Quest Journals
Journal of Research in Applied Mathematics
Volume 5 ~ Issue 2 (2019) pp: 47-52
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

An Optimal Family of Eighth-Order Iterative Methods for Solving Nonlinear Equations

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ABSTRACT: *Dependence on King's fourth order methods, we develop a family of optimal eighth-order methods for finding a simple zero of nonlinear equation. The proposed family is developed by combining King's fourth-order method and Newton's method as a third step. Using the divided difference and two real-valued weight functions in the third step to increase the order and decreasing the number of function evaluations to be optimal. Numerical examples are given to demonstrate the performance of the new method.*
KEYWORDS: *Efficiency index, Convergence order, Optimal method, Iterative methods*

Received 28 March, 2019; Accepted 08 April, 2019 © The author(s) 2019.
Published with open access at www.questjournals.org

I. INTRODUCTION

Finding the root of nonlinear equations is the most and classical problems in science. In this work, we develop iterative methods to find a simple root r of a nonlinear equation, $f(r) = 0$ where, $f: I \subseteq \mathbb{R} \to \mathbb{R}$ for an open interval I. One of the famous iterative methods for solving nonlinear equation is Newton–Raphson method (NM),

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \; f'(x_n) \neq 0, \tag{1}$$

Newton's method is convergence quadratic in some neighborhood of r [7]. The efficiency index (EI) of an iterative method can be defined as $EI = p^{\frac{1}{n}}$ such that p is the order of the method and n is the number of total function or derivative evaluations per iteration [11]. According to the optimality, optimal order of any iterative method is given by 2^{n-1} [4]. In the recent years there are numerous of the optimal three-step methods with eighth-order convergence to solve nonlinear equations developed, for example see [1,3,5,11] and its References. One-parameter family of fourth-order methods have created by King methods (KM)[9], which is written as : $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$,

$$x_{n+1} = y_n - \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)} \frac{f(y_n)}{f'(x_n)}, \tag{2}$$

where $\beta \epsilon \mathbb{R}$, this family is optimal fourth-order of convergence, and $EI = 4^{\frac{1}{3}} \approx 1.5874$.

Theorem1[8].: Let $\delta_1(x),\ \delta_2(x), \dots, \delta_s(x)$ be iterative functions with the orders $p_1, p_2, \dots, p_f$ respectively. Then the installation of iterative functions $\delta_1\left(\delta_2\left(\dots\left(\delta_s(x)\right)\dots\right)\right)$, defines the iterative method of the order $p_1, p_2, \dots, p_f$.

Using theorem 1, adding the Newton's method as a third step to Kings method (2), gives

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = y_n - \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)} \frac{f(y_n)}{f'(x_n)},$$
$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \tag{3}$$

The method (3) has $EI = 8^{\frac{1}{5}} \approx 1.5157$, and is not optimal. To decrease the number of functions evaluation of method (3) to four, by substituting the approximation of $f'(z_n)$ by $\frac{f[x_n,z_n]f[y_n,z_n]}{f[x_n,y_n]}$ using the divided difference [5] to decrease the numbers of function evaluations. Recently Al-Oufi and Al-Subaihi, developed king's method by addition Newton's method as a third step and using the divided difference, also three weighted functions to obtain a new family of optimal methods with order of convergence equals eight, as the following

$A(0) = 1\,,A'(0) = 1,$

$B(0) = -1\,,B'(0) = B''(0) = 0, B'''(0) = -12\beta,$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}\frac{f(y_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n)f[x_n,y_n]}{f[x_n,z_n]f[y_n,z_n]} \times \{A(t_1) + B(t_2) + C(t_3)\}\,, \tag{4}$$

where $A(t_1)$, $B(t_2)$ and $C(t_3)$ are three real-valued weight function and $t_1 = \frac{f(z)}{f(x)}$, $t_2 = \frac{f(y)}{f(x)}$, $t_3 = \frac{f(z)}{f(y)}$.

The method (4) has $EI = 8^{\frac{1}{4}} \approx 1.682$.

II. THE METHOD AND CONVERGENCE ANALYSIS

To build an optimal eighth-order method without using more weighted functions, we will develop the method (4) by decreases the number of weighted function from three to two as follows

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}\frac{f(y_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n)f[x_n,y_n]}{f[x_n,z_n]f[y_n,z_n]} \times \{A(v) + B(u)\}\,, \tag{5}$$

where $A(v)$, $B(u)$ are two real-valued weight function, and $v = \frac{f(z)}{f(x)}$,$u = \frac{f(y)}{f(x)}$.

Theorem2.: Let $r \in I$ be a simple zero of an sufficiently differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently closed to r, then the method (5) gives optimal eighth-order of convergence if satisfies the following conditions:

$A(0) = \frac{1}{2}\,,A'(0) = 1,$

$B(0) = \frac{1}{2}\,,B'(0) = B''(0) = 0\,,B'''(0) = -12\beta\,,|B^{(4)}(0)| < \infty.$

Proof: Let $e_n = x_n - r$ be the error. Expanding $f(x)$ about r by Taylor expansion, we have

$$f(x_n) = f'(r)(e + c_2e^2 + c_3e^3 + c_4e^4 + c_5e^5 + c_6e^6 + c_7e^7 + c_8e^8 + c_9e^9)\,, \tag{6}$$

where $c_k = \frac{f^{(k)}(r)}{k!f'(r)}$, $k = 2,3, \dots$.

$$f'(x_n) = f'(r)(1 + 2c_2e + 3c_3e^2 + 4c_4e^3 + 5c_5e^4 + 6c_6e^5 + 7c_7e^6 + 8c_8e^7 + 9c_9e^8)\ . \tag{7}$$

Dividing (6) by (7), we get

$$\frac{f(x_n)}{f'(x_n)} = e - c_2e^2 + 2(c_2^2 - c_3)e^3 + \cdots + (-64c_2^7 + 304c_2^5c_3 - 176c_2^4c_4 + \cdots + 31c_4c_5 - 7c_8)e^8 + O(e^9). \tag{8}$$

Substituting the equation (8), into first step of (5), we obtain

$$y_n = r + c_2e^2 + 2(-c_2^2 + c_3)e^3 + \cdots + (64c_2^7 - 304c_2^5c_3 + \cdots - 31c_4c_5 + 7c_8)e^8 + O(e^9)\ . \tag{9}$$

Expanding $f(y_n)$ about r to get

$$f(y_n) = f'(r)[(c_2e^2 + 2(-c_2^2 + c_3)e^3 + \cdots + (144c_2^7 - 552c_2^5c_3 \dots - 31c_4c_5 + 7c_8)e^8] + O(e^9)\ , \tag{10}$$

substituting (6), (7), (9) and (10) into the second step of (5), we have

$$z_n = r + (2\beta c_2^3 + c_2^3 - c_2c_3)e^4 + \cdots + (2\beta^5c_2^7 + 18\beta^4c_2^7 + \cdots - 13c_3c_6 - 17c_4c_5)e^8 + O(e^9). \tag{11}$$

From (11), we obtain

$$f(z_n) = f'(r)[(2\beta c_2^3 + c_2^3 - c_2c_3)e^4 + \cdots + (-209c_2^2c_3c_4 + \cdots + 50c_3^2c_4 + 37c_2^7)e^8 + O(e^9)]. \tag{12}$$

By expanding $A(v)$, $B(u)$ using Taylor series expansion, we have

$$A(v) = A(0) + A'(0)v + \frac{1}{2}A''(0)v^2 + \frac{1}{3!}A'''(0)v^3 + \frac{1}{4!}A^{(4)}(0)v^4 \dots + O(v^9), \tag{13}$$

$$B(u) = B(0) + B'(0)u + \frac{1}{2}B''(0)u^2 + \frac{1}{3!}B'''(0)u^3 + \frac{1}{4!}B^{(4)}(0)u^4 \dots + O(u^9). \tag{14}$$

Furthermore,

$$f[x_n, y_n] = f'(r)(1 + c_2 e + (c_2^2 c_3)e^2 + (-2c_2^3 + 3c_2c_3 + c_4)e^3 + \cdots + O(e^9)), \tag{15}$$
$$f[y_n, z_n] = f'(r)(1 + c_2^2 e^2 + (-2c_2^3 + 2c_2c_3)e^3 + \cdots + O(e^9)), \tag{16}$$
$$f[x_n, z_n] = f'(r)(1 + c_2 e + c_3 e^2 + c_4 e^3 + \cdots + O(e^9)). \tag{17}$$

Lastly, using (11) – (17) and the following conditions:

$A(0) = \frac{1}{2}$, $A'(0) = 1$,

$B(0) = \frac{1}{2}$, $B'(0) = B''(0) = 0$, $B'''(0) = -12\beta$, $|B^{(4)}(0)| < \infty$,

we get the error expression

$$e_{n+1} = r + \left(\frac{1}{24}B^{(4)}(0)c_2^5c_3 - \frac{1}{24}B^{(4)}(0)c_2^7 - \frac{1}{12}\beta c_2^7 B^{(4)}(0) + 2\beta c_2^4 c_4 + \cdots + 4c_2^3c_3^2\right)e^8 + O(e^9). \tag{18}$$

Which show that the order of a family (5) is completely eighth for every $\beta \epsilon \mathbb{R}$. This finishes the proof.

It is established that the next new methods have convergence of order eighth, which will indicate by Abdulkarim and Al-Subaihi Methods (ASM1-ASM5):

ASM1.: Let

$A(v) = v + \frac{1}{2}$,

$B(u) = \frac{1}{2} - 2\beta u^a$, $a \geq 3$, $a \epsilon \mathbb{R}$.

Then (ASM1) will be as

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = y_n - \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)} \frac{f(y_n)}{f'(x_n)},$$
$$x_{n+1} = z_n - \frac{f(z_n) f[x_n, y_n]}{f[x_n, z_n] f[y_n, z_n]} \times \{1 + v - 2\beta u^a\}. \tag{19}$$

ASM2.: choosing

$A(v) = e^v - \frac{1}{2}$,

$B(u) = \frac{1}{2} + u^a \sin(u) - 2\beta u^a$, $a \geq 3$, $a, \beta \epsilon \mathbb{R}$.

Hence, we obtain a family of eighth-order of these method

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = y_n - \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)} \frac{f(y_n)}{f'(x_n)},$$
$$x_{n+1} = z_n - \frac{f(z_n) f[x_n, y_n]}{f[x_n, z_n] f[y_n, z_n]} \times \{e^v + u^a \sin(u) - 2\beta u^a\}. \tag{20}$$

ASM3.:

If the functions $A(v)$, $B(u)$ are define by:

$A(v) = \frac{1}{2}\cos(v) + \sin(v)$,

$B(u) = \frac{1}{2} - 2\beta u^3 + du^{a+1}$, $a, d, \beta \epsilon \mathbb{R}$.

These methods can be written as

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = y_n - \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)} \frac{f(y_n)}{f'(x_n)},$$
$$x_{n+1} = z_n - \frac{f(z_n) f[x_n, y_n]}{f[x_n, z_n] f[y_n, z_n]} \times \left\{\frac{1}{2}\cos(v) + \sin(v) + \frac{1}{2} - 2\beta u^3 + du^{a+1}\right\}. \tag{21}$$

ASM4. : Let

$A(v) = \frac{1}{2} + \sin(v)$,

$B(u) = \frac{1}{2} - 2\beta u^3 e^u$.

We get another eighth order methods (ASM4)

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}\frac{f(y_n)}{f'(x_n)},$$
$$x_{n+1} = z_n - \frac{f(z_n)f[x_n,y_n]}{f[x_n,z_n]f[y_n,z_n]} \times \{1+\sin(v) - 2\beta u^3 e^u\}. \qquad (22)$$

ASM5. : Choosing
$A(v) = \frac{1}{2} + v + mv^t, t > 1, t, m \epsilon \mathbb{R},$
$B(u) = \frac{1}{2} + \frac{u^4}{2} - 2\beta u^3.$
A new family can be can be obtained as

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = y_n - \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}\frac{f(y_n)}{f'(x_n)},$$
$$x_{n+1} = z_n - \frac{f(z_n)f[x_n,y_n]}{f[x_n,z_n]f[y_n,z_n]} \times \left\{1+v+mv^t+\frac{u^4}{2} - 2\beta u^3\right\}. \qquad (23)$$

III. NUMERICAL RESULTS

In this section, we use the methods (ASM1-ASM5) to solve several nonlinear equations and compare with the some order methods. We specifically take $a = 3, d = m = 1, t = p = 2$ and $\beta = 1$ for the methods (ASM1-ASM5). All calculations were done using MATLAB. The stopping criteria is $|x_{n+1} - x_n| \leq 10^{-300}$ and $|f(x_n)| \leq 10^{-300}$.

Table 1. Test functions

Functions	**Roots (r)**
$f_1(x) = \cos(x) - x$	0.73908513321561
$f_2(x) = xe^x + \log(1+x+x^4)$	0.0
$f_3(x) = \sin(x) - \frac{x}{3}$	2.27886266007583
$f_4(x) = x^3 + 4x^2 - 10$	1.3652300134141

Table 1 shows the functions and its root. We compared with the Newton method (NM), (1), the King method (KM), (2), Al-Oufi and Al-Subaihi method (OSM1), (4), and Sharma optimal method of order eighth (SHM) [10], given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)}\frac{f(y_n)}{f'(x_n)},$$
$$x_{n+1} = z_n - \left[1 + \frac{f(z_n)}{f(x_n)} + \left(\frac{f(z_n)}{f(x_n)}\right)^2\right]\frac{f(z_n)f[x_n,y_n]}{f[x_n,z_n]f[y_n,z_n]}.$$

Liu and Wang optimal method of order eighth (LWM) [6] , given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)}\frac{f(y_n)}{f'(x_n)},$$
$$x_{n+1} = z_n - \left[\left(\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}\right)^2 + \frac{f(z_n)}{f(y_n)-\theta f(z_n)} + \frac{4f(z_n)}{f(x_n)+\beta f(z_n)}\right]\frac{f(z_n)}{f'(x_n)},$$

where $\beta = 0, \theta = 1$, Sharma and Arora's method of optimal order eight (SAM8) [11]

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = y_n - \frac{f(y_n)}{2f[x_n,y_n]-f'(x_n)},$$
$$x_{n+1} = z_n - \frac{f(z_n)}{2f[y_n,z_n]-f[x_n,z_n]}\frac{f[y_n,z_n]}{f[x_n,z_n]}.$$

Shown in Table 2, the number of iterations indicate by $(Iter)$ and calculate the value of $|f(x_n)|$ and $|x_{n+1} - x_n|$. Further, computational order of convergence (COC) is defined by [13] as well displayed in the Table 2,

$$COC \approx \frac{\ln|(x_{n+1}-r)/(x_n-r)|}{\ln|(x_n-r)/(x_{n-1}-r)|}.$$

Table 2. Numerical Comparison

Method	$Iter$	$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	COC
$f_1(x) = \cos(x) - x,\ x_0 = 0.6$				
NM	8	3.00558e-379	1.79587e-379	2
KM, $\beta = 1$	4	1.4947e-323	8.93096e-324	4
SHM	3	9.49798e-674	5.67514e-674	8
LWM	3	2.57724e-626	1.53992e-626	8
SAM8	3	1.9598e-699	1.171e-699	8
OSM1	3	7.00448e-626	4.18525e-626	8
ASM1	3	9.06714e-654	5.41771e-654	8
ASM2	3	8.03215e-663	4.79929e-663	8
ASM3	3	1.08517e-662	6.48401e-663	8
ASM4	3	2.5518e-641	1.52473e-641	8
ASM5	3	4.44253e-658	2.65446e-658	8
$f_2(x) = xe^x + \log(1 + x + x^4),\ x_0 = 0.25$				
NM	9	5.41196e-552	2.70598e-552	2
KM, $\beta = 1$	4	5.84058e-312	2.92029e-312	4
SHM	3	9.33255e-441	4.66628e-441	8
LWM	3	1.57085e-388	7.85424e-389	8
SAM8	3	8.37774e-433	4.18887e-433	8
OSM1	3	4.07059e-520	2.03529e-520	8
ASM1	3	1.93237e-521	9.66186e-522	8
ASM2	3	1.2588e-517	6.29398e-518	8
ASM3	3	1.27728e-517	6.38639e-518	8
ASM4	3	2.17545e-533	1.08772e-533	8
ASM5	3	2.04002e-519	1.02001e-519	8
$f_3(x) = \sin(x) - \frac{x}{3},\ x_0 = 2$				
NM	9	1.4018e-454	1.42503e-454	2
KM, $\beta = 1$	5	4.61651e-681	4.693e-681	4
SHM	3	9.83341e-386	9.99636e-386	8
LWM	3	1.91876e-334	1.95055e-334	8
SAM8	3	2.93893e-452	2.98763e-452	8
OSM1	3	2.45792e-310	2.49865e-310	8
ASM1	3	2.06175e-394	2.09592e-394	8
ASM2	3	7.3894e-365	7.51185e-365	8
ASM3	3	6.33357e-370	6.43852e-370	8
ASM4	3	1.53091e-336	1.55628e-336	8
ASM5	3	4.15597e-383	4.22483e-383	8
$f_4(x) = x^3 + 4x^2 - 10,\ x_0 = 5.8$				
NM	12	4.02455e-479	2.43714e-480	2
KM, $\beta = 1$	6	1.62415e-326	9.83535e-328	4
SHM	4	3.08048e-444	1.86544e-445	8
LWM	4	1.72791e-345	1.04637e-346	8
SAM8	4	2.20293e-857	1.33403e-858	8
OSM1	4	7.33079e-383	4.4393e-384	8
ASM1	4	2.82783e-334	1.71245e-335	8
ASM2	4	4.7573e-345	2.88087e-346	8
ASM3	4	7.29367e-342	4.41682e-343	8
ASM4	4	3.59248e-320	2.1755e-321	8
ASM5	4	2.29045e-342	1.38703e-343	8

IV. CONCLUSION

In this work, a family of optimal eighth-order iterative method has been developed for solving nonlinear equations. New proposed families are obtained by replacing $f'(z)$ using divided difference. We reduce the number of weighted functions from sum of three to two, so to decrease the flops at each iteration. The new families methods have $IE = 8^{\frac{1}{4}} \approx 1.682$ and four functions evaluation. Lastly, comparing other methods using numerical examples to illustrate the convergence of the new methods.

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I.A. Al-Subaihi" An Optimal Family of Eighth-Order Iterative Methods for Solving Nonlinear Equations" Quest Journals Journal of Research in Applied Mathematics , vol. 05, no. 02, 2019, pp. 47-52