Quest Journals
Journal of Research in Applied Mathematics
Volume 9 ~ Issue 8 (2023) pp: 44-47
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

Optimum Linear MPC with Input-Output Equations

Endre Nagy
Member, SICE

Abstract: *A new method is proposed in this paper for solution of the linear model predictive control problem with I/O equations. Settling the task may be obtained through application of the principle "direct stochastic optimum tracking" with a simple algorithm, which can be derived from a previously developed optimization procedure. The final result is obtained through iterations. An example shows the applicability and advantages of the method.*

Keywords: *optimum control, stochastic control, predictive control.*

Received 16 August, 2023; Revised 31 August, 2023; Accepted 03 September, 2023 © The author(s) 2023. Published with open access at ***www.questjournals.org***

I. INTRODUCTION

The objective of Model Predictive Control (MPC) is solution (or approximate solution) of an optimum stochastic control problem using a process model, predictor equation and a known future reference trajectory. MPC is probably the most general way to formulate the process control problem in the time domain [2]. The performance index (PI) usually quadratic and costs future process output errors. Early MPC methods were established by Richalet and al. [8]. Improvement soon followed by the development of Dynamic Matrix Controller [6]. Over the years various MPC methods (Generalized Predictive Control (GPC) [4], [5] etc.) evolved. The main differences among the methods are the ways by which future predictions are computed, the type of applied model, how the PI is defined and the applied assumptions. Traditional Model Predictive Controllers have the disadvantages that there are several parameters to choose. Moreover, the solution usually is based on some assumptions and there are no guarantied stability properties, not regarding special conditions.

The proposed new solution has a rather different approach: optimum solution of the problem (minimization of a suitable stochastic PI) without assumptions (other than specification of the stochastic process), not regarding computational inaccuracies. The resultant control may be called Optimum Predictive Control (OPC).

II. MAIN COMPONENTS OF THE SOLUTION

2.1. Models and Performance Index

All types of process models may be used for MPC; the most frequently used are transfer function model, impulse response model, step response model and state space model [2]. In this paper transfer function model will be used. Classic MPC algorithms use multistep predictions into the future and minimize a quadratic PI, thus minimum of a typical MPC PI for a SISO plant would be

$$\min_{u(i)} J_{1,N} = \min_{u(i)} E\{[\sum_{i=0}^{N-1}[(\hat{y}(t+i+1/t)\text{-r}(t+i+1))^2 + \lambda_i u^2(t+i)]]/t\} \quad (1)$$

(provided, there is no extra time delay in the process). In (1) $y(i)$ is the output, $u(i)$ is the control signal, $r(i)$ is the reference, λ_i is the control weight and N is the number of time steps on the finite horizon, t is bottom of the horizon, $E\{\cdot/t\}$ denotes the conditional expectation operator. Different or additional constraints may be treated e.g. with the method of Lagrange multipliers. The (1) N – stage control problem can be derived from the one – stage generalized minimum variance tracking problem. However, the above PI is often modified, because MPC techniques usually apply the free and forced response concept [2] and use different cost and control horizons. In practice the MPC problem may be formulated, too, as [2]

$$\min_{u(i)} J_{1,N} = \min_{u(i)} \sum_{i=0}^{N-1}\{[\hat{y}(t+i+1/t)\text{-}r(t+i+1)]^2 + \lambda_i u^2(t+i)\}. \quad (2)$$

In this paper mainly the (2) PI is used with the difference that the multistep ahead predictions are replaced with a sequence of one–step ahead predictions, resulting in

$$\min_{u(i)} J_{1,N} = \min_{u(i)} \sum_{i=0}^{N-1} \{[\hat{y}(t+i+1/t+i) - r(t+i+1)]^2 + \lambda_i u^2(t+i)\}. \tag{3}$$

Computation with the (3) PI becomes possible through application of an earlier developed optimization method, which estimates an optimum trajectory through step – by – step optimum extensions of the trajectory. Additional advantage is that accuracy of the computation – despite simpler algorithm – is increased because optimum trajectory is computed.

2.2. Prediction with I/O Equations

To solve the (3) problem, predictions are necessary at each sampling time point. Predictions may be made in different ways, depending on the process, the used criterion and the admissible predictors [1]. The process is assumed to be given with a suitable stochastic model, e.g. with the ARMAX transfer function model [3] (ARMA autoregressive moving average model [1] with exogenous inputs), which for one step delay in the control signal is

$$G(q^{-1})y(t) = q^{-1}H(q^{-1})u(t) + L(q^{-1})e(t). \tag{4}$$

In (4) $G(q^{-1})$, $H(q^{-1})$ and $L(q^{-1})$ are polynomials in the backward hift operator,

$$G(q^{-1}) = 1 + g_1 q^{-1} + \ldots + g_{ng} q^{-ng}, \tag{5}$$

$$H(q^{-1}) = h_1 + \ldots + h_{nh} q^{-nh}, \tag{6}$$

$$L(q^{-1}) = 1 + l_1 q^{-1} + \ldots + l_{nl} q^{-nl}, \tag{7}$$

and $e(t)$ is white noise sequence of independent random variables with zero mean; $e(t)=0$ beyond the bottom of computational horizon. Assume the last measurement happened at t. In this case only the $\hat{y}(t+1)$, $\hat{y}(t+2)$,…estimates give information on the probable future development of the process. Introduce the variable y' with the definition

$$y'(t+i) = \begin{cases} y(t+i), i = 0, -1, -2\ldots, \\ \hat{y}(t+i), i > 0. \end{cases} \tag{8}$$

With this notation

$$y'(t+i) = f(y'(t+i-1), y'(t+i-2)\ldots, u(t+i-1), \ldots). \tag{9}$$

(8), (9) refer to estimation of future outputs: the not measurable outputs are replaced with their estimates.
Consider the one – step ahead prediction at $t+i$. From (4) and (8)

$$y'(t+i+1/t+i) = \frac{q^{-1}H(q^{-1})}{G(q^{-1})} u(t+i+1) + \frac{L(q^{-1})}{G(q^{-1})} e(t+i+1). \tag{10}$$

The control signal is obtained from solution of the whole control problem through optimization on the horizon. The one – stage ahead prediction is suitable for computations which are based on a series of two – stage optimizations. This type of predictor may be called "(output) predictor for MPC", MPC_PRED. Another possibility is application of the standard optimum multistep ahead output prediction method with (2).

2.3. Optimization with "Optimized Stochastic Trajectory Tracking"

Optimum solution of the stochastic control problem may be based on the principle "direct stochastic optimum tracking" (referring to direct computation of the optimum stochastic trajectory, without comparison of different trajectories). The principle claims that a predicted optimum stochastic trajectory (or output sequence) can be computed by step – by – step optimum extension of a part of the predicted optimum trajectory (output sequence). The optimization method based on this principle is called "optimized stochastic trajectory / output sequence tracking" (OSTT). OSTT is an extension of the deterministic method "optimized trajectory tracking" (OTT) [7] for stochastic systems. Consider a slightly modified version of the (3) problem and the (4) model, where d – step delays are in the control signal. Assume an optimum output sequence has been computed on the time interval $[t+d, t+d+i]$. The next unknown predicted optimum output can be obtained from

$$\min_{u(t+i)} J_{t+d+i,t+d+i+1} = \min_{u(t+i)} \sum_{j=i}^{i+1}\{[y'(t+d+j)\text{-}r(t+d+j)]^2 + \lambda_{t+j} u^2(t+j)\}. \quad (11)$$

Necessary condition for minimum is

$$\frac{\partial J_{t+d+i,t+d+i+1}}{\partial u(t+i)} = 0. \quad (12)$$

For evaluation of (11), $y'(t+d+i)$ and $y'(t+d+i+1)$ are substituted with system equations. The unknown control signal can be computed as

$$u(t+i+1)=f(u(t+i),u(t+i\text{-}1),\ldots y'(t+d+i\text{-}1),y'(t+d+i\text{-}2),\ldots e(t),e(t\text{-}1),\ldots r(t+d+i),r(t+d+i+1)),\ i\geq 0. \quad (13)$$

From comparison of the measured and estimated outputs $e(t)$, $e(t\text{-}1)$,…can be computed. However, $e(t+1)$, $e(t+2)$…is considered zero. With (13) and the formerly computed values $y'(t+d+i+1)$ can be obtained. The whole optimum output sequence can be computed similarly with OSTT. At the start of the computation, estimation has to be made on $u(t)$, which can be done e.g. as

$$y'(t+d) \approx r(t+d).$$

Through completing the optimization $N - 1$ times on a section of two – stages, $y'(t+d+N-1) \neq r(t+d+N-1)$ is reached. The final solution is obtained by iterations. For application of the Newton method, deviation in the final value for a SISO system can be written as

$$r(t+d+N-1) - y'(t+d+N-1) \approx \frac{\partial y'(t+d+N-1)}{\partial u(t)} \Delta u(t). \quad (14)$$

From (14)

$$\Delta u(t) \approx \frac{r(t+d+N-1) - y'(t+d+N-1)}{\partial y'(t+d+N-1)/\partial u(t)}. \quad (15)$$

The new starting control signal is chosen as

$$u(t)_{new} = u(t)_{old} + \Delta u(t). \quad (16)$$

The derivative in (15) can be approximated as

$$\frac{\partial y'(t+d+N-1)}{\partial u(t)} \approx [y'(t+d+N-1,u(t)+h) - y'(t+d+N-1,u(t))]/h, \quad (17)$$

where h is an appropriate small value. The Newton method, which is a local one, can be applied for multivariable systems too, through use of the Jacobian matrix. Several other methods can be used for iteration, like regula falsi, secant method etc. Regula falsi is a global method; however, it cannot be extended for multivariable systems. If the receding horizon principle is used, only variables on the first stage are kept, the others are discarded, and the procedure is repeated at the next sampling instant. The method can be generalized for multivariable systems through multivariable prediction and multivariable iteration. This is the OPC solution for linear SISO plants described with I/O equations.

III. EXAMPLE

This example concerns computing the control signal for MPC. Consider a plant given with the model

$$y(i+1)=ay(i)+bu(i)+e((i+1)+ce(i), \quad (18)$$

where $y(i)$ – output, a, b, c – parameters, e(i) – sequence of independent random variables with zero expectation and $\sigma^2 = 0.075$ variance. To the simulation a = 0.9, b = 0.8 and c = 0.95 has been selected. The PI for two stages is given as

$$J_{i+1}^{i+2} = \sum_{j=1}^{i+1}\{(r(j+1) - y'(j+1))^2 + \lambda u^2(j)\}. \quad (19)$$

The necessary condition for minimum is given in (12). With (12), (18) and (19)

$$u(i+1)=-\{b^2(1+a^2)+\lambda\}u(i)/(ab^2)-(1+a^2)y'(i)/b+r(i+1)/(ab)+r(i+2)/b-c(1+a^2)e(i)/(ab). \quad (20)$$

$e(i)$ can be computed from the preceding estimation for $y(i)$; however, only on the first stage for MPC. Therefore, $e(i+1)$ is taken into consideration with its expectation on all stages. Fig. 1 shows the result of simulation in combination with receding horizon control on an 8 step finite horizon with the (20) control law and with $\lambda = 0.1$:

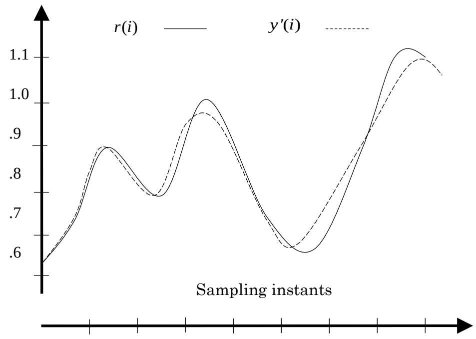

FIGURE 1. Simulation result for receding horizon control

IV. CONCLUSIONS

There are several approaches to solve the MPC problem. However, the existing solutions are based on simplifications and assumptions and there are no guarantied stability properties. The proposed new method solves the problem optimally without assumptions (other than made for disturbance specification), therefore the result may be more accurate. Source of inaccuracies is computational in character. The paper employs the previously developed optimization method "optimized stochastic trajectory / output sequence tracking" in combination with a new output predictor, named MPC_PRED. When applying it, an optimum predicted output sequence is computed through a series of two – stage optimizations in function of a selected starting control signal, and the final solution is obtained through iterations. Various simulations show the accuracy and applicability of the presented method. The used method can be extended for linear and nonlinear state space design, too.

REFERENCES

[1]. K.J. Aström and B. Wittenmark, Computer Controlled Systems: Theory and Design (Prentice – Hall International, Englewood Cliffs, N.J., 1984).

[2]. E.F. Camaho and C. Bordons, Model Predictive Control (Springer - Verlag, London, 1999).

[3]. J.V. Candy, Signal Processing (McGraw - Hill Book Company, New York, 1987).

[4]. D. W. Clarke, C. Mohtadi and P. S. Tuffs, Generalized Predictive Control. Part I. The Basic Algorithm. Automatica, **23**(2), 137-148, 1987.

[5]. D.W. Clarke, C. Mohtadi and P.S. Tuffs, Generalized Predictive Control. Part II. Extensions and Interpretations. Automatica **23**(2), 149-160, 1987.

[6]. C.R Cutler and B.C. Ramaker, "Dynamic Matrix Control – A Computer Control Algorithm", JACC Conference Proceedings, San Francisco, C.A., 1980, paper WP5-B.

[7]. E. Nagy, "Optimum Control of Sampling Systems through Optimized Trajectory Tracking", ACC Conference Proceedings, Albuquerque, 1997, pp. 3897 – 3901.

[8]. J. Richalet, A. Rault, J.L. Testaud and J. Papon, Model Predictive Heuristic Control: Applications to Industrial Processes. Automatica, **14**(2), 413 – 428, 1978.