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Review Paper

Certain Integral Representations and Summation Formulae For Extended Hurwitz - Lerch Zeta Function of Three Variables

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ABSTRACT: *This paper is divided in two sections: Section A, Section B.*
In section A, the generalization of Hurwitz-Lerch Zeta Function of two variables is given in three variables.
In section B, we have given the further extension of Hurwitz-Learch Zeta Function in three variables defined in section A.
Hence forth we will denote "Hurwitz - Lerch Zeta Function of three variables" by "3V - H - L Z F."
Certain Integral representations and Summation formulae are obtained for 3V - H - L Z F. We have also represented 3V - H - L Z F in terms of generalised Hypergeometric Function ${}_pF_q$.
All the main results are given in the form of five theorems.Corollaries of three theorems are also discussed .Special cases are also discussed.

KEY WORDS: *Generalized Hurwitz-Lerch Zeta Function, Gamma Function, Beta Function, Hypergeometric Function, Binomial Series, Eulerian Integral.*

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I. INTRODUCTION

The Hurwitz − Lerch Zeta Function (H − L Z F) ϕ (z, s, a) is defined as [5,6]:

$$\phi\,(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s} \tag{1}$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^- ;\ s \in \mathbb{C}) \quad \text{when } |z| < 1$$

$$\text{and} \quad R\,(s) > 1 \quad \text{when } |z| = 1$$

Various types of generalizations, extensions, and properties of the H - L Z F can be found in [7 – 14].

Recently, Chai and Parmar [16] have introduced two variables generalization of the function defined in eq^n no. (1) by:

$$\phi_{\mu,\eta,\eta' ;\text{ט}}\,(z, t, s, a) = \sum_{k,l=0}^{\infty} \frac{(\mu)_{k+l}\,(\eta)_k\,(\eta')_l}{(\text{ט})_{k+l}\quad k!\,l!}\ \frac{z^k t^l}{(k+l+a)^s} \tag{2}$$

$(\mu,\eta,\eta' \in \mathbb{C}\,;\; a,\upsilon \in \mathbb{C}\setminus\mathbb{Z}_0^-\,;\; s,z,t\in\mathbb{C})$

when $|z| < 1$ and $|t| < 1$

And $\Re(s+\upsilon-\mu-\eta-\eta') > 0$

when $|z| = 1$ and $|t| = 1$

Also very recently, Nisar [1] has further extended the H - L Z F of two variables defined in eq^{n} no. (2) as:

$$\phi_{\mu,\eta,\eta',\delta,\delta';\upsilon,\xi,\xi'}(z,t,s,a) = \sum_{k,l=0}^{\infty} \frac{(\mu)_{k+l}\,(\eta)_k\,(\eta')_l\,(\delta)_k(\delta')_l}{(\upsilon)_{k+l}\,(\xi)_k(\xi')_l\; k!\; l!}\,\frac{z^k t^l}{(k+l+a)^s} \tag{3}$$

$(\mu,\eta,\eta',\delta,\delta' \in \mathbb{C}\,;\; a,\upsilon,\xi,\xi' \in \mathbb{C}\setminus\mathbb{Z}_0^-\,;\; s,z,t\in\mathbb{C})$

when $|z| < 1$ and $|t| < 1$

And $\Re(s+\upsilon+\xi+\xi'-\mu-\eta-\eta'-\delta-\delta') > 0$

when $|z| = 1$ and $|t| = 1$

Motivated by the above recent works, in the present note, we have first generalized the H - L Z F of two variables in three variables which is given in section A and in section B, we have given further extension of 3V - H - L Z F defined in section A by eq^{n} no. (4).

Section A

The H - L Z F of two variables [introduced by Chia and Parmar [16] in eq^{n} no. (2)] is generalized in three variables as:

$$\phi_{\mu,\eta,\eta',\eta'';\upsilon}(z,t,\omega,s,a) = \sum_{k,l,m=0}^{\infty} \frac{(\mu)_{k+l+m}\,(\eta)_k\,(\eta')_l(\eta'')_m}{(\upsilon)_{k+l+m}\; k!\; l!\; m!}\,\frac{z^k t^l \omega^m}{(k+l+m+a)^s} \tag{4}$$

$(\mu,\eta,\eta',\eta'' \in \mathbb{C}\,;\; a,\upsilon \in \mathbb{C}\setminus\mathbb{Z}_0^-\,;\; s,z,t,\omega\in\mathbb{C})$

when $|z| < 1$ and $|t| < 1$ and $|\omega| < 1$

And $\Re(s+\upsilon-\mu-\eta-\eta'-\eta'') > 0$

when $|z| = 1$ and $|t| = 1$ and $|\omega| = 1$

Section B

The extension of H - L Z F of three variables defined in eq^{n} no. (4) is given as:

$$\phi_{\mu,\eta,\eta',\eta'',\delta,\delta',\delta'';\upsilon,\xi,\xi',\xi''}(z,t,\omega,s,a) = \sum_{k,l,m=0}^{\infty} \frac{(\mu)_{k+l+m}\,(\eta)_k\,(\eta')_l(\eta'')_m(\delta)_k(\delta')_l(\delta'')_m}{(\upsilon)_{k+l+m}\,(\xi)_k(\xi')_l(\xi'')_m\; k!\; l!\; m!}\,\frac{z^k t^l \omega^m}{(k+l+m+a)^s} \tag{5}$$

$(\mu,\eta,\eta',\eta'',\delta,\delta,\delta'' \in \mathbb{C}\,;\; a,\upsilon,\xi,\xi',\xi'' \in \mathbb{C}\setminus\mathbb{Z}_0^-\,;\; s,z,t,\omega\in\mathbb{C})$

when $|z| < 1$ and $|t| < 1$ and $|\omega| < 1$

And $\Re\,(s+\upsilon+\xi+\xi'+\xi''-\mu-\eta-\eta'-\eta''-\delta-\delta-\delta'')>0$

when $|z|=1$ and $|t|=1$ and $|\omega|=1$

Now we discuss below some of the cases of above eq^n no. (5):

Case I .

If we put $\delta=\xi$, $\delta'=\xi'$, $\delta''=\xi''$ in eq^n no. (5) we get the following result:

$$\phi_{\mu,\eta,\eta'\eta'';\upsilon}\,(z,t,\omega,s,a)=\sum_{k,l,m=0}^{\infty}\frac{(\mu)_{k+l+m}\,(\eta)_k\,(\eta')_l(\eta'')_m}{(\upsilon)_{k+l+m}\quad k!\;l!\;m!}\,\frac{z^k t^l\omega^m}{(k+l+m+a)^s}\,, \tag{6}$$

$(\mu,\eta,\eta'\eta''\in\mathbb{C};\ a,\upsilon\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ s,z,t,\omega\in\mathbb{C})$

when $|z|<1$ and $|t|<1$ and $|\omega|<1$

And $\Re\,(s+\upsilon-\mu-\eta-\eta'-\eta'')>0$

when $|z|=1$ and $|t|=1$ and $|\omega|=1$

Case II .

If $\mu=\upsilon$, $\delta=\xi$, $\delta'=\xi'$, $\delta''=\xi''$ in eq^n no. (5) then we get :

$$\phi_{\eta,\eta'\eta''}\,(z,t,\omega,s,a)=\sum_{k,l,m=0}^{\infty}\frac{(\eta)_k\,(\eta')_l(\eta'')_m}{k!\;l!\;m!}\,\frac{z^k t^l\omega^m}{(k+l+m+a)^s}\,, \tag{7}$$

$(\eta,\eta'\eta''\in\mathbb{C};\ a\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ s,z,t,\omega\in\mathbb{C})$

when $|z|<1$ and $|t|<1$ and $|\omega|<1$

And $\Re\,(s-\eta-\eta'-\eta'')>0$

when $|z|=1$ and $|t|=1$ and $|\omega|=1$

Case III.

If $\eta'\to\infty$ in eq^n no. (5) then we get:

$$\phi_{\mu,\eta,\eta''\,\delta,\delta',\delta'';\upsilon,\xi,\xi',\xi''}\,(z,t,\omega,s,a):=\lim_{\eta'\to\infty}\{\,\phi_{\mu,\eta,\eta'\eta''\,\delta,\delta',\delta'';\upsilon,\xi,\xi',\xi''}\,(z,t/\eta',\omega,s,a)\,\}$$

$$=\sum_{k,l,m=0}^{\infty}\frac{(\mu)_{k+l+m}\,(\eta)_k(\eta'')_m(\delta)_k(\delta')_l(\delta'')_m}{(\upsilon)_{k+l+m}\;(\xi)_k(\xi')_l(\xi'')_m\;k!\;l!\;m!}\,\frac{z^k t^l\omega^m}{(k+l+m+a)^s} \tag{8}$$

$(\mu,\eta,\eta'',\delta,\delta,\delta''\in\mathbb{C};\ a,\upsilon,\xi,\xi',\xi''\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ s,z,t,\omega\in\mathbb{C})$

when $|z|<1$ and $|t|<1$ and $|\omega|<1$

And $\mathbb{R}\,(s+\upsilon+\xi+\xi'+\xi''-\mu-\eta-\eta''-\delta-\delta-\delta'') > 0$

when $|z| = 1$ and $|t| = 1$ and $|\omega| = 1$

Case IV.

If $\mu \to \infty$ in eq^n no. (5) then we get:

$$\phi_{\eta,\eta',\eta''\,\delta,\delta',\delta'';\upsilon,\xi,\xi',\xi''}(z,t,\omega,s,a) := \lim_{\mu\to\infty}\{\phi_{\mu,\eta,\eta',\eta''\,\delta,\delta',\delta'';\upsilon,\xi,\xi',\xi''}(z/\mu,t/\mu,\omega/\mu,s,a)\}$$

$$= \sum_{k,l,m=0}^{\infty} \frac{(\eta)_k(\eta')_l(\eta'')_m(\delta)_k(\delta')_l(\delta'')_m}{(\upsilon)_{k+l+m}(\xi)_k(\xi')_l(\xi'')_m\,k!\,l!\,m!}\frac{z^k t^l \omega^m}{(k+l+m+a)^s}, \tag{9}$$

$(\eta,\eta',\eta'',\delta,\delta,\delta'' \in \mathbb{C};\ a,\upsilon,\xi,\xi',\xi'' \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ s,z,t,\omega \in \mathbb{C})$

when $|z| < 1$ and $|t| < 1$ and $|\omega| < 1$

And $\mathbb{R}\,(s+\upsilon+\xi+\xi'+\xi''-\eta-\eta'-\eta''-\delta-\delta-\delta'') > 0$

when $|z| = 1$ and $|t| = 1$ and $|\omega| = 1$

Case V.

If $\min(|\mu|,|\eta'|,|\eta''|) \to \infty$ in eq^n no.(5) then we get:

$$\phi_{\eta,\delta,\delta',\delta'';\upsilon,\xi,\xi',\xi''}(z,t,\omega,s,a) := \lim_{\min(|\mu|,|\eta'|,|\eta''|)\to\infty}\{\phi_{\mu,\eta,\eta',\eta''\,\delta,\delta',\delta'';\upsilon,\xi,\xi',\xi''}(z/\mu,t/\mu\eta',\omega/\mu\eta'',s,a)\}$$

$$= \sum_{k,l,m=0}^{\infty} \frac{(\eta)_k(\delta)_k(\delta')_l(\delta'')_m}{(\upsilon)_{k+l+m}(\xi)_k(\xi')_l(\xi'')_m\,k!\,l!\,m!}\frac{z^k t^l \omega^m}{(k+l+m+a)^s}, \tag{10}$$

$(\eta,\delta,\delta,\delta'' \in \mathbb{C};\ a,\upsilon,\xi,\xi',\xi'' \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ s,z,t,\omega \in \mathbb{C})$

when $|z| < 1$ and $|t| < 1$ and $|\omega| < 1$

And $\mathbb{R}\,(s+\upsilon+\xi+\xi'+\xi''-\eta-\delta-\delta-\delta'') > 0$

when $|z| = 1$ and $|t| = 1$ and $|\omega| = 1$

Various Integral representations, Summation Formulae and Representation in terms Generalized Hypergeometric Function are obtained in the present note.

II. PRILIMINARIES

2.1. The Eulerian Integral is given as {[1] page 3, eq^n no. (17)}**:**

$$\frac{1}{(k+l+m+a)^s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(k+l+m+a)t} \, dt \tag{11}$$

$$(\min \mathbb{R}(s), \mathbb{R}(a) > 0 , \quad k, l \in \mathbb{N}_0)$$

2.2. Eulerian Beta Integral [1]:

$$B(a, b\text{-}a) = \frac{\Gamma(a) \; \Gamma(b-a)}{\Gamma(b)} = \int_0^\infty \frac{y^{a-1}}{(1+y)^b} \, dy, \tag{12}$$

$$\mathbb{R}(b) > \mathbb{R}(a) > 0,$$

2.3. The Binomial Series:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \tag{13}$$

where, $\binom{n}{k} = \frac{n!}{k! \; (n-k)!}$

2.4. The generalized hypergeometric function F (−) is given as [1]:

$$pFq[(\beta_1, \ldots, \beta_p; \delta_1, \ldots, \delta_q; z] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\beta_i)_n z^n}{\prod_{j=1}^{q} (\beta_j)_n n!} \tag{14}$$

where $p, q \in Z^+$; $b_j \neq 0, -1, -2 \ldots$

2.5. The following identity {[17] page 56, eq^n no. (1)}:

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} A(l,k) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} A(l, k-l) \tag{15}$$

III. MAIN RESULT

III (a). INTEGRAL REPRESENTATIONS

Theorem 1. The following integral representation $\phi_{\mu,\eta,\eta'\eta'' \, \delta,\delta',\delta'';\text{ט},\xi,\xi',\xi''}(z, t, \omega, s, a)$ holds true:

$$\phi_{\mu,\eta,\eta'\eta'' \, \delta,\delta',\delta'';\text{ט},\xi,\xi',\xi''}(z, t, \omega, s, a) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} \sum_{k,l,m=0}^{\infty} \frac{(\mu)_{k+l+m} (\eta)_k (\eta')_l (\eta'')_m (\delta)_k (\delta')_l (\delta'')_m}{(\text{ט})_{k+l+m} (\xi)_k (\xi')_l (\xi'')_m \; k! \; l! \; m!}$$

$$\times \; z^k t^l \omega^m (e^{-x})^k (e^{-x})^l (e^{-x})^m dx \tag{16}$$

Proof: Denoting the left – hand side of eq^n no. (16) by **I_1** and then using Eulerian Integral given in eq^n no. (11) on right – hand side of eq^n no. (5) we get:

$$I_1 = \sum_{k,l,m=0}^{\infty} \frac{(\mu)_{k+l+m}\,(\eta)_k\,(\eta')_l(\eta'')_m(\delta)_k(\delta')_l(\delta'')_m}{(\upsilon)_{k+l+m}\,(\xi)_k(\xi')_l(\xi'')_m\;k!\;l!\;m!} \times z^k t^l \omega^m \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1}\, e^{-(k+l+m+a)x}\, dx \qquad (17)$$

Interchanging the order of integration and summation, which is verified by uniform convergence of the involved series under the given condition, on the right – hand side of last equation we get:

$$I_1 = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1}\, e^{-ax} \sum_{k,l,m=0}^{\infty} \frac{(\mu)_{k+l+m}\,(\eta)_k\,(\eta')_l(\eta'')_m(\delta)_k(\delta')_l(\delta'')_m}{(\upsilon)_{k+l+m}\,(\xi)_k(\xi')_l(\xi'')_m\;k!\;l!\;m!} \times z^k t^l \omega^m\,(e^{-x})^k(e^{-x})^l(e^{-x})^m dx \qquad (18)$$

which is the desired result i,e eq^n no. (16) which we wanted to prove .

Corollary 1.

When we use (11) and $\delta=\xi$, $\delta'=\xi'$, $\delta''=\xi''$ in equation no. (8) we get:

$$\phi_{\mu,\eta,\eta'';\upsilon}\,(z, t, \omega, s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1}\, e^{-ax}\; \phi_1\,(\mu, \eta, \eta''\,; \upsilon\,; ze^{-x}, te^{-x}, \omega e^{-x})\, dx \qquad (19)$$

$$\min\{\mathbb{R}\,(s), \mathbb{R}(a)\} > 0 \text{ when } |z| \le 1 \;\; (z \ne 1),$$

$$|t| \le 1\; (t \ne 1), |\omega| \le 1\; (\omega \ne 1)$$

$$\mathbb{R}(s) > 1 \text{ when } z = 1, t = 1, \omega = 1$$

Corollary 2.

When we use (11) and $\delta=\xi$, $\delta'=\xi'$ $\delta''=\xi''$ in eq^n no. (9) we get:

$$\phi_{\eta,\eta',\eta'';\upsilon}\,(z, t, \omega, s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1}\, e^{-ax}\, \phi_1\,(\eta, \eta', \eta''\,; \upsilon\,; ze^{-x}, te^{-x}, \omega e^{-x})\, dx \qquad (20)$$

$$\min\{\mathbb{R}\,(s), \mathbb{R}(a)\} > 0 \text{ when } |z| \le 1 \;\; (z \ne 1),$$

$$|t| \le 1\; (t \ne 1), |\omega| \le 1\; (\omega \ne 1)$$

$$\mathbb{R}(s) > 1 \text{ when } z = 1, t = 1, \omega = 1$$

Corollary 3.

When we use (11) and $\delta=\xi$, $\delta'=\xi'$, $\delta''=\xi''$ in eq^n no. (10) we get:

$$\phi_{\eta;\upsilon}\,(z, t, \omega, s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1}\, e^{-ax}\;\phi_1\,(\eta\,; \upsilon\,; ze^{-x}, te^{-x}, \omega e^{-x})\, dx \qquad (21)$$

$$\min\{\mathbb{R}\,(s), \mathbb{R}(a)\} > 0 \text{ when } |z| \le 1 \;\; (z \ne 1),$$

$$|t| \le 1\; (t \ne 1), |\omega| \le 1\; (\omega \ne 1)$$

$$\mathbb{R}(s) > 1 \text{ when } z = 1, t= 1, \omega=1$$

Theorem 2. The following integral for $\phi_{\mu,\eta,\eta',\eta'' \delta,\delta',\delta'';\upsilon,\xi,\xi',\xi''}$ (z, t, ω, s, a) holds true:

$$\phi_{\mu,\eta,\eta',\eta'' \delta,\delta',\delta'';\upsilon,\xi,\xi',\xi''}(z,t,\omega,s,a) = \int_0^\infty \frac{y^{\mu-1}}{(1+y)^{\upsilon}} \phi_{\eta,\eta',\eta'' \delta,\delta',\delta'';\xi,\xi',\xi''}\left(\frac{zy}{1+y},\frac{ty}{1+y},\frac{\omega y}{1+y},s,a\right) \quad (22)$$

Proof: Setting $a=\mu+k+l+m$, $b=\upsilon+k+l+m$ in eq^n no. (12), we get:

$$\frac{\Gamma(\mu+k+l+m)\,\Gamma(\upsilon-\mu)}{\Gamma(\upsilon+k+l+m)} = \int_0^\infty \frac{y^{(\mu+k+l+m)-1}}{(1+y)^{(\upsilon+k+l+m)}}\,dy, \quad (23)$$

Or,

$$\frac{(\mu)_{k+l+m}}{(\upsilon)_{k+l+m}}\,\frac{\Gamma(\mu)\,\Gamma(\upsilon-\mu)}{\Gamma(\upsilon)} = \int_0^\infty \frac{y^{(\mu+k+l+m)-1}}{(1+y)^{(\upsilon+k+l+m)}}\,dy,$$

Or,

$$\frac{(\mu)_{k+l+m}}{(\upsilon)_{k+l+m}} = \frac{\Gamma(\upsilon)}{\Gamma(\mu)\,\Gamma(\upsilon-\mu)}\int_0^\infty \frac{y^{(\mu+k+l+m)-1}}{(1+y)^{(\upsilon+k+l+m)}}\,dy, \quad (24)$$

$$\mathbb{R}(\upsilon) > \mathbb{R}(\mu) > 0,\ k,l,m \in N$$

Denoting the left – hand side of eq^n no. (22) by $\mathbf{I_2}$ and using eq^n no. (24) on right – hand side of eq^n no. (5) we get:

$$\mathbf{I_2} = \frac{\Gamma(\upsilon)}{\Gamma(\mu)\,\Gamma(\upsilon-\mu)} \sum_{k=0}^{\infty}\int_0^\infty \frac{y^{(\mu+k+l+m)-1}}{(1+y)^{(\upsilon+k+l+m)}} \frac{(\eta)_k(\eta')_l(\eta'')_m(\delta)_k(\delta')_l(\delta'')_m}{(\xi)_k(\xi')_l(\xi'')_m\,k!\,l!\,m!} \frac{z^k t^l \omega^m}{(k+l+m+a)^s}\,dy \quad (25)$$

On interchanging the signs of integration and summation on the right – hand side of the last eq^n ,we get:

$$\mathbf{I_2} = \frac{\Gamma(\upsilon)}{\Gamma(\mu)\,\Gamma(\upsilon-\mu)}\int_0^\infty \frac{y^{(\mu)-1}}{(1+y)^{(\upsilon)}} \sum_{k=0}^{\infty} \frac{(\eta)_k(\eta')_l(\eta'')_m(\delta)_k(\delta')_l(\delta'')_m}{(\xi)_k(\xi')_l(\xi'')_m\,k!\,l!\,m!} \left(\frac{y}{1+y}\right)^k \left(\frac{y}{1+y}\right)^l \left(\frac{y}{1+y}\right)^m \frac{z^k t^l \omega^m}{(k+l+m+a)^s}\,dy \quad (26)$$

Or,

$$\mathbf{I_2} = \frac{\Gamma(\upsilon)}{\Gamma(\mu)\,\Gamma(\upsilon-\mu)}\int_0^\infty \frac{y^{(\mu)-1}}{(1+y)^{(\upsilon)}} \sum_{k=0}^{\infty} \frac{(\eta)_k(\eta')_l(\eta'')_m(\delta)_k(\delta')_l(\delta'')_m}{(\xi)_k(\xi')_l(\xi'')_m\,k!\,l!\,m!} \left(\frac{zy}{1+y}\right)^k \left(\frac{ty}{1+y}\right)^l \left(\frac{\omega y}{1+y}\right)^m \frac{1}{(k+l+m+a)^s}\,dy \quad (27)$$

Or,

$$\mathbf{I_2} = \frac{\Gamma(\upsilon)}{\Gamma(\mu)\,\Gamma(\upsilon-\mu)}\int_0^\infty \frac{y^{\mu-1}}{(1+y)^{\upsilon}} \phi_{\eta,\eta',\eta'' \delta,\delta',\delta'';\xi,\xi',\xi''}\left(\frac{zy}{1+y},\frac{ty}{1+y},\frac{\omega y}{1+y},s,a\right)dy \quad (28)$$

which is the right -hand side of eq^n (22) i,e the desired result .

Theorem 3. The following integral for $\phi_{\mu,\eta,\eta',\eta'' \delta,\delta',\delta'';\upsilon,\xi,\xi',\xi''}$ (z, t, ω, s, a) holds true:

$$\phi_{\mu,\eta,\eta',\eta'' \delta,\delta',\delta'';\upsilon,\xi,\xi',\xi''}(z,t,\omega,s,a) = \frac{\Gamma(\upsilon)}{\Gamma(s)\,\Gamma(\mu)\,\Gamma(\upsilon-\mu)}\int_0^\infty\int_0^\infty \frac{x^{s-1}e^{-ax}y^{\mu-1}}{(1+y)^{\upsilon}} \sum_{k=0}^{\infty}\frac{(\eta)_k(\delta)_k}{k!(\xi)_k}\left(\frac{zy\,e^{-x}}{1+y}\right)^k \sum_{l=0}^{\infty}\frac{(\eta)_l(\delta)_l}{l!(\xi)_l}\left(\frac{ty\,e^{-x}}{1+y}\right)^l \sum_{m=0}^{\infty}\frac{(\eta)_m(\delta)_m}{m!(\xi)_m}\left(\frac{\omega y\,e^{-x}}{1+y}\right)^m dx\,dy, \quad (29)$$

Proof: Denoting the left – hand side of eq^n no. (29) by $\mathbf{I_3}$ and the using eq^n no. (16) on right- hand side we get:

$$I_3 = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} \sum_{k,l,m=0}^\infty \frac{(\mu)_{k+l+m} (\eta)_k (\eta')_l (\eta'')_m (\delta)_k (\delta')_l (\delta'')_m}{(\upsilon)_{k+l+m} (\xi)_k (\xi')_l (\xi'')_m \; k! \; l! \; m!} z^k t^l \omega^m (e^{-x})^k (e^{-x})^l (e^{-x})^m dx, \qquad (30)$$

Or,

$$I_3 = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} \sum_{k,l,m=0}^\infty \frac{(\mu)_{k+l+m} (\eta)_k (\eta')_l (\eta'')_m (\delta)_k (\delta')_l (\delta'')_m}{(\upsilon)_{k+l+m} (\xi)_k (\xi')_l (\xi'')_m} \frac{(ze^{-x})^k}{k!} \frac{(te^{-x})^l}{l!} \frac{(\omega e^{-x})^m}{m!} dx \qquad (31)$$

,

Using eq^n no. (24) on right – hand side of above eq^n no. (31) we get:

$$I_3 = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} \sum_{k,l,m=0}^\infty \frac{\Gamma(\upsilon)}{\Gamma(\mu)\ \Gamma(\upsilon-\mu)} \int_0^\infty \frac{y^{(\mu+k+l+m)-1}}{(1+y)^{(\upsilon+k+l+m)}} dy \frac{(\eta)_k (\eta')_l (\eta'')_m (\delta)_k (\delta')_l (\delta'')_m}{(\xi)_k (\xi')_l (\xi'')_m}$$

$$\frac{(ze^{-x})^k}{k!} \frac{(te^{-x})^l}{l!} \frac{(\omega e^{-x})^m}{m!} dx \qquad (32)$$

Or,

$$I_3 = \frac{\Gamma(\upsilon)}{\Gamma(s)\ \Gamma(\mu)\ \Gamma(\upsilon-\mu)} \int_0^\infty \int_0^\infty \frac{x^{s-1} e^{-ax} y^{\mu-1}}{(1+y)^{\upsilon}} \sum_{k=0}^\infty \frac{(\eta)_k (\delta)_k}{k! (\xi)_k} \left(\frac{zy\, e^{-x}}{1+y}\right)^k \sum_{l=0}^\infty \frac{(\eta)_l (\delta)_l}{l! (\xi)_l} \left(\frac{ty\, e^{-x}}{1+y}\right)^l$$

$$\sum_{m=0}^\infty \frac{(\eta)_m (\delta)_m}{m! (\xi)_m} \left(\frac{\omega y\, e^{-x}}{1+y}\right)^m dx\, dy \qquad (33)$$

which is the right -hand side of eq^n no. (29) it is desired result.

Corollary 4. If $\delta=\delta'=\delta''=1$ and $\xi=\xi'=\xi''=1$ in eq^n no. (29) then we get,

$$\phi_{\mu,\eta,\eta',\eta'';\upsilon}(z, t, \omega, s, a) = \frac{\Gamma(\upsilon)}{\Gamma(s)\ \Gamma(\mu)\ \Gamma(\upsilon-\mu)} \int_0^\infty \int_0^\infty \frac{x^{s-1} e^{-ax} y^{\mu-1}}{(1+y)^{\upsilon}} \left(1 - \frac{zy\, e^{-x}}{1+y}\right)^{-\eta} \left(1 - \frac{ty\, e^{-x}}{1+y}\right)^{-\eta'}$$

$$\left(1 - \frac{\omega y\, e^{-x}}{1+y}\right)^{-\eta''} \qquad (34)$$

$$\mathbb{R}(\upsilon) > \mathbb{R}(\mu) > 0 \ ; \ \min\{\mathbb{R}(s), \mathbb{R}(a)\} > 0$$

III (b). SUMMATION FORMULA

Theorem 4. The following summation formula hold true.

$$\sum_{r=0}^\infty \frac{(s)_r}{r!} \phi_{\mu,\eta,\eta',\eta''\,\delta,\delta',\delta'';\upsilon,\xi,\xi',\xi''}(z, t, \omega, s+r, a)\, x^r = \phi_{\mu,\eta,\eta',\eta''\,\delta,\delta',\delta'';\upsilon,\xi,\xi',\xi''}(z, t, \omega, s, a-x) \qquad (35)$$

$$(|x| < |a| \ ; \ s \neq 1)$$

Proof : Using eq^n no. (5) in right -hand side of eq^n no. (35) we get:

Right -hand side of eq^n no. (35) $= \sum_{k,l,m=0}^{\infty} \frac{(\mu)_{k+l+m} (\eta)_k (\eta')_l (\eta'')_m (\delta)_k (\delta')_l (\delta'')_m}{(\upsilon)_{k+l+m} (\xi)_k (\xi')_l (\xi'')_m \; k! \; l! \; m!} \frac{z^k t^l \omega^m}{(k+l+m+a-x)^s}$

$$= \sum_{k,l,m=0}^{\infty} \frac{(\mu)_{k+l+m} (\eta)_k (\eta')_l (\eta'')_m (\delta)_k (\delta')_l (\delta'')_m}{(\upsilon)_{k+l+m} (\xi)_k (\xi')_l (\xi'')_m} \frac{z^k t^l \omega^m}{k!l!m!(k+l+m+a)^s} \left(1 - \frac{x}{k+l+m+a}\right)^{-s}$$

$$= \sum_{r=0}^{\infty} \frac{(s)_r}{r!} \left\{ \sum_{k,l,m=0}^{\infty} \frac{(\mu)_{k+l+m} (\eta)_k (\eta')_l (\eta'')_m (\delta)_k (\delta')_l (\delta'')_m}{(\upsilon)_{k+l+m} (\xi)_k (\xi')_l (\xi'')_m \; k! \; l! \; m!} \frac{z^k t^l \omega^m}{(k+l+m+a)^{s+r}} \right\} x^r$$

(Using Binomial theorem)

$$= \sum_{r=0}^{\infty} \frac{(s)_r}{r!} \{\phi_{\mu,\eta,\eta',\eta'' \delta,\delta',\delta'';\upsilon,\xi,\xi',\xi''} (z, t, \omega, s+r, a)\} x^r$$

(In view of eq^n no. (5))

= left – hand side of equation no. (35).

Hence**,** Theorem (4) is proved.

III (c). REPRESENTATION IN TERMS GENERALIZED HYPERGEOMETRIC FUNCTION

Theorem 5 For a ≠ {-1, -2, . . . } and z ≠ 0 , the following explicit series representation hold true .

$$\phi_{\mu,\eta,\eta',\eta'' \delta,\delta',\delta'';\upsilon,\xi,\xi',\xi''} (z, t, \omega, s, a) = \sum_{k=0}^{\infty} \frac{(\mu)_k (\eta)_k (\delta)_k}{(\upsilon)_k (\xi)_k \, k!} \frac{z^k}{(k+a)^s}$$

$$\times \; F\{\eta',\eta'', \delta', \delta'' \, (1 - \xi - k), (-k) \; ; \; \xi', \xi'' \, (1 - \eta - k), (1 - \delta - k) \; ; \frac{t}{z}, \frac{\omega}{z}\} \qquad (36)$$

where F (−) is the generalized Hypergeometric Function $_pF_q$ defined in eq^n no. (15)

Proof: Eq^n no. (15) may also be expressed as:

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} A(m, l, k) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} \sum_{m=0}^{l} A(m, l, k - l - m) \qquad (37)$$

Using eq^n no. (37) on right-hand side of eq^n no. (5)

$$\phi_{\mu,\eta,\eta',\eta'' \delta,\delta',\delta'';\upsilon,\xi,\xi',\xi''} (z, t, \omega, s, a) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} \sum_{m=0}^{l} \frac{(\mu)_k (\eta)_{k-l-m} (\eta')_l (\eta'')_m (\delta)_{k-l-m} (\delta')_l (\delta'')_m}{(\upsilon)_k (\xi)_{k-l-m} (\xi')_l (\xi'')_m \; (k-l-m)! \; l! \; m!} \frac{z^{k-l-m} t^l \omega^m}{(k+a)^s} \qquad (38)$$

Since:

$$(k - l - m)! = \frac{(-1)^{l+m}}{(-k)_{l+m}} \; k! , \quad 0 \le m \le l \le k \qquad (39)$$

$$(\eta)_{k-l-m} = \frac{(-1)^{l+m} (\eta)_k}{(1-\eta-k)_{l+m}} , \quad 0 \le m \le l \le k \qquad (40)$$

Therefore in light of eq^{n}s (39) and (40) , eq^n (38) reduces To

$\phi_{\mu,\eta,\eta',\eta'' \delta,\delta',\delta'';\upsilon,\xi,\xi',\xi''} (z, t, \omega, s, a) =$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{k} \sum_{m=0}^{l} \frac{(\mu)_k (\eta')_l (\eta'')_m (\delta')_l (\delta'')_m}{(\upsilon)_k (\xi')_l (\xi'')_m \; l! \; m!} \frac{(-1)^{l+m} (\eta)_k}{(1-\eta-k)_{l+m}} \frac{(-1)^{l+m} (\delta)_k}{(1-\delta-k)_{l+m}} \frac{(1-\xi-k)_{l+m}}{(-1)^{l+m} (\xi)_k} \frac{z^{k-l-m} t^l \omega^m}{(k+a)^s} \frac{(-k)_{l+m}}{(-1)^{l+m} \, k!}$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{k} \sum_{m=0}^{l} \frac{(\mu)_k (\eta')_l (\eta'')_m (\delta')_l (\delta'')_m}{(\upsilon)_k (\xi')_l (\xi'')_m \; l! \; m!} \frac{(\eta)_k}{(1-\eta-k)_{l+m}} \frac{(\delta)_k}{(1-\delta-k)_{l+m}} \frac{(1-\xi-k)_{l+m}}{(\xi)_k} \frac{z^{k-l-m} t^l \omega^m}{(k+a)^s} \frac{(-k)_{l+m}}{k!}$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{k} \sum_{m=0}^{l} \frac{(\mu)_k (\eta)_k (\eta')_l (\eta'')_m (\delta)_k (\delta')_l (\delta'')_m}{(\upsilon)_k (\xi)_k (\xi')_l (\xi'')_m \, k! \; l! \; m!} \frac{(-k)_{l+m} (1-\xi-k)_{l+m}}{(1-\eta-k)_{l+m} (1-\delta-k)_{l+m}} \frac{z^{k-l-m} t^l \omega^m}{(k+a)^s}$$

$$= \sum_{k=0}^{\infty} \frac{(\mu)_k (\eta)_k (\delta)_k}{(\upsilon)_k (\xi)_k \, k!} \frac{z^k}{(k+a)^s} \;\; X \;\; F\{\eta'\eta'', \delta', \delta'' \, (1-\xi-k), (-k) \; ; \; \xi', \xi'' \, (1-\eta-k), (1-\delta-k) \, ; \frac{t}{z}, \frac{\omega}{z}\} \qquad (41)$$

where F (-) is the generalized Hypergeometric Function.

= Right – hand side of eq^n no. (36).

Hence, Theorem (5) is proved.

Corollary 5. If we set $\delta=\xi$ in eq^n no. (37), then we get

$$\phi_{\mu,\eta,\eta'\eta'' \, \xi,\delta',\delta'';\upsilon,\xi,\xi',\xi''} \, (z, t, \omega, s, a) = \sum_{k=0}^{\infty} \frac{(\mu)_k (\eta)_k}{(\upsilon)_k \, k!} \frac{z^k}{(k+a)^s} \;\; X \;\; F\{\eta'\eta'', \delta', \delta'' \, , (-k) \; ; \; \xi', \xi'' \, (1-\eta-k) \, ; \frac{t}{z}, \frac{\omega}{z}\}$$

(24)

Corollary 6. If we set $\upsilon=\eta$, $\delta=\xi$, $\delta'=\xi'$ and $\delta''=\xi''$ in eq^n no. (37), then we get:

$$\phi_{\mu,\eta,\eta'\eta''\xi,\xi',\xi'';\eta,\xi,\xi',\xi''} \, (z, t, \omega, s, a) = \sum_{k=0}^{\infty} \frac{(\mu)_k}{k!} \frac{z^k}{(k+a)^s} \;\; X \;\; F\{\eta'\eta'', (-k) \; ; \; (1-\eta-k) \; ; \frac{t}{z}, \frac{\omega}{z}\} \qquad (43)$$

NOTE: In the next paper certain integral representation and summation formula for extended Hurwitz - Lerch Zeta function of 'n – variables.

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