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Research Paper

A Characterization of theZero-One inflated Binomial Distribution

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ABSTRACT:*In this paper, we introduce a characterization of the zero-one inflated binomial distribution through a linear differential equation of its probability generating function.*

Keywords*: Binomial Distribution,Zero-One Inflated Binomial Distribution, Probability Generating Function, Linear Differential Equation.*

I. INTRODUCTION

The binomial distribution (BD) is a well-known discrete distribution. It arises as the distribution of the number of successes in a sum of identically and independent trials. For further details of the BD see Johnson et al [1] pp 108-155, and pp 109, in particular, for historical remarks and genesis of the BD. For recent real-life applications of the standard BD and its inflated form, see Banik and Kibria[2]. Recently, Nanjundan and Pasha [3] characterized the zero-inflated BD via a linear differential equationof its pgf.

In this paper, we introduce in Section 2, the definition of the BD and its zero-one inflated formwith their probability generating function (pgf), followed in Section 3by acharacterizationof the zero-one inflated binomial distribution (ZOIBD)through a linear differential equation.

II. THE BINOMIAL DISTRIBUTIONS AND ITS ZERO-ONE INFLATED FORM

Let $n \in \{1, 2, 3 \dots\}$ and $p \in (0,1)$, then the discrete random variable (rv) X having probability mass function (pmf);

$$P(X = x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, 2, 3, \dots, n \\ 0, & \text{otherwise}, \end{cases} \quad (1)$$

is said to have a BD with parameters n and p. We will denote that by writing $X \sim BD(n, p)$.

The pgf of the rv X, $G_X(t)$, can be shown to be;

$$G_X(t) = E(t^X) = \sum_{x=0}^{n} \binom{n}{x} (pt)^x (1-p)^{n-x}$$
$$= (1 - p + pt)^n$$

Let $X \sim BD(n, p)$ as given in (1), let $\alpha \in (0,1)$ be a proportion of zero added to the rv X, and let $\beta \in (0,1)$ be an extra proportion added to the proportion of ones of the rv X, such that $0 < \alpha + \beta < 1$, then the rv Z defined by;

$$P(Z=z)=\begin{cases}\alpha+(1-\alpha-\beta)(1-p)^n, & z=0\\ \beta+n(1-\alpha-\beta)p\,(1-p)^{n-1}, & z=1\\ (1-\alpha-\beta)\binom{n}{z}p^z(1-p)^{n-z}, & z=2,3,\dots,n\\ 0\,, & \text{otherwise,}\end{cases} \tag{2}$$

is said to have a ZOIBD, and we will denote that by writing $Z \sim \mathrm{ZOIBD}(n,p;\alpha,\beta)$.

Note that, if $\beta\to 0$, then (2) reduces to the form of the zero-inflated BD. Similarly, the case with $\alpha\to 0$ and $\beta\to 0$, reduces to the standard case of BD.

The pgf of the rv Z can be shown to be;

$$\begin{aligned} G_Z(t) &= \alpha+\beta t+(1-\alpha-\beta)G_X(t)\\ &= \alpha+\beta t+(1-\alpha-\beta)(1-p+pt)^n \end{aligned} \tag{3}$$

III.CHARACTERIZATION OF THE ZERO-ONE INFLATED BINOMIAL DISTRIBUTION

We give below the main result in this paper.

Theorem 1: The discrete rv Z taking non-negative integer values, has a ZOIBD if its pgf, $G(t)$, satisfies for some arbitrary number c, a positive number b and non-zeros numbers a, f and h, that;

$$(a+bt)\frac{\partial}{\partial t}G(t)=c+ht+fG(t) \tag{4}$$

Proof: Without loss of generality, let us assume that $b=1$, hence (4) becomes as;

$$(a+t)\frac{\partial}{\partial t}G(t)=c+ht+fG(t) \tag{5}$$

Assume first that $f\neq 1$.Now, (5), by using of the rule of the derivative of two product functions, can be written in the following equivalent form;

$$\frac{\partial}{\partial t}\left[\frac{G(t)}{(a+t)^f}\right]=(c+ht)(a+t)^{-f-1}$$

Hence,

$$\frac{G(t)}{(a+t)^f}=\int (c+ht)(a+t)^{-f-1}\,dt \tag{6}$$

Now by making the substitutingx= $a+t$in the integral given in (6) and evaluatedit, we get that;

$$\frac{G(t)}{(a+t)^f}=\left(-\frac{c}{f}-\frac{ah}{f(f-1)}-\frac{ht}{f-1}\right)(a+t)^{-f}+k$$

where k is a an arbitrary constant. Hence;

$$G(t)=\left(-\frac{c}{f}-\frac{ah}{f(f-1)}-\frac{ht}{f-1}\right)+k(a+t)^f$$

Since 1 = $G(1)$; we get that;

$$k=\left[1+\frac{c}{f}+\frac{ah}{f(f-1)}+\frac{h}{f-1}\right](a+1)^{-f}$$

Therefore;

$$G(t)=\left(-\frac{c}{f}-\frac{ah}{f(f-1)}-\frac{ht}{f-1}\right)+\left[1+\frac{c}{f}+\frac{ah}{f(f-1)}+\frac{h}{f-1}\right](a+1)^{-f}(a+t)^f$$

Or equivalently,

$$G(t) = -\left(\frac{c}{f} + \frac{ah}{f(f-1)}\right) - \frac{ht}{f-1} + \left[1 + \frac{c}{f} + \frac{ah}{f(f-1)} + \frac{h}{f-1}\right]\left(\frac{a+t}{a+1}\right)^{f} \quad (7)$$

Let;

$$p = \frac{1}{a+1} \quad (8)$$

$$\alpha = -\frac{c}{f} - \frac{ah}{f(f-1)} \quad (9)$$

$$\beta = -\frac{h}{f-1} \quad (10)$$

Then, G(t) given in (7) can be written in the following form;

$$G_Z(t) = \alpha + \beta t + (1-\alpha-\beta)(1-p+pt)^{f} \quad (11)$$

which is the same form given (3).

Now we need to check possible values of the parameters of (11); namely, f, p, α and β.

Consider the case thatf$\in \{2,3 \ldots\}$. If $a > 1$, then p, given by (8) satisfies that $0 < p < 1$. Therefore, if $0 < \alpha < 1, 0 < \beta < 1$ and$0 < \alpha + \beta < 1$, then G(t) given in (11) is a pgf of $\mathrm{ZOIBD}(f, p; \alpha, \beta)$ as given in (3).

If $-(f-1) < h < 0$ then $0 < -\frac{h}{f-1} < 1$, hence $0 < \beta < 1$. If; $-f - \frac{ah}{f-1} < c < -\frac{ah}{f-1}$, then $0 < -\frac{c}{f} - \frac{ah}{f(f-1)} < 1$, and hence $0 < \alpha < 1$. If c is also satisfying that $-f - \frac{h(a+f)}{f-1} < c < -\frac{h(a+f)}{f-1}$, then $0 < -\frac{c}{f} - \frac{ah}{f(f-1)} - \frac{h}{f-1} < 1$ and hence $0 < \alpha + \beta < 1$. Since the intervals $(-f - \frac{ah}{f-1}, -\frac{ah}{f-1})$ and $(-f - \frac{h(a+f)}{f-1}, -\frac{h(a+f)}{f-1})$ is overlapping and their intersection is $(-f - \frac{h(a+f)}{f-1}, -\frac{ah}{f-1})$, it follows that if c is satisfying that$-f - \frac{h(a+f)}{f-1} < c < -\frac{ah}{f-1}$, then $0 < \alpha < 1,\ 0 < \beta < 1$ and$0 < \alpha + \beta < 1$, and therefore G(t) is the pgf of $\mathrm{ZOIBD}(f, p; \alpha, \beta)$.

Now if $f = 1$, then the solution of (5), on the same lines as given above, can be shown to be;

$$G(t) = h(a+t)\log(a+t) - c + ah + k(a+t) \quad (12)$$

where k is a an arbitrary constant. Therefore;

$$\frac{\partial^{(z)}}{\partial t^{z}} G(t) = (z-2)!\, h(-1)^{z-2}(a+t)^{-(z-1)}, \qquad z = 2, 3, \ldots (13)$$

Since for any positive integer number z, $P(Z = z) = \frac{1}{z!}\frac{\partial^{(z)}}{\partial t^{z}} G_Z(0)$, we have from (13) that;

$$P(Z = z) = \frac{(-1)^{z-2}}{z(z-1)}\frac{h}{a^{z-1}} \qquad z = 2, 3, \ldots$$

which is negative for some z = 2, 3,…, and therefore it is not a pmf, implying that G given in (12) is not a pgf and hence the case that $f = 1$ is not possible.

Other cases of f, namely, f is negative or it is not an integer, then we have from (7) that;

$$\frac{\partial^{(z)}}{\partial t^{z}} G(t) = f(f-1)\ldots(f-z+1)A(a+t)^{f-z}, \qquad z = 2, 3, \ldots$$

where;

$$A = \left[1 + \left(\frac{c + cf + ah + fh}{f(f-1)}\right)\right]\left(\frac{1}{a+1}\right)^{f}$$

Hence, also $P(Z = z) = \frac{1}{z!}\frac{\partial^{(z)}}{\partial t^{z}} G_Z(0)$ will be negative for some z = [f]+2, [f]+3, …, where [f] is the integral part of f, implying that G is not a pgf and therefore this case is not possible.

Finally, if c = 0, then we arrive simply, either by solving (5) directly or by letting $c \to 0$ in the above proof, which is straightforward, to the same conclusion that the pgf is given (7) with p and β given by (8) and

(10), respectively, and that $\alpha = -\frac{ah}{f(f-1)}$, and on the same lines as above, whenthat $f \in \{2,3 \dots\}$and also $a > 1$,it can be shown that if $-\frac{f(f-1)}{f+a} < h < 0$, then $0 < \alpha < 1$, $0 < \beta < 1$ and$0 < \alpha + \beta < 1$.This completes the proof.

Theorem 1 leads to the following.

Theorem 2: Let Z be a discrete rv taking non-negative integer values, then $Z \sim ZOIBD(f, p; \alpha, \beta)$, for some non-zero f, p, α and βif and only if its pgf,G(t), satisfying (4) for some arbitrary number c, a positive number b and non-zeros numbers a, f and h.

Theorem 2 leads to the following conclusion obtained byNanjundan and Pasha[3].

Theorem3: Let Z be a discrete rv taking non-negative integer values, then $Z \sim ZIBD(f, p; \alpha)$, for some non-zero f, p and αif and only if its pgf G(t) satisfying;

$$(a + bt)\frac{\partial}{\partial t}G(t) = c + fG(t)$$

for somepositive number b and non-zeros numbers a, c and f.

Proof: Just let $h \rightarrow 0$ in Theorem 2.

IV. CONCLUSIONS

We introduceda characterization of the zero-one inflated binomialdistribution through a linear differential equation of its probability generating function. We would propose an extension of this results to other forms and distributions.

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