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Research Paper

Applications of Leibniz Rule for Fractional Derivative

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ABSTRACT: *This paper mainly studies the applications of Leibniz rule for fractional derivative based on the Jumarie's modified Riemann-Liouville fractional derivative. Some examples are proposed to illustrate the applications.*
KEYWORDS: *Applications, Leibniz Rule for Fractional Derivative, Jumarie's Modified Riemann-Liouville Fractional Derivative, Examples*

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I. INTRODUCTION

In the second half of the 20th century, a considerable number of studies on fractional calculus were published in the engineering literature. In fact, fractional calculus has many applications in modern calculus, differential and integral equations, physics, signal processing, hydrodynamics, viscoelasticity, mathematical biology and electrochemistry [1-16]. There is no doubt that fractional calculus has become an exciting new mathematical method to solve diverse problems in mathematics, science, and engineering.

In this article, we mainly evaluate the fractional derivatives of three factional analytic functions:

$$\left({}_0D_x^\alpha\right)^{20}\left[\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}\otimes E_\alpha(x^\alpha)\right], \tag{1}$$

$$\left({}_0D_x^\alpha\right)^{4}\left[E_\alpha(x^\alpha)\otimes cos_\alpha(x^\alpha)\right], \tag{2}$$

$$\left({}_0D_x^\alpha\right)^{5}\left[\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}\otimes sin_\alpha(x^\alpha)\right], \tag{3}$$

where $0<\alpha\leq 1$, E_α, cos_α, sin_α are α-fractional exponential function, cosine function, and sine function respectively. Using Leibniz rule for fractional derivative, these fractional derivatives of fractional analytic functions can be easily obtained.

II. DEFINITIONS AND PROPERTIES

At first, we introduce the fractional calculus used in this paper.

Definition 2.1: Let α be a real number and m be a positive integer, then the modified Riemann-Liouville fractional derivative of Jumarie type ([18]) is defined by

$$\left({}_{x_0}D_x^\alpha\right)[y(x)]=\begin{cases}\frac{1}{\Gamma(-\alpha)}\int_{x_0}^{x}(x-\tau)^{-\alpha-1}y(\tau)d\tau, & \text{if } \alpha<0\\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}(x-\tau)^{-\alpha}[y(\tau)-y(a)]\,d\tau & \text{if } 0\leq\alpha<1\\ \frac{d^m}{dx^m}\left({}_{x_0}D_x^{\alpha-m}\right)[y(x)], & \text{if } m\leq\alpha<m+1\end{cases} \tag{1}$$

where $\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}dt$ is the gamma function defined on $s>0$. If $\left({}_{x_0}D_x^\alpha\right)^n[y(x)]=\left({}_{x_0}D_x^\alpha\right)\left({}_{x_0}D_x^\alpha\right)\cdots\left({}_{x_0}D_x^\alpha\right)[y(x)]$ exists, then $y(x)$ is called n-th order α-fractional differentiable function, and $\left({}_{x_0}D_x^\alpha\right)^n[y(x)]$ is the n-th order α-fractional derivative of $y(x)$. We note that $\left({}_{x_0}D_x^\alpha\right)^n\neq {}_{x_0}D_x^{n\alpha}$ in general.

Definition 2.2 ([19])**:** Suppose that x, x_0 and a_n are real numbers, $x_0\in(a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as a α-fractional power series, that is, $f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval (x_0-r, x_0+r), then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0, where r is the radius of convergence about x_0. If $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and is α-fractional analytic at every point in open interval (a,b), then we say that f_α is an α-fractional analytic function on $[a,b]$.

Definition 2.3 ([17])**:** The Mittag-Leffler function is defined by

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)},\tag{2}$$

where α is a real number, $\alpha>0$, and z is a complex variable.

Definition 2.4 ([17])**:** Assume that $0<\alpha\le 1$ and x is a real variable. Then $E_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}$ is called α-fractional exponential function, and the α-fractional cosine and sine function are defined by

$$cos_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)},\tag{3}$$

and

$$sin_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)}.\tag{4}$$

Proposition 2.5: *Suppose that α,β,c are real numbers, $0<\alpha\le 1$, and $\beta\ge\alpha$. Then*

$$\left({}_0D_x^\alpha\right)\left[x^\beta\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha},\tag{5}$$

$$\left({}_0D_x^\alpha\right)[c]=0,\tag{6}$$

$$\left({}_0D_x^\alpha\right)[E_\alpha(x^\alpha)]=E_\alpha(x^\alpha),\tag{7}$$

$$\left({}_0D_x^\alpha\right)[sin_\alpha(x^\alpha)]=cos_\alpha(x^\alpha),\tag{8}$$

$$\left({}_0D_x^\alpha\right)[cos_\alpha(x^\alpha)]=-sin_\alpha(x^\alpha).\tag{9}$$

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.6 ([20])**:** Let $0<\alpha\le 1$, $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ be two α-fractional analytic functions,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}x^{n\alpha},\tag{10}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}x^{n\alpha}.\tag{11}$$

Then we define

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)\\&=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}x^{n\alpha}\otimes\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}x^{n\alpha}\\&=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)x^{n\alpha}.\end{aligned}\tag{12}$$

Theorem 2.7 ([17]) (Leibniz rule for fractional derivative)**:** *Let f_α, g_α be fractional analytic functions, and m be a positive integer, then*

$$\left({}_0D_x^\alpha\right)^m[f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)]=\sum_{k=0}^{m}\binom{m}{k}\left({}_0D_x^\alpha\right)^{m-k}[f_\alpha(x^\alpha)]\otimes\left({}_0D_x^\alpha\right)^{k}[g_\alpha(x^\alpha)].\tag{13}$$

III. MAIN RESULTS

In the following, we will make use of Leibniz rule for fractional derivative to find the fractional derivatives of some fractional analytic functions.

Example 3.1: Let $0<\alpha\le 1$. Then the 20-th order fractional derivative

$$\begin{aligned}&\left({}_0D_x^\alpha\right)^{20}\left[\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}\otimes E_\alpha(x^\alpha)\right]\\&=\sum_{k=0}^{20}\binom{20}{k}\left({}_0D_x^\alpha\right)^{20-k}[E_\alpha(x^\alpha)]\otimes\left({}_0D_x^\alpha\right)^{k}\left[\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}\right]\\&=E_\alpha(x^\alpha)\otimes\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}+\binom{20}{1}E_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)\left[\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}\right]+\binom{20}{2}E_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)^2\left[\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}\right]\\&=E_\alpha(x^\alpha)\otimes\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}+20\cdot E_\alpha(x^\alpha)\otimes\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+190\cdot E_\alpha(x^\alpha)\\&=E_\alpha(x^\alpha)\otimes\left[\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}+20\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+190\right].\end{aligned}\tag{14}$$

Example 3.2: Suppose that $0<\alpha\le 1$. Then the following 4-th order fractional derivative

$$\begin{aligned}&\left({}_0D_x^\alpha\right)^{4}[E_\alpha(x^\alpha)\otimes cos_\alpha(x^\alpha)]\\&=\sum_{k=0}^{4}\binom{4}{k}\left({}_0D_x^\alpha\right)^{4-k}[E_\alpha(x^\alpha)]\otimes\left({}_0D_x^\alpha\right)^{k}[cos_\alpha(x^\alpha)]\\&=E_\alpha(x^\alpha)\otimes cos_\alpha(x^\alpha)+\binom{4}{1}E_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)[cos_\alpha(x^\alpha)]+\binom{4}{2}E_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)^2[cos_\alpha(x^\alpha)]\\&\quad+\binom{4}{3}E_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)^3[cos_\alpha(x^\alpha)]+\binom{4}{4}E_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)^4[cos_\alpha(x^\alpha)]\\&=E_\alpha(x^\alpha)\otimes cos_\alpha(x^\alpha)+4\cdot E_\alpha(x^\alpha)\otimes[-sin_\alpha(x^\alpha)]+6\cdot E_\alpha(x^\alpha)\otimes[-cos_\alpha(x^\alpha)]\\&\quad+4\cdot E_\alpha(x^\alpha)\otimes sin_\alpha(x^\alpha)+E_\alpha(x^\alpha)\otimes cos_\alpha(x^\alpha)\\&=-4\cdot E_\alpha(x^\alpha)\otimes cos_\alpha(x^\alpha).\end{aligned}\tag{15}$$

Example 3.3: If $0<\alpha\le 1$, then the 5-th order fractional derivative

$$\left({}_0D_x^\alpha\right)^{5}\left[\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}\otimes sin_\alpha(x^\alpha)\right]$$

$$= \sum_{k=0}^{5}\binom{5}{k}\left({}_{0}D_{x}^{\alpha}\right)^{50-k}[sin_{\alpha}(x^{\alpha})] \otimes \left({}_{0}D_{x}^{\alpha}\right)^{k}\left[\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}\right]$$
$$= \left({}_{0}D_{x}^{\alpha}\right)^{5}[sin_{\alpha}(x^{\alpha})] \otimes \frac{1}{\Gamma(2\alpha+1)}x^{2\alpha} + \binom{5}{1}\left({}_{0}D_{x}^{\alpha}\right)^{4}[sin_{\alpha}(x^{\alpha})] \otimes \left({}_{0}D_{x}^{\alpha}\right)\left[\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}\right]$$
$$+ \binom{5}{2}\left({}_{0}D_{x}^{\alpha}\right)^{3}[sin_{\alpha}(x^{\alpha})] \otimes \left({}_{0}D_{x}^{\alpha}\right)^{2}\left[\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}\right]$$
$$= cos_{\alpha}(x^{\alpha}) \otimes \frac{1}{\Gamma(2\alpha+1)}x^{2\alpha} + 5 \cdot sin_{\alpha}(x^{\alpha}) \otimes \frac{1}{\Gamma(\alpha+1)}x^{\alpha} + 10 \cdot [-cos_{\alpha}(x^{\alpha})]$$
$$= \frac{1}{\Gamma(2\alpha+1)}x^{2\alpha} \otimes cos_{\alpha}(x^{\alpha}) + 5 \cdot \frac{1}{\Gamma(\alpha+1)}x^{\alpha} \otimes sin_{\alpha}(x^{\alpha}) - 10 \cdot cos_{\alpha}(x^{\alpha}). \tag{16}$$

IV. CONCLUSION

Based on Jumarie type of Riemann-Liouville fractional derivative, Leibniz rule for fractional derivative is a very important theorem in fractional calculus. It is different from the Leibniz rule for other fractional derivative, and is more natural. In the future, we will also study the fractional differential equations and fractional calculus problems.

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