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Review Paper

Triangular Numbers and Squares: an Asymptotic Result

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I. Squares

Given the positive real number, x, let $f(x)$ equal the number of squares less than or equal to x. It is not hard to see that $f(x) = \lfloor \sqrt{x} \rfloor$.

Example: $f(10) = \lfloor \sqrt{10} \rfloor = 3$. The three squares less than or equal to 10 are 1, 4, and 9.

Squares satisfy many beautiful identities:

1. $1 + 3 + 5 + \ldots + (2n - 1) = n^2$
2. $\dfrac{(n-1)^2 + (n+1)^2}{2} = n^2 + 1$
3. $\left\lfloor \dfrac{(n-1)^2 + n^2 + (n+1)^2}{3} \right\rfloor = n^2$

II. Triangular Numbers

The n-th triangular number, $t_n = 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$. See [1]. Triangular numbers satisfy many beautiful identities:

1. $t_n + t_{n-1} = n^2$
2. $t_n^2 - t_{n-1}^2 = n^3$
3. $[3(2n + 1)]^2 = t_{9n+4} - t_{3n+1}$
4. $n^2 - (n - 1)^2 + (n - 2)^2 - (n - 3)^2 + \ldots + (-1)^{n+1} = t_n$
5. $\left\lfloor \dfrac{t_{n-1} + t_{n+1}}{2} \right\rfloor = t_n$
6. $\left\lfloor \dfrac{t_{n-1} + t_n + t_{n+1}}{3} \right\rfloor = t_n$
7. $t_{3n+1} = t_{2n+1} + t_{2n} + t_n$
8. $t_n + t_{n+a} = n^2 + (a+1)n + t_a$

III. Main Result

Given the positive real number, x, let $g(x)$ equal the number of triangular numbers less than or equal to x. In other words, we seek the greatest natural number, n, such that $\dfrac{n(n+1)}{2} \le x$. Then we have $n(n + 1) \le 2x$

which becomes $n^2 + n - 2x < 0$. Now the graph of $h(n) = n^2 + n - 2x$ is an upright parabola with positive n-intercept, $\frac{-1+\sqrt{1+8x}}{2}$. Then $g(x) = \left\lfloor \frac{-1+\sqrt{1+8x}}{2} \right\rfloor$. We examine the limit, as x goes to infinity, of $\frac{g(x)}{f(x)}$. We will use asymptotics for $g(x)$ and $f(x)$, and then take their ratio. We have $f(x) \sim \sqrt{x}$ and $g(x) \sim \frac{\sqrt{8x}}{2} = \sqrt{2x}$. Then

$$\boxed{\lim_{x\to\infty} \frac{g(x)}{f(x)} = \lim_{x\to\infty} \frac{\sqrt{2x}}{\sqrt{x}} = \sqrt{2}}$$

Reference

[1]. M.Lewinter, J.Meyer, *Elementary Number Theory with Programming*, Wiley & Sons, 2015.