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Review Paper

On some nonlinear fractional integro-partial differential equations with respect to functions

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Abstract
Some nonlinear fractional integro-partial differential equations with respect to functions are considered. The solutions of the considered equations are given. With the help of the theory parabolic partial differential equations, we obtain exact solutions and the uniqueness of the solutions.

Keywords: Cauchy problem, Parabolic partial differential equations- Fractional integral equations with respect to functions- Integro partial differential equations- Nonlinear integral differential equations.

Mathematics Subject Classifications: 34A12- 34A40- 37D60- 47D62- 43G20.

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1-Introduction

Consider the following nonlinear fractional integro-partial differential equations with respect to functions:

$$u(x,t) = \varphi(x) + \frac{1}{\Gamma(\alpha)} \int_0^t [\{\psi(t)-\psi(s)\}^{\alpha-1} \frac{d\psi(s)}{ds} \big(Lu(x,s)+v(x,s)\big)]\, ds \ , \qquad (1.1)$$

Where$0 < \propto \leq 1$, $\Gamma(.)$ is the gamma function, ψ is a given bijective function, continuous and nonnegative on an interval $J=[0,T], T>0$, such that $\psi(0)=0$ and the derivative $\frac{d\psi(t)}{dt}$ is continuous and positive on J,
$Lu(x,t) = \sum_{q=2m} a_q(x) D_x^q u(x,t)$, $q=(q_1,\dots,q_n)$ is a multi-index,

$$v(x,t) = f(x,t,w_1(x,t),\dots,w_k(x,t)), \qquad (1.2)$$

$$w_i(x,t) = \sum_{q<2m} b_{qi}(x,t) D_x^q u(x,t),$$

f is defined on $R^n X\, J\, X\, S$, where S is a bounded and closed subset of R^k.

We suppose that f is continuous and bounded on $R^n XJXS$ and satisfies the following Lipchitz condition:

$|f(x,t,w_1,\dots,w_k) - f(x,t,w_1^*,\dots,w_k^*| \leq K|\sum_{i=1}^{k}|w_i - w_i^*|$, (1.3)

for all $w_i, w_i^* \in S$, $(x,t) \in R^n XJ$, K is a positive constant , (K is independent of x,t,w_i,w_i^*).

The linear partial differential operator L is supposed to be uniformly elliptic on R^n , i.e. there is a positive number c such that for every $x \in R^n$ and for every $y \in R^n$, the following inequality is satisfied:

$(-1)^{m-1}\sum_{|q|=2m} a_q(x)y^q \geq c|y|^{2m}$, (1.4)

($|y|^2 = y_1^2 + \cdots + y_n^2$) , the number c is independent of x, t *and* y, φ is a given bounded continuous function on R^n.

Let suppose the following conditions

(C1): All the coefficients a_q are continuous and bounded on R^n, $|q| = 2m$,

($C2$): *All the coefficients a_q satisfy a $H\ddot{o}lder$ condition ,namely there exists a positive constant K and a constant $\beta\epsilon(0,1)$ such that*

$|a_q(x) - a_q(y)| \leq K|x-y|^\beta$, *for all* $x, y \in R^n$, $t \in J$,

($C3$): *The partial derivatives $D_x^q a_q(x)$ are continuous and bounded on* R^n,

($C4$): *All the coefficients $b_{qi}(x,t)$ and all the partial derivatives $D_x^q b_{qi}(x,t)$ are bounded and continuous on $R^n XJ$, $i = 1,\dots,k$, $|q| < 2m$.*

In section 2, we shall write some results about the Cauchy problem of parabolic partial differential equation. In section 3, we shall solve equation (1.1)

2- A Cauchy problem

Consider the following Cauchy problem:

$\frac{\partial u(x,t)}{\partial t} = Lu(x,t)$, **(2.1)**

$u(x,0) = \varphi(x)$, **(2.1)**

Where $\boldsymbol{\varphi(x)}$ is a continuous and bounded functions defined on $\boldsymbol{R^n}$.

According to the conditions (C1),…,(C3), there is a Green function $G(x,y,t)$ such that the unique solution of the Cauchy problem (2.1), (2.2) is given by:

$u(x,t) = \int_{R^n} G(x,y,t)\varphi(y)dy$.

The Green function G satisfies the following properties:

(P1): $D_x^q G$, $D_y^q G$ **are continuous and bounded on $R^n X R^n XJ$, $|q| \leq 2m$,**

(P2): $\}D_x^q D_y^p G(x,y,t) \leq \frac{K\eta(x-y,t)}{t^{\nu_1}}$, where $\eta(x,t) = \exp[-\nu_2 \, |x|^{2m}/t^{1/2m-1}]$, $\nu_{1=} \frac{1}{2m(|p|+|q|+n)}$, K, ν_2 are positive constants and $|q| < 2m, |p| \leq 2m$, $p = (p_1,\dots,p_n)$ is a multi-index,

see [1].

3- Uniqueness and existence of solutions

Theorem 3.1.
Let u , $D_x^q u$ be continuous and bounded functions on $R^n XJ$. If u is a solution of (1.1), then that solution is unique.

Proof. According to the results in [2-4], we can write

$$u(x,t) = \int_0^\infty \int_{R^n} G\big(x,y,\theta\psi^\alpha(t)\big)\varphi(y)dyd\theta +$$

$$\int_0^t \int_0^\infty \int_{R^n} \Lambda(t,s,\Theta) G(x,y,\theta(\psi(t)-\psi(s)^\alpha)v(y,s)dyd\theta ds, \quad (3.1)$$

Where $\Lambda(t,s,\theta) = \alpha\theta\big(\psi(t)-\psi(s)\big)^{\alpha-1}\zeta_\alpha(\theta)$, ζ_α is a probability density function defined on $(0,\infty)$.

Let u_1, u_2 be two solutions of equation (1.1). Using (3.1), we can write:

$$u_1(x,t) - u_2(x,t) = \int_0^t \int_0^\infty \int_{R^n} \Lambda(t,s,\Theta) G(x,y,\theta(\psi(t)-\psi(s)^\alpha)(v_1(x,t)-v_2(x,t)dyd\theta ds,$$

Where $v_j(x,t) = f\left(x,t,w_{1j}(x,t),\dots,w_{kj}(x,t)\right), j = 1,2,$

$$w_{ij}(x,t) = \sum_{q<2m} b_{qi}(x,t) D_x^q u_j(x,t).$$

According to the conditions of the function ψ and the properties (P1), (P2) of the Green function G and the condition (1.2), we can find a constant $K > 0$ and a constant $\gamma \in (0,1)$ such that

$$Max_{x\in R^n}|u_1(x,t)-u_2(x,t)| \le K \int_0^t (t-s)^{\gamma-1} Max_{x\in R^n}|u_1(x,s)-u_2(x,s)|ds,$$

See [2-6]

This complete the proof of the theorem..

Theorem 3.2.

Equation (1.2) has a unique solution v such that $D_x^q v$ is continuous and bounded on $R^n XJ$, for all $|q| < 2m$.

Proof. We shall use the method of successive approximations Let $\{v_n\}$ be a sequence defined by :

$$v_{n+1}(x,t) = f(x,t,\textstyle\sum_{q<2m} b_{q1}(x,t) D_x^q u_n(x,t),\dots,\sum_{q<2m} b_{qk}(x,t) D_x^q u_n(x,t)),$$

Where:

$$u_n(x,t) = \int_0^\infty \int_{R^n} G\big(x,y,\theta\psi^\alpha(t)\big)\varphi(y)dyd\theta + \int_0^t \int_0^\infty \int_{R^n} \Lambda(t,s,\Theta) G(x,y,\Theta\big(\psi(t)-\psi(s)\big)^\alpha v_n(y,s)dyd\theta ds$$

Thus we can find a positive constant K and a constant $\gamma \in (0,1)$ such that:

$$Max_{x\in R^n}|v_{n+1}(x,t)-v_n(x,t)| \le K \int_0^t (t-s)^\gamma Max_{x\in R^n}|v_n(x,s)-v_{n-1}(x,s)ds$$

It is easy to get:

$$Max_{x\epsilon R^n}|v_{n+1}(x,t)-v_n(x,t)| \le \frac{MK^n\Gamma^n(\gamma)t^{n\gamma}}{\Gamma(n\gamma+1)},$$

Thus the sequence $\{v_n(x,t)\}$ uniformly converges to a function $v(x,t)$ on R^nXJ. It is clear that $D_x^q v(x,t)$, $D_x^q u(x,t)$ are continuous and bounded on R^nXJ for all $|q|<2m$.

Hence the required result. (See [7-20]).

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