Quest Journals
Journal of Research in Applied Mathematics
Volume 9 ~ Issue 8 (2023) pp: 29-34
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

Forcing Total Outer Independent Monophonic Domination Number of a Graph

P. Arul Paul Sudhahar[1] and T.Premala[2]
[1]*Department of MathematicsGovernment Arts & Science College, Nagercoil, Tamilnadu, India.*
[2]*Department of MathematicsNarayanaguru College of Engineering, Manjalumoodu, Tamilnadu, India*

ABSTRACT
We initiate the study of forcing total outer independent monophonic domination number of a graph. Let G be a connected graph and M a minimum total outer independent monophonic dominating set of G. A subset $T \subseteq M$ is called a forcing subset of M if M is the unique minimum total outer independent monophonic dominating set containing T. A forcing subset for M of minimum cardinality is a minimum forcing subset of M. The forcing total outer independent monophonic domination number of M, denoted by$f_{\Upsilon_{m_t}^{0i}}(M)$, is the cardinality of a minimum forcing subset of M. The forcing total outer independent monophonic domination number of G, denoted by $f_{\Upsilon_{m_t}^{0i}}(G)$, is $f_{\Upsilon_{m_t}^{0i}}(G) = min\left\{f_{\Upsilon_{m_t}^{0i}}(M)\right\}$, where the minimum is taken over all minimum total outer independent monophonic dominating sets M in G.Some of its general properties are studied. It is shown that for every paira, b of integers with $0 \leq a \leq b-4$ and $b \geq 5$, there exists a connected graph G such that $f_{\Upsilon_{m_t}^{0i}}(G) = a$ and $\Upsilon_{m_t}^{0i}(G) = b$.

AMS Subject Classification: 05C12
Keywords: total monophonic number, total outer independent monophonic number, total outer independent monophonic domination number, forcing total outer independent monophonic domination number.

R**eceived 08 August, 2023; **R**evised 21 August, 2023; **A**ccepted 23 August, 2023 © The author(s) 2023. **P**ublished with open access at **www.questjournals.org

I. INTRODUCTION

Let $G = (V, E)$ be a graph and n be the number of vertices and $m$be the number of edges. Thus the cardinality of $V(G) = m$ and the cardinality of $E(G) = n$. We consider a finite undirected graph without loops or multiple edges. For the basic graph theoretic notations and terminology we refer to Buckley and Harary.For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic.

The neighbourhood of a vertex v is the set $N(v)$ consisting of all vertices which are adjacent with$v$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighbourhood. A vertex v is an extreme vertex if the subgraph induced by its neighbourhood is complete. A vertex v in a connected graph G is a cut vertex of G, if $G - v$ is disconnected. A vertex v in a connected graph G is said to be a semi-extreme vertex if$\Delta(< N(v) >) = |N(v)| - 1$. A graph G is said to be semi-extreme graph if every vertex of G is a semi-extreme vertex. A connected acyclic graph is called a tree.

A subset of $V(G)$ is independent if there is no edge between any two vertices of this set. The independence number of a graph$G$, denoted by $\alpha(G)$, is the maximum cardinality of an independent subset of the set of vertices of G. A *monophonic* set of G is a set $M \subseteq V(G)$ such that every vertex of G is contained in a monophonic path joining some pair of vertices in M. The *monophonic number*$m(G)$ of$G$ is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a *minimum monophonic* set of G. A total monophonic set of a graph G is a monophonic set M such that the subgraph induced by M has no isolated vertex. The minimum cardinality of a total monophonic set of G is its total monophonic number and is denoted by $m_t(G)$. A total monophonic set of size $m_t(G)$ is said to be a m_t-set.

A subset $M \subseteq V(G)$ is *independent monophonic set* if M is a monophonic set and every two vertices from M are not adjacent in G. The minimum cardinality of a independent monophonic set of G is the *independent monophonic number* of G and is denoted by $im(G)$.

*Corresponding Author: Onwukwe Ijioma.

A subset$M \subseteq V(G)$ is *outer independent monophonic set* if M is a monophonic set and the set $V(G) - M$ is independent. The minimum cardinality of a independent monophonic set of G is the *outer independent monophonicnumber* of G and is denoted by $m_{oi}(G)$.

A *total outer-independent dominating set* of a graph G is a set D of vertices of G such that every vertex of G has a neighbor in D, the set of all vertices in D is not isolated and the set $V(G) - D$ is independent. The *total outer-independent domination number* of a graph G is the minimum cardinality of a total outer-independent dominating set of G and is denoted by $\gamma_t^{oi}(G)$.

A monophonic set $M \subseteq V$ is said to be *total outer independent monophonicdominating set*, abbreviated TOIMDS if it is a monophonic dominating set and $\langle V - M \rangle$ is an independent set. The minimum cardinality of a total outer independent monophonic dominating set, denoted by $\Upsilon_{m_t}^{0i}(G)$ is called the *total outer independent monophonic domination number* of G.

The following theorem will be used in the sequal.

Theorem 1.1 Each extreme vertexof a connected graph Gbelongs to every total outerindependent monophonicdomination number of G.

Definition 2.1 Let G be a connected graph and M a minimum total outer independent monophonic dominating set of G. A subset $T \subseteq M$ is called a forcing subset for M if M is the unique minimum total outer independent monophonic dominating set containing T. A forcing subset for M of minimum cardinality is a minimum forcing subset of M. The forcing total outer independent monophonic domination number of M denoted by$f_{\Upsilon_{m_t}^{0i}}(M)$, is the cardinality of a minimum forcing subset of M. The forcing total outer independent monophonic domination number of G, denoted by $f_{\Upsilon_{m_t}^{0i}}(G)$ is $f_{\Upsilon_{m_t}^{0i}}(G) = \min\left\{f_{\Upsilon_{m_t}^{0i}}(M)\right\}$, where the minimum is taken over all minimum total outer independent monophonic dominating sets M in G.

Example 2.2

For the graph G given in Figure 2.1 $M_1 = \{v_1, v_2, v_4, v_5, v_7, v_9\}$ and$M_2 = \{v_1, v_3, v_4, v_5, v_6, v_8, v_9\}$are the only two total outer independent monophonic domination number of G. It is clear that$f_{\Upsilon_{m_t}^{0i}}(M_1) = 1$ and$f_{\Upsilon_{m_t}^{0i}}(M_2) = 1$ so that$f_{\Upsilon_{m_t}^{0i}}(G) = 1$

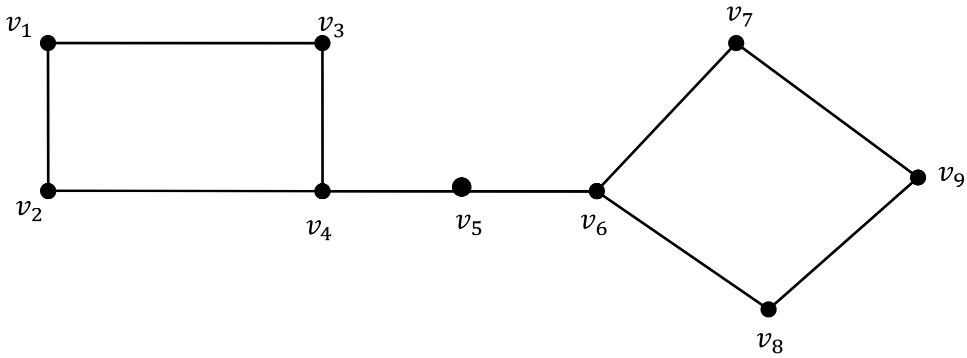

Figure 2.1

Theorem 2.3 For any connected graph G,$0 \leq f_{\Upsilon_{m_t}^{0i}}(G) \leq \Upsilon_{m_t}^{0i}(G) \leq p$.

Definition 2.4 A vertex v of a connected graph G is said to be a total outer independent monophonic dominating vertex of G if v belongs to every minimum total outer independent monophonic dominating set of G.

Theorem 2.5 Let G be a connected graph. Then

a) $f_{\Upsilon_{m_t}^{0i}}(G) = 0$ if and only if G has a unique minimum total outer independent monophonic dominating set

b) $f_{\Upsilon_{m_t}^{0i}}(G) = 1$ if and only if G has at least two minimum total outer independent monophonic dominating sets, one of which is unique minimum total outer independent monophonic dominating set containing one of its elements, and

c) $f_{\Upsilon_{m_t}^{0i}}(G) = \Upsilon_{m_t}^{oi}(G)$ if and only if no minimum total outer independent monophonic dominating set of G is the unique minimum total outer independent monophonic dominating set containing any of its proper subsets.

Proof. (a) Let $f_{\Upsilon_{m_t}^{0i}}(G) = 0$. Then by definition $f_{\Upsilon_{m_t}^{0i}}(M) = 0$ for some total outer independent monophonic dominating set M of G so that the empty set φ is the minimum forcing subset for M. Since the empty set φ is a subset of every set, it follows that M is the unique minimum total outer independent monophonic dominating set of G. The converse is clear.

b) Let $f_{\Upsilon^{oi}_{m_t}}(G) = 1$. Then by Theorem 2.5 (a), G has atleast two total outer independent monophonic dominating sets. Also, since $f_{\Upsilon^{oi}_{m_t}}(G) = 1$, there is a singleton subset T of a total outer independent monophonic dominating set M of G such that T is not a subset of any other total outer independent monophonic dominating set of G. Thus M is the unique total outer independent monophonic dominating set containing one of its elements. The converse is clear.

c) Let$f_{\Upsilon^{oi}_{m_t}}(G) = \Upsilon^{oi}_{m_t}(G)$.Then $f_{\Upsilon^{oi}_{m_t}}(M) = \Upsilon^{oi}_{m_t}(M)$ for every total outer independent monophonic dominating sets$M$ in G. Also, it is clear $\Upsilon^{oi}_{m_t}(G) \geq 2$ and hence $f_{\Upsilon^{oi}_{m_t}}(G) \geq 2$.Then by theorem 2.5 (a) G has atleast two total outer independent monophonic dominating set and so that the empty set φ is not a forcing subset for any total outer independent monophonic dominating set of G. Since$f_{\Upsilon^{oi}_{m_t}}(M) = \Upsilon^{oi}_{m_t}(G)$, no proper subset of M is a forcing subset of M. Thus no total outer independent monophonic dominating set of G is the unique total outer independent monophonic dominating set containing any of its proper subsets. Conversely, the data implies that G contains more than one total outer independent monophonic dominating set and no subset of any total outer independent monophonic dominating set M other than M is a forcing subset for M. Hence it follows that$f_{\Upsilon^{oi}_{m_t}}(G) = \Upsilon^{oi}_{m_t}(G)$.

Theorem 2.6 Let G be a connected graph and let $\Im$be the set of relative complements of the minimum forcing subsets in their respective minimum total outer independent monophonic dominating sets in G. Then $\cap_{F\in\Im} F$ is the set of total outer independent monophonic dominating vertices of G.

Proof. Let W be the set of all total outer independent monophonic dominating vertices of G. We have to show that $W = \cap_{F\in\Im} F$. Let $v \in W$. Then v is a total outer independent monophonic dominating vertex of G that belongs to every minimum total outer independent monophonic dominating set M of G. Let $T \subseteq M$ be any minimum forcing subset for any minimum total outer independent monophonic dominating set M of G. We claim that $v \notin T$. If $v \in T$, then $T' = T - \{v\}$ is a proper subset of T such that M is the unique minimum total outer independent monophonic dominating set containingT' so that T' is a forcing subset for M with $|T'| < |T|$, which is a contradiction to T is a minimum forcing subset for M. Hence $v \in\cap_{F\in\Im} F$ so that $W \subseteq\cap_{F\in\Im} F$.

Conversely, let $v \in\cap_{F\in\Im} F$. Then v belongs to the relative complement of$T$ in M for every T and every M such that $T \subseteq M$, where T is a minimum forcing subset for M. Since$F$ is the relative complement of T in M, we have $F \subseteq M$ and thus $v \in M$ for every M, which implies that v is a total outer independent monophonic dominating vertex of G. Thus $v \in W$ and so $\cap_{F\in\Im} F \subseteq W$. Hence $W = \cap_{F\in\Im} F$.

Corollary 2.7 Let G be a connected graph and Ma minimum total outer independent monophonic dominating set of G. Then no total outer independent monophonic dominating vertex of Gbelongs to any minimum forcing set of M.

Theorem 2.8 Let G be a connected graph and W be the set of all total outer independent monophonic dominating vertices of G. Then $f_{\Upsilon^{oi}_{m_t}}(G) \leq \gamma^{oi}_{m_t}(G) - |W|$.

Proof. Let M be any minimum total monophonic dominating set of G. Then $\gamma^{oi}_{m_t}(G) = |M|, W \subseteq M$ and M is the unique minimum forcing total outer independent monophonic dominating set containing $M - W$. Then $f_{\Upsilon^{oi}_{m_t}}(G) \leq |M - W| = |M| - |W| = \Upsilon^{oi}_{m_t}(G) - |W|$.

Theorem 2.9 For any complete graph $G = K_p$, $p \geq 2$, $f_{\Upsilon^{oi}_{m_t}}(G) = 0$.

Proof. For $G = K_p$, it follows from Theorem 1.1, that the set of all vertices of G is the unique total outer independent monophonic dominating set. Now, by Theorem 2.5 (a), $f_{\Upsilon^{oi}_{m_t}}(G) = 0$.

Theorem 2.10 For the complete bipartite graph $G = K_{r,\ s}(r , s \geq 2)$,

$$f_{\Upsilon^{oi}_{m_t}}(G) = \begin{cases} 0 & \text{if } r < s \\ 1 & \text{if } r = s \end{cases}$$

Proof. Let $U = \{u_1 , u_2 , \dots , u_r\}$ and $V = \{v_1 , v_2 , \dots , v_s\}$ be the two bipartite sets of G. Let us consider $r < s$. Clearly r is the total outer independent monophonic domination number of G. Therefore $S = \{u_1 , u_2 , \dots u_r\}$ is the unique minimum total outer independent monophonic dominating set of G . Hence it follows from Theorem 2.5 (a), that $f_{\Upsilon^{oi}_{m_t}}(G) = 0$.

If $r = s$, then $S_1 = \{u_1 , u_2 , \dots , u_s\} S_2 = \{v_1 , v_2 , \dots , v_s\}$ are the two minimum total outer independent monophonic dominating set of G so that $f_{\Upsilon^{oi}_{m_t}}(S_1) = 1$ and $f_{\Upsilon^{oi}_{m_t}}(S_2) = 1$.Hence,$f_{\Upsilon^{oi}_{m_t}}(G) = 1$.

Corollary 2.11 If G is a connected graph with k simplicial vertices and l cut-vertices, then $f_{\Upsilon_{m_t}^{0i}}(G) \leq \Upsilon_{m_t}^{0i}(G) - (k+l)$.

Theorem 2.12 For every pair a, b of integers with $0 \leq a \leq b-4$ and $b \geq 5$, there exists a connected graph G such that $f_{\Upsilon_{m_t}^{0i}}(G) = a$ and $\Upsilon_{m_t}^{0i}(G) = b$.

Proof. We prove this theorem by considering three cases. For $b = 4$, we have $a = 0$. Hence for $G = K_4$, we have $f_{\Upsilon_{m_t}^{0i}}(G) = 0$ and $\Upsilon_{m_t}^{0i}(G) = 4 = b$. Now, take $b \geq 5$.

Case 1.$a = 0$. Let $G = K_b$. Then by Theorem 2.9 and Theorem 1.1, we have $f_{\Upsilon_{m_t}^{0i}}(G) = 0$ and $\Upsilon_{m_t}^{0i}(G) = b$.

Case 2.$a = 1$. Let $C_6: t, u, v, w, x, y, t$ be a cycle of order of 6 . Let H be the graph obtained fromC_6 by adding $b-5$ new vertices $x_1, x_2, \dots, x_{b-5}$ and joining each $x = (1 \leq i \leq b-5)$ to v . Let G be the graph obtained from H by adding a new vertex z and join z to both u and w . The graph G is given in Figure2.2 . Let $S = [x_1, x_2, \dots, x_{b-5}]$ be the set of all extreme vertices of G . Then by Theorem 1.1, every total outer independent monophonic dominating set contains S. G contains exact by two minimum total outer independent monophonic dominating sets namely $S_1 = S \cup \{v, u, w, x, y, z\}$ and $S_2 = S \cup \{v, x, y, t, z\}$ .Then $\Upsilon_{m_t}^{0i}(G) = |s| + 5 = b$. Clearly $f_{\Upsilon_{m_t}^{0i}}(G) = 1 = a$.

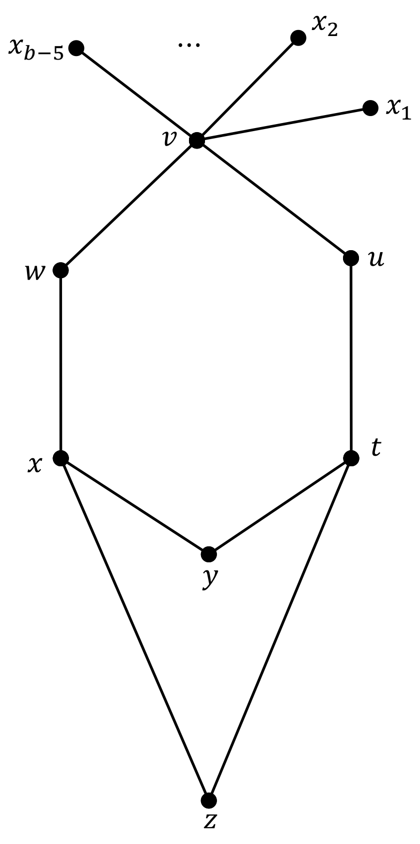

Figure 2.2

Case 3.$a \geq 2$.Let $P_4: y_1, y_2, y_3, y_4$ be a path of order 4 and $C_6: v_1, v_2, v_3, v_4, y_3, y_2, v_1$ be a cycle of order of 6. Let H be a graph obtained from P_4 and C_6 by adding new vertices $x_1, x_2, \dots, x_{b-a-4}$ and joining each $x_i(1 \leq i \leq b-a-4)$ with y_2. Let G be the graph obtained from H by adding new vertices $z_1, z_2, \dots, z_{a-2}$ and joining each $z_i(1 \leq i \leq a-2)$ to both $y_3$and v_3. The graph G is given in Figure 2.3

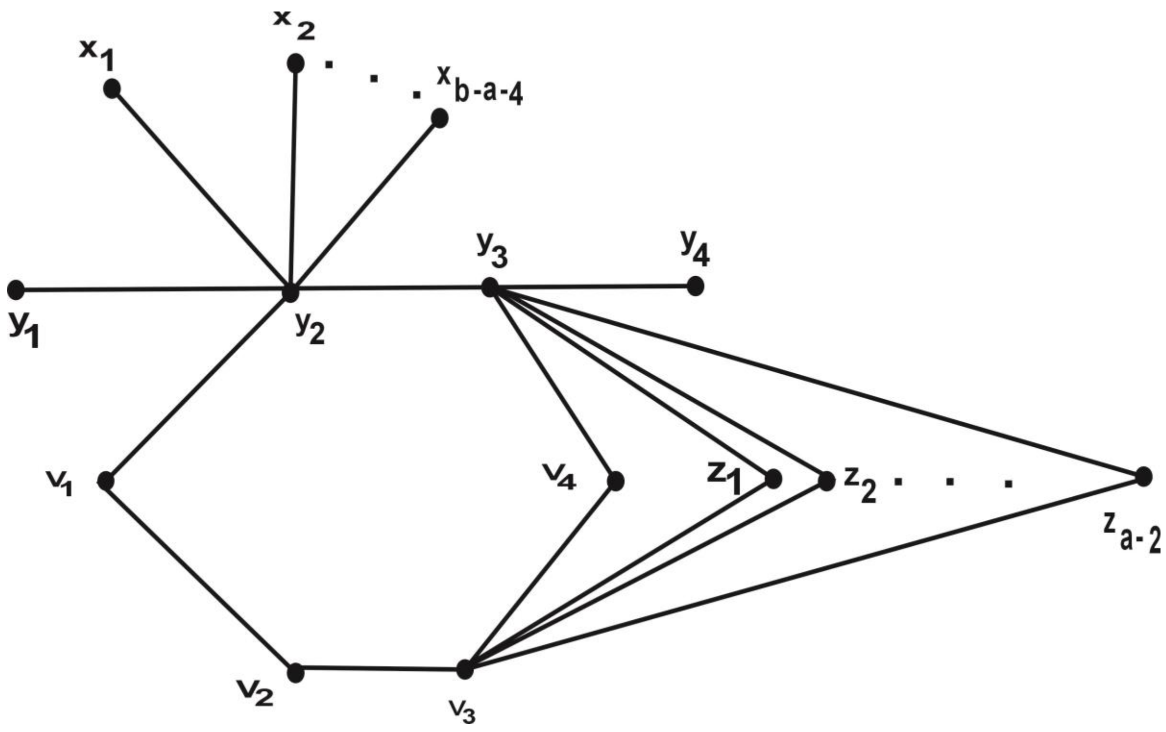

Figure 2.3

Let $M_1 = \{y_1, y_4, x_1, x_2, \ldots, x_{b-a-4}\}$ be the set of all extreme vertices of G. By Theorem 1.1, every total outer independent monophonic dominating set of G contains M_1. It is clear that $M_1$is not a total outer independent monophonic dominating set of G. Let $M_2 = M_1 \cup \{v_1, v_3, y_2, y_3, z_1, z_2, \ldots, z_{a-2}\}$ is the minimum total outer independent monophonic dominating set of G, so that$\Upsilon_{m_t}^{0i}(G) = b$.

Next to show that $f_{\Upsilon_{m_t}^{0i}}(G) = a$. Since every minimum total outer independent monophonic dominating set of G contain $M_1 \cup \{y_2, y_3\}$, it follows from Theorem 2.8, that $f_{\Upsilon_{m_t}^{0i}}(G) \leq \Upsilon_{m_t}^{0i}(G) - M_1 \cup \{y_2, y_3\} = a$. Moreover, the set M_2 is the minimum total outer independent monophonic dominating set of G if and only if M_2 is of the form $M_1 \cup \{v_1, v_3, y_2, y_3, z_1, z_2, \ldots, z_{a-2}\}$. If $f_{\Upsilon_{m_t}^{0i}}(G) < a$, let M_3 be a subset of M_2 with $|M_3| < a$. Then there is a vertex y_j such that $y_j \notin M_3$. Let w_j be a vertex different from y_j. Then $M_2' = M_2 - (\{y_j\} \cup \{w_j\})$ is the minimum total outer independent monophonic dominating set of G different from M_2 such that M_2' contain M_3. Thus$f_{\Upsilon_{m_t}^{0i}}(G) \geq a$. Then $f_{\Upsilon_{m_t}^{0i}}(G) = a$.

II. Conclusion

We can extend the concept of forcing total outer independent monophonic domination number to find the forcing total outer independent edge monophonic domination number , forcing upper total outer independent monophonic domination number of a graph and so on.

References

[1]. Arul Paul Sudhahar P, Bertilla Jaushal A.J, Vijayan A,'The total monophonic domination number of a graph', Asian Journal of Mathematics and Computer Research, 2016;14(4):290-295.
[2]. Arul Paul Sudhahar P, Bertilla Jaushal A.J, Vijayan A, 'The total edge monophonic domination number of a graph', Proceedings on International Conferences on Applicable Mathematics; 2018.
[3]. Arul Paul Sudhahar P, Kalaivani V.K, Vijayan A,'The total edge monophonic number of a graph', International Journal of Computational and Applied Mathematics.2016;11(2):159-166.
[4]. Arul Paul Sudhahar. P and Bertilla Jaushal A.J, 'The Total Outer Independent Monophonic Dominating Parameters in Graph', Asian Research Journal of Mathematics, 2019;Vol.14,PP.1-8.
[5]. Buckley , Harary F,'Distance in graphs',Addision-Wesley, Redwood City;1990.
[6]. Caro Y, Yuster R,'Dominating a family of graphs with small connected graphs', Combin. Probab. Comput.2000;9:309-313.
[7]. Cockayne E.J, Dawes R.M, Hedetniemi S.T, 'Total domination in graphs', Networks. 1980;10:211-219.
[8]. Ersin Aslan, Alpay Kirlangic,'The average lower domination number of graphs', Bulletin of International Mathematical Virtual Institute. 2013;3(2):155-160.
[9]. Harary.F,'Graph Theory', Narosa Publishing House;1998.
[10]. Haynes T.W, Hedetniemi S.T, Slater P.J,'Fundamentals of domination in graphs', Marcel Dekker Inc.., New York. 1998;208.
[11]. Michael A. Henning,'A survey of selected recent results on total domination in graphs', Discrete Mathematics. 2009;309:32-63.

[12]. Santhakumaran AP, Titus P, Ganesamoorthy K,'On the monophonic number of a graph', J. Appl. Math.& Informatics. 2014;32(1-2):255-266.
[13]. 'Total Outer-independent domination in Graphs', marcin.krzywkowski@gmail.com
[14]. Thomasse S, Yeo A,'Total domination of graphs and small transversals of hypergraphs', Combinatorica.2007;27:473-487.
[15]. Tufan Turaci, Ersin Aslan,'The average lower reinforcement number of a graph', RAIRO, Theoretical Informatics and Applications, 2016;50(2):135-144.
[16]. Venkanagouda M. Goudar, Bhavyavenu KI., 'Total outer independent Geodetic Number of a Graph', J. Ultrascientist of Physical Sciences. 2018;30(4);247-256.