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Research Paper

Limit in Dual Space

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ABSTRACT : *In this paper we have studied the behavior of the different limits in some of the dual spaces of function spaces. A good number of results have been established. We have also observed that in few cases we have to take the of a set suitable conditions. We also used the notion of a perfect function space in establishing a few results .As a matter of facts the notions of which are used in this paper are parametric convergent ,parametric limit, projective convergent , projective limit, dual space of a function space and perfect function space in addition of the suitably defined and constructed some of the function spaces.*

Keywords : *Dual space of a function space , Parametric limit , Parametric convergent, Perfect function space, Projective limit ,Projective convergent .*

I. INTRODUCTION

The concept of limit and convergence play vital role in the study of both sequence and function spaces. Peterson,(1) and Cooke,(1) in chapter 10 , have given ample results on limit and convergence but for sequence space only . Later on some of students of the school of mathematics extended the theory of limits in their for function spaces. Still there is a vast scope for further work by extending results on limit for different function spaces . Some of the results established in Sharan,(1) have been extended to the case of dual spaces of the function spaces considered by Sharan himself .

Essential Preliminaries And Definitions

2.1 Function Space -A linear space whose elements are functions is called a function space . Thus a set V of functions is a functions space if it contains the origin and for f , g in V and for every scalar α , f + g and αf are in **V .** Here we consider only real functions of real variables. so α is taken to be real scalar so that our purpose is served .Definitions of some special function spaces are being given below making the use of which some results have been established .Moreover , the integration has been taken through in Lebesgue sense in the interval $[0, \infty)$. We denote the set $[0, \infty)$ by E .

2.1.1 Γ : It denotes the space of all convergent and bounded functions.

2.1.2 L□ : It denotes the space of all functions f such that $|f(x)| < K$ for almost all $x \geq 0$ where K is constant .

2.1.3 L_1: It denotes the space of integrable functions, that is L_1 is the space of all functions f such that $\int_E |f(x)|\, dx <$ □

2.1.4 □: It denotes the space of all functions continuous and bounded in $[0, \infty)$. Clearly $\zeta < L_\infty$

2.1.5 Φ : Let E = $[0.\infty)$.Let E^1 be a subset of E such that $m(E^1)$ is finite . Then the set of all functions f such that f(x) is finite and bounded for almost all x in E^1 and is zero in the compliment of E^1 , is defined to be the space of finite functions and is denoted by Φ.

2.2 Convergent Function - A function f(x) which is (i) essentially bounded in $[0, \infty)$,and (ii) tends to a definite finite limit as x tends to ∞ is called a convergent function. As π denotes the space of all convergent and bounded functions , thus clearly □ **< L□**

2.3 Dual Space :α^* of a function space α is the space of all functions of f such that $\int_E |f(x)g(x)|\,dx <$ □ for every function g(x) in α.Also α^* is a function space.Also $\Gamma^* = L_1$; L□$^* = L_1$; □$^* = L_1$;$L_1^* =$ L□ ..

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2.4 Perfect Space : A function space α is said to be perfect when $\alpha^{**}=\alpha$.Also L_1,L_∞ are perfect.See Sharan , (1)
2.5 Parametric Convergent (or t – convergent) :Let $f_t(x)$ be a family of functions of x defined for all t in $[0,\infty)$, where t is a parameter .If to every $\epsilon > 0$, there corresponds a positive number $T(\epsilon)$, idependent of x , such that , for almost all $x \geq 0$,$|$ **$f_t(x) - f_{t^1}(x)$** $|$ □ □for all t , $t^1 \geq T(\epsilon)$, then the family $f_t(x)$ is said to be parametric convergent (t – cgt). [See Sharan , (1)] .
2.6 Parametric Limit (t–limit):If , to given any $\epsilon > 0$, there corresponds a number $T(\epsilon)$, independent of x , such that for almost all $x \geq 0$, **$|f_t(x) -$ □ $(x)|$** □ □for all $t \geq T(\epsilon)$, then $\psi(x)$ is called the parametric limit (t–limit) of $f_t(x)$ and we write t–limit of $f_t(x) = \psi(x)$.Here we observe that any function equal to $\psi(x)$, for almost all $x \geq 0$, is also a t–limit of $f_t(x)$.Therefore when we say that $\psi(x)$ is the parametric limit (t-limit) of $f_t(x)$, we mean that $\psi(x)$ is a t-limit of $f_t(x)$ and all functions equivalent to $\psi(x)$ in $[0,\infty)$ are t-limits of $f_t(x)$.[A function θ is said to be equivalent to $\psi(x)$ in $[0,\infty)$. when $\theta(x) = \psi(x)$ almost everywhere in $[0,\infty)$.

2.7 Projective Convergence (or $\alpha\beta$-convergence or p-convergence):Let $\alpha^* \supseteq \beta$ and $F_g(t) = \int_E f_t(x)g(x)dx$
Where $f_t(x)$ is in α ang $g(x)$ is in β then if $F_g(t)$ tends to a definite finite limit as t – tends to ∞ for every $g(x)$ in β then we say that $f_t(x)$ is projective convergent (or p-convergent) relative to β , or $f_t(x)$ is $\alpha\beta$-convergent and $f_t(x)$ is simply called p-convergent in α or α-convergent when $\beta = \alpha^*$.
2.7.1 A necessary and sufficient condition for $\alpha\beta$- convergence of $f_t(x)$ is that to every g in β and to every $\epsilon > 0$, there corresponds a positive number $T(\epsilon,g)$ such that , for all t , $t^1 \geq T(\epsilon,g)$, $| \int_E g(x)\{f_t(x) - f_{t^1}(x)\}dx | \leq \epsilon$.
2.8 Projective Limit (p– limit or $\alpha\beta$ – limit):A function ψ , in α or outside α ,is called a projective limit (p-limit) of $f_t(x)$ in α relative to β and we write $\psi(x) = \alpha\beta$ - limit of $f_t(x)$ when $\int_E |g(x)\psi(x)|dx <$ □ for every g in β , and $\lim_{t\to\infty} \int_E f_t(x)g(x)\,dx = \int_E \psi(x)g(x)dx$ for every g in β .
When $\beta = \alpha^*$, ψ is called a projective limit (p - limit of $f_t(x)$) in α and we write , $\psi(x) = \alpha$ – limit of $f_t(x)$. Different $\alpha\beta$ – limits of $f_t(x)$ can differ only in a set of x of measure zero. Hence when we say that $\psi(x)$ is the $\alpha\beta$ – limits of $f_t(x)$ we mean that $\psi(x)$ is an $\alpha\beta$-limit of $f_t(x)$ and other $\alpha\beta$-limit of $f_t(x)$ are equivalent to $\psi(x)$. It follows from the definitions that every $\alpha\beta$ – limit belong to β^* .

3 In this section we established some of the results using the notions above

Theorem (3.1): Let $f_t(x)$ in Γ^{**} be a family of functions of x defined for all t in E where t is a parameter then Every parametric convergent family of $f_t(x)$ in Γ^{**} is $\Gamma^{**}\Gamma^*$ - convergent and Every parametric limit of $f_t(x)$ is $\Gamma^{**}\Gamma^*$ - limit of $f_t(x)$.
Proof : Let f_t be a family of functions in function space $\Gamma^{**}\Gamma^*$. Also let f_t be parametric convergent in $\Gamma^{**}\Gamma^*$.Then to every $\epsilon > 0$,there exists a positive number $T(\epsilon)$, independent of x , such that, for almost all $x \geq 0$,$|f_t(x) - f_{t^1}(x)| \leq \epsilon$ …….(1) for all t, $t^1 \geq T(\epsilon)$,Now let g(x) be any function in Γ^* .Hence g(x) must be in L_1 Thus $\int_E |g(x)|dx < \infty$ ….(2).Now since we see that , $| \int_E g(x)\{f_t(x) - f_{t^1}(x)\}dx | \leq \int_E |g(x)\{f_t(x) - f_{t^1}(x)\}|dx$ $\leq \epsilon \int_E |g(x)|dx$ [By (1)] $\leq \epsilon K(g)$ [By (2)]for all t , $t^1 \geq T(\epsilon)$ every $\epsilon > 0$, where K(g) is a constant depending on g but independent of t in $E = [0,\infty)$ .But then by the necessary and sufficient conditions for f_t to be$\Gamma^{**}\Gamma^*$-convergent . f_t is $\Gamma^{**}\Gamma^*$-convergent .Or simply f_t is $\Gamma^{**}\Gamma^*$-convergent .
For the proof of part (ii) we proceed as follow : Let $f_t(x)$ be a family of functions of x in Γ^{**} .Also let parametric limit of $f_t(x) = \psi(x)$ .Then for a given any $\epsilon > 0$, there corresponds a number $T(\epsilon)$, independent of x , such that for almost all $x \geq 0$, $|f_t(x) - \psi(x)| \leq \epsilon$…….(3) for all $t \geq T(\epsilon)$,Again let $E = [0, \infty)$ .Also let g(x) be any function of x in Γ^*. Then find that , $\int_E |g(x)|dx < \infty$……. (4). Now $\int_E |g(x)\psi(x)|dx$ for g(x) is in Γ^*, $\epsilon > 0$, $t \geq T(\epsilon) = \int_E |g(x)||\psi(x) - f_t(x) + f_t(x)|dx \leq \int_E |g(x)||\psi(x) - f_t(x)|dx + \int_E |g(x)f_t(x)|dx < \epsilon K(g) + \int_E |g(x)f_t(x)|dx$ Also $\int_E |g(x)f_t(x)|dx < \infty$ ……..(5) for f_t is in Γ^{**} and g in Γ^*,Also, $|\int_E f_t(x)g(x)dx - \int_E \psi(x)g(x)dx| \leq \int_E |g(x)|f_t(x) - \psi(x)|dx < \epsilon.K(g)$.Thus $\lim_{t\to\infty} \int_E f_t(x)g(x)\,dx = \int_E \psi(x)g(x)dx$ …(6).Thus by [(3.13) and (3.14)] it follows that $\psi(x)$ is $\Gamma^{**}\Gamma^*$ - limit of $f_t(x)$ .Hence the proof of the theorem .

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Theorem (3.2) :If Γ is a perfect function space then every parametric limit of $f_t(x)$, a family of functions of x in Γ^{**} is Γ-limit .

Proof :Let $f_t(x)$ be a family of function of x in Γ^{**}.But Γ is perfect by hypothesis .Hence $\Gamma = \Gamma^{**}$ .Thus the family $f_t(x)$ of functions of x is in Γ^{**} implies that $f_t(x)$ is in Γ .lso let $\psi(x)$ = parametric limit of $f_t(x)$. Then for a given any $\epsilon > 0$, there exists a positive number $T(\epsilon)$, independent of x , such that for almost all $x \geq 0$, $|f_t(x) - \psi(x)| \leq \epsilon$ ….(7) for all $t \geq T(\epsilon)$,Again let $g(x)$ be any function of x in Γ^*.Then earlier $\int_E |g(x)|dx < \infty$… (8) Let $E = [0 , \infty)$ .Then $\int_E g(x)\psi(x)\,dx$ for g in Γ^*, $\epsilon > 0$, $t \geq T(\epsilon) = \int_E |g(x)||\psi(x) - f_t(x) + f_t(x)|dx \leq \int_E |g(x)||\psi(x) - f_t(x)|dx + \int_E |g(x)f_t(x)|dx < \epsilon.K(g) + \int_E |g(x)f_t(x)|dx$ Also , $\int_E |g(x)f_t(x)|dx < \infty$ …(9) for $f_t(x)$ is in Γ^{**} implies $f_t(x)$ is in Γ and g is in $\Gamma^* = L_1$ .Also $| \int_E f_t(x)g(x)\,dx - \int_E \psi(x)g(x)dx | \leq \int_E |g(x)||f_t(x) - \psi(x)|dx < \epsilon K(g)$ [By (7) and (8)]Hence $\lim_{t\to\infty} \int_E f_t(x)g(x)\,dx = \int_E \psi(x)g(x)dx$…(10) Thus by [(9) and (10)]$\psi(x)$ is $\Gamma\,\Gamma^*$ - limit of $f_t(x)$. that is $\psi(x)$ is Γ-limit of $f_t(x)$ .Thus theorem is established .

Theorem (3.3) :Let $f_t(x)$, in a function space α , be a family of functions of x defined for all t in $[0,\infty)$, where t is a parameter .Then (i) every parametric convergent family $f_t(x)$ in α is αL_1 – convergent .and (ii)every parametric limit of $f_t(x)$ is αL_1 –limit of $f_t(x)$, provided $L^* \supseteq \alpha$.

Proof :Let $f_t(x)$ be a family of functions of x defined for all t in $[0,\infty)$, where t is a parameter .
Also let , $f_t(x)$ is parametric convergent in α.Then by definition of parametric convergent , to every $\epsilon > 0$, there exists a positive number $T(\epsilon)$, independent of x , such that , for almost all $x \geq 0$,$| f_t(x) - f_{t^1}(x) | \leq \epsilon$ ..... (11) for all $t , t^1 \geq T(\epsilon)$.Also let $g(x)$ be any function in L_1 and let $E = [0,\infty)$.Hence, $\int_E |g(x)|dx < \infty$….. (12) Now we see that, $\int_E g(x)\{f_t(x) - f_{t^1}(x)\}dx | \leq \int_E |g(x)\{f_t(x) - f_{t^1}(x)\}|dx \leq \epsilon \int_E |g(x)|dx$ [By (11)] $< \epsilon.K(g)$ [By (12)]for all $t , t^1 \geq T(\epsilon)$ for every $\epsilon>0$,where $K(g)$ is a constant depending on g but independent of t in E. But then by a set of necessary and sufficient conditions,$f_t(x)$ is αL_1-convergent .Now let $\psi(x)$ = the parametric limit of $f_t(x)$. Then for a given any $\epsilon > 0$, there exists a positive number $T(\epsilon)$, independent of x , such that for almost all $x \geq 0$,$|f_t(x) - \psi(x)| \leq \epsilon$….(13) Since $\int_E |g(x)\psi(x)|dx$ for g in L_1,$\epsilon > 0$, $t \geq T(\epsilon)$,$= \int_E |g(x)||\psi(x) - f_t(x) + f_t(x)|dx \leq \int_E |g(x)||\psi(x) - f_t(x)|dx + \int_E |g(x)f_t(x)|dx < \epsilon.K(g) + \int_E |g(x)f_t(x)|dx$ Also , $\int_E |g(x)f_t(x)|dx < \infty$ ……(14) .for $f_t(x)$ is in α and g in L_1 .Also $| \int_E f_t(x)g(x)\,dx - \int_E \psi(x)g(x)dx | \leq \int_E |g(x)||f_t(x) - \psi(x)|dx < \epsilon\, K(g)$[By (12) and (13)]Therefore $\lim_{t\to\infty} \int_E f_t(x)g(x)\,dx = \int_E \psi(x)g(x)dx$ ….(15).Thus by [(14) and (15)] $\psi(x)$ is αL_1 - limit of $f_t(x)$ .Thus the theorem is established .

II. CONCLUSION

We observed that the behavior of dual space as for the function space . we also observed that the technique of establishing the results for function spaces are quite different from those of the technique of establishing the results to the case of sequence space. In course of doing so technique has to be changed a bit. As a result of which the look of the results for limit in dual spaces differ from the look of the results for limits in function spaces although we recall that the dual space of a function space is it self a function space , in addition of the suitably defined and constructed some of the function spaces.On the basis of above result we can suggest all the result for function space also holds good by dual space of function space. Its also extend one step ahead the theory of dual space of function space by different suitably defined function spaces.

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