constructed as below,

$I = [\dim(G_1 \times G_2), \dim(G_1 \times G_2) + O(G_2)(y - \dim(G_2) + 1) - 2]$

This can be used not more than one time. Clearly it indicates that the interval having $O(G_2)(y - \dim(G_2) + 1) - 1$ colours. For that reason, there exists G-fiber, say G^h, that uses not more than $y - \dim(G_2)$ colours from the interval which mentioned above. But then the G- fiber coloured with not more than $(\dim(G_1) + \dim(G_2) - 1) + (y - \dim(G_2)) = (\dim(G_1) + \dim(G_2) - 1 + y - \dim(G_2) = \dim(G_1) + y - 1) = \chi_{pc}(G_1) - 1$ colours.

This is contradiction to the fact that the fiber G^h is isomorphic to G_1.Hence the claim is proved for $y \geq 1$.

From case (i) and (ii), claim is proved.

From the above, $\dim(G_1) + y - 1 = \chi_{pc}(G_1) - 1$

$$y = \chi_{pc}(G_1) - 1 - \dim(G_1) + 1$$
$$y = \chi_{pc}(G_1) - \dim(G_1)$$

Then put $y = \chi_{pc}(G_1) - \dim(G_1)$ in the claim, we have

$$\begin{aligned}&\chi_{pc}(G_1 \times G_2) \geq dim(G_1 \times G_2) + O(G_2)[y - \dim(G_2) + 1] - 1 \\ &\geq \dim(G_1) + \dim(G_2) + O(G_2)\left[\chi_{pc}(G_1) - \dim(G_1) - \dim(G_2) + 1\right] - 1 \\ &\geq \dim(G_1) + \dim(G_2) + \chi_{pc}(G_1)\, O(G_2) - [\dim(G_1) + \dim(G_2)]O(G_2) + O(G_2) - 1 \\ &\geq \chi_{pc}(G_1)\, O(G_2) + O(G_2) + dim(G_1 \times G_2) - dim(G_1 \times G_2)O(G_2) - 1 \\ &\geq \left[\chi_{pc}(G_1) + 1\right]O(G_2) - dim(G_1 \times G_2)[O(G_2) - 1] - 1\end{aligned}$$

Hence the theorem

We clearly know that, Cartesian product of two graphs satisfies commutative property, therefore the above result can also be written as,

$\chi_{pc}(G_1 \times G_2) \geq \left[\chi_{pc}(G_2) + 1\right]O(G_1) - dim(G_1 \times G_2)[O(G_1) - 1] - 1$

III. UPPER BOUND FOR PACKING CHROMATIC NUMBER OF CARTESIAN PRODUCT OF GRAPHS.

Next we compute the upper bounds for packing chromatic number for any two graphs G_1 & $G_{2.}$

$(i)\ \chi_{pc}(G_1 \times G_2) \leq O(G_1).O(G_2) - \delta(G_1 \times G_2) + 1$

Due to the complexity of finding the independence number of Cartesian product, also see[1,3,4,5], the above result is more convenient after substituting lower bound for $\delta(G_1 \times G_2)$**.** By the result of Vizing [**6**];

$\delta(G_1 \times G_2) \geq \delta(G_1)\delta(G_2) + min\{O(G_1) - \delta(G_1)\, , O(G_2) - \delta(G_2)\}$, it resulting

$\chi_{pc}(G_1 \times G_2) \leq O(G_1).O(G_2) - \delta(G_1)\delta(G_2) - min\{O(G_1) - \delta(G_1)\, , O(G_2) - \delta(G_2)\} - 1$

The above inequality for $2 \leq x \leq y$ gives

$\chi_{pc}(K_x \times K_y) \leq x(y-1)+1$. Instead Theorem 1 gives the following results,
$\chi_{pc}(K_x \times K_y) \geq y(x-1)+1$and $\chi_{pc}(K_x \times K_y) \geq x(y-1)+1$ .Finally we bring to an end that
$\chi_{pc}(K_x \times K_y) = x(y-1)+1$

By [3], which states that for any graph G_1 and any bipartite graph G_2,

$$\delta(G_1 \times G_2) \geq \alpha(G_1).O(G_2)/2.$$

Here $\alpha(G_1)$ indicates the 2 – independence number of G_1, which means that the proportion of the largest union of 2 independence sets in $V(G_1)$.

$\therefore$ (ii) $\chi_{pc}(G_1 \times G_2) \leq O(G_1).O(G_2) - \alpha(G_1).O(G_2)/2.$

The above inequality carries for any two graphs provided one is bipartite graph.

IV. LOWER AND UPPER BOUND FOR PACKING CHROMATIC NUMBER OF SUBDIVISION GRAPH.

The subdivision graph $S_d(H)$ of a graph H is procured from H by subdividing every edge of H. Then $V(S_d(H)) = V(H) \cup E(H)$. The following result perceived in [**2**] for a connected bipartite graph H with minimum of two edges, that is $\chi_{pc}(S_d(H)) = 3$.
Here we observe the subdivision graph of any random graph. Let us consider the complete graph with at least 3 vertices.

Proposition: 4.1 $\chi_{pc}(S_d(K_x)) = x+1$ for any $x \geq 3$

Proof: Suppose $x = 3$, we clearly know that $S_d(K_3) = C_6$ and we know that the packing chromatic number of the graph C_6 is 4. This satisfies the claim.
Now let us consider the value of x is greater than or equal to 4, then $V(K_x) = \{u_1, u_2, \dots, u_n\}$ and for the edge $u_i u_j$, $1 \leq i, j \leq x$, let u_{ij} indicate the vertex of the subdivision graph of the complete graph, which is acquired by subdividing $u_i u_j$.

Let $\pi : V(S_d(K_x)) \rightarrow \{1,2,3,\dots\}$ be a colouring. We need to prove that there exists $u \in S_d(K_x)$ such that $\pi(u) \geq x+1$.

If $\pi(u_i) > 1$ for all the values of i then $\pi(u_i) \neq \pi(u_j)$ for $i \neq j$, for some l, $\pi(u_l) > x$. Let us assume that $\pi(u_1) = 1$ then it can be proceeded with $\pi(u_{1j}) \neq \pi(u_{1l})$ for $j \neq l$ which proves the claim unless $\{\pi(u_{12}), \pi(u_{13}), \dots, \pi(u_{1x})\} = \{2,3,\dots,x\}$

On the other hand let us assume that $\pi(u_{12}) = 2$, clearly we know that $d(u_{1j}, u_2) \leq 3$, we deduce that $\pi(u_2) \notin \{2,3,\dots,x\}$ and so $\pi(u_2) = 1$.

Let us consider the set $T = \{u_{23}, u_{24}, u_{2x}\}$ and more precisely $|T| \geq 2$. We can have that they can only get colour 3 because the distance between the vertices x and u_{1j} are at most 4, otherwise a colour bigger than x is used. But we already have $\pi(u_{23}) = \pi(u_{24}) = 3$, which is contradiction to our fact. Hence $\chi_{pc}(S_d(K_x)) = x + 1$ for any $x \geq 3$.

Theorem 4.2: Let H be any connected graph with minimum of three vertices then

$$\omega(H) + 1 \leq \chi_{pc}(S_d(H)) \leq \chi_{pc}(H) + 1$$

Proof: Let us define the packing chromatic colouring of the graph H as follows
$\pi : V(H) \rightarrow \{1,2,\dots,m\}$. Also define $\tilde{\pi} : V(S_d(H)) \rightarrow \{1,2,\dots,m+1\}$ by assigning $\tilde{\pi}(e) = 1$ for any edge $e \in E(H)$ and $\tilde{\pi}(u) = \pi(u) + 1$ for every vertex $u \in V(H)$.
Suppose $\tilde{\pi}(x) = \tilde{\pi}(y) = \gamma$ for some $x, y \in V(S_d(H))$. Also note that $\pi(x) = \pi(y)$ since $\tilde{\pi}(x) = \tilde{\pi}(y)$. And therefore we denote $d_{S_d(H)}(x,y) = 2d_H(x,y) \geq 2\gamma \geq \gamma + 1$.This is clearly proving the upper bound. By the above proposition, stated as $\chi_{pc}(S_d(K_x)) = \chi_{pc}(K_x) + 1$ for any $x \geq 3$, which satisfies the upper bound of our claim.
For proving the lower bound let us consider two cases.

Case (i): If $\omega(H) = 2$

We clearly know that , if the packing chromatic number is 2, then the connected graph would be only of Stars, that is $K_{1,m}$.
As $S_d(H)$ can be a star only when $H = K_2$, then we conclude that $\chi_{pc}(S_d(H)) > 2$ for every connected graph H with greater than of order 2.

Case (i): If $\omega(H) = n > 3$

Let H' be the subgraph of H, then $\chi_{pc}(H) \geq \chi_{pc}(H')$, then by the above proposition, stated that $\chi_{pc}(S_d(H)) \geq \chi_{pc}\big(S_d(K_x)\big) = x + 1$

From case (i) and case(ii), the lower bound is satisfied.

Hence $\omega(H) + 1 \leq \chi_{pc}(S_d(H)) \leq \chi_{pc}(H) + 1$ for any connected graph with minimum of three vertices.

V. CONCLUSION

In this paper we described on the upper bound of the packing chromatic number of Cartesian product of any two graphs and the lower bound of the packing chromatic number of Cartesian product of any two graphs also the upper and lower bounds of the packing chromatic number of subdivision graphs.
More precisely we have derived that the complete graph with more than 3 vertices has the packing chromatic number exceeds by one with the number of vertices on it.

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