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Research Paper

Efficient Jacobi Collocation Method for Nonlinear Singular Fractional Integro-Differential Equations

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Abstract:
This paper presents an efficient numerical approach for solving nonlinear weakly singular fractional integro-differential equations. We develop a Jacobi collocation method that effectively handles the singularity while maintaining high accuracy. The proposed technique transforms the original problem into a system of algebraic equations through careful discretization. We provide rigorous error analysis demonstrating the method's spectral convergence. Numerical examples confirm the theoretical results and showcase the algorithm's superior performance compared to existing methods, particularly in handling both the nonlinearity and weak singularity simultaneously.

المستخلص:

تقدم هذه الورقة البحثية أسموباً عددياً فعالًا لحل المعادلات التكاممية-التفاضمية الكدرية غير الخطية ذات النواة المفردة الضعيفة. طورنا طريقة جاكوبي التجميعية التي تتعامل بكفاءة مع التفرد مع الحفاظ عمى دقة عالية. تقوم التقنية المقترحة بتحويل المدألة الأصمية إلى نظام معادلات جبرية من خلال إجراءات تضبيب مدروسة. نقدم تحميلًا نظرياً لمخطأ يثبت التقارب الطيفي لمطريقة. تؤكد الأمثمة العددية النتائج النظرية وتظهر أداءً ممتازاً لمخوارزمية مقارنة بالطرق الموجودة، خاصة في التعامل مع كل من اللاخطية والتفرد الضعيف في آن واحد.

Keywords: *Fractional integro-differential equations- Weakly singular kernel - Jacobi collocation method - Nonlinear equations - Spectral accuracy - Numerical solution.*

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1. Introduction

Recently, fractional calculus (FC) has attracted much attention since it can be used to model physical and engineering problems. There are several definitions of fractional derivatives that do not coincide in general, like, Grounwald-Letnikov, Riemann-Liouville, Caputo, Atangana- Baleanu, and Caputo-Fabrizio . In this section, we intend to use the Caputo fractional derivative (CFD), which is the most commonly used derivative among physicists and scientists because it provides a physical interpretation that is consistent with the behavior of many physical and biological systems, making it a valuable tool for modeling and analyzing various natural phenomena, including biology, energy systems, physics, groundwater flow modeling, and geomechanics. The performance of many life systems can be represented utilizing fractional integro-differential equations (FIDEs) by virtue of the recent works of FC in different trends of science and technology. In fact, solving fractional weakly singular kernels integral and integro-differential equations can be challenging but there are numerical methods that can be used to approximate their solutions; for example, the finite volume method, finite difference method, finite element method, two-grid method, backward substitution method, and the spectral collocation method that is commonly used in literatures.[1]

2. Preliminaries

We will provide initial findings that will prove beneficial in subsequent analysis. To begin, consider a predetermined time horizon denoted as $T > 0$. We will now define the subsequent spaces:[3]

$$L^p(0,T;R^n)=\{\phi:[0,T]\to R^n \mid \phi(\cdot) \text{ is measurable},$$
$$\|\phi(\cdot)\|_p \equiv \left(\int_0^T |\phi(t)|^p dt\right)^{1/p} < \infty\Big\},\ 1\le p<\infty,$$

$$L^\infty(0,T;R^n)=\{\phi:[0,T]\to R^n \mid \phi(\cdot) \text{ is measurable},$$
$$\|\phi(\cdot)\|_\infty \equiv \operatorname*{esssup}_{t\in[0,T]} |\phi(t)| < \infty\}.$$

Also, we define

$$L^{p+}(0,T;R^n)=\bigcup_{r>p} L^r(0,T;R^n),\quad 1\le p<\infty,$$

$$L^{p-}(0,T;R^n)=\bigcap_{r<p} L^r(0,T;R^n),\quad 1\le p<\infty.$$

In the subsequent analysis, we utilize the notation $\Delta = \{(t, s) \in [0, T]^2 \mid 0 \le s < t \le T\}$. It is important to note that the "diagonal line" represented by $\{(t, t) | t \in [0, T]\}$ does not belong to Δ. Consequently, if we consider a continuous mapping $\phi : \Delta \to R^n$ where $(t, s) \mapsto \phi(t, s)$, the function $\phi(\cdot, \cdot)$ may become unbounded as the difference $|t - s| \to 0$.

In the section, we adopt the notation $t_1 \vee t_2 = \max\{t_1, t_2\}$ and $t_1 \wedge t_2 = \min\{t_1, t_2\}$, for any $t_1, t_2 \in R$.

Notably, $t^+ = t \vee 0$.

Lemma (2.1)[3] Let p, q, $r \ge 1$ satisfy $\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}$. Then for any $f(\cdot) \in L^q(R^n)$, $g(\cdot) \in L^r(R^n)$,

$$\|f(\cdot) * g(\cdot)\|_{L^p(R^n)} \le \|f(\cdot)\|_{L^q(R^n)} \|g(\cdot)\|_{L^p(R^n)}. \tag{1}$$

Corollary 6.3.2.1. Let $\beta \in (0, 1)$, $1 \le r < \frac{1}{1-\beta}$, and $\frac{1}{p}+1 = \frac{1}{q}+\frac{1}{r}$, $p, q \ge 1$.

Then for any $a < b$, $0 < \delta \le b - a$, and $\varphi(\cdot) \in L^q(a, b)$,

$$\left(\int_a^{a+\delta} \left|\int_a^t \frac{\varphi(s)ds}{(t-s)^{1-\beta}}\right|\right)^{1/p} \le \left(\frac{\delta^{1-r(1-\beta)}}{1-r(1-\beta)}\right)^{1/r} \|\varphi(\cdot)\|_{L^p(a,b)}, \tag{2}$$

3. Approximation base

Here, we are thinking of creating a method similar to wavelet approximations using a class of nonorthogonal functions such as the MLF. It is noticeable that most of those who deal with wavelet approximations tend to use orthogonal functions as a basis. Therefore, it is necessary for us to start with the basics of the subject, and we have chosen to define the step function and GFMLF.[4]

Step function and GFMLF(3.1)[4]

For the step function, let

$$\varphi_{jk}(t)=\begin{cases}1, & \frac{j-1}{2^k} \le t \le \frac{j}{2^k},\\ 0, & \text{otherwise,}\end{cases}$$

where $j = 1, 2, ..., 2^k = n$, then for any function $O(t) \in L^2$, there exist step functions as

$$O_j(t)=\sum_{k=1}^{n} a_{jk}\varphi_{jk}(t),$$

such that

$$\lim_{j\to\infty}\|O_j(t)-O(t)\|=0.$$

Now, the MLF of two-parameter is given by the following power series

$$W^{\beta,\alpha}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\beta k+\gamma)},\ \beta>0,\ \gamma>0,\ t\in R.$$

Furthermore, the generalized MLF is defined as

$$W^{\alpha,\gamma}(t)=\sum_{k=0}^{\lceil\alpha\rceil}\frac{t^{\alpha k}}{\Gamma(\alpha k+\gamma)},\ \alpha,\gamma>0,\ t\in R.$$

In our work, we shall define the GFMLF as

$$W^{\alpha\beta,\gamma}(t)=\sum_{k=0}^{M}\frac{t^{k}}{\Gamma(\beta k+\gamma)},\quad 0<\alpha\leq 1,\ \beta>0,\ \gamma>0,\ t\in R. \qquad (3)$$

Seeking the fractional differentiation and integration of Eq (3) with the use of Eqs (1) and (1), we have

$$D^{F}W_{M}^{\alpha\beta,\gamma}(t)=\sum_{k=0}^{M}\frac{\Gamma(\alpha k+\gamma)}{\Gamma(\beta k+\gamma)}\frac{t^{\alpha k-F}}{\Gamma(\alpha k+1-F)} \qquad (4)$$

and

$$I^{F}W_{M}^{\alpha\beta,\gamma}(t)=\sum_{k=0}^{M}\frac{\Gamma(\alpha k+\gamma)}{\Gamma(\beta k+\gamma)}\frac{t^{\alpha k+F}}{\Gamma(\alpha k+1+F)}. \qquad (5)$$

Approximation via GFSMLF (3.2)[4]

We now define GFSMLF, which can be described as a new function for approximation

$$Q_{j}^{\alpha\beta,\gamma}(t)=\begin{cases}W_{j}^{\alpha\beta,\gamma}, & \frac{j-1}{2^{k}}\leq t\leq\frac{j}{2^{k}},\\ 0, & \text{otherwise,}\end{cases}$$

where $j = 1, 2, ..., 2^{k} = n$. Its fractional differentiation and integration are defined with the use of Eqs. (6.1.6) and (6.1.7) by

$$D^{F}Q_{j}^{\alpha\beta,\gamma}(t)=\begin{cases}D^{F}W_{j}^{\alpha\beta,\gamma}, & \frac{j-1}{2^{k}}\leq t\leq\frac{j}{2^{k}},\\ 0, & \text{otherwise,}\end{cases}$$

and

$$I^{F}Q_{j}^{\alpha\beta,\gamma}(t)=\begin{cases}I^{F}W_{j}^{\alpha\beta,\gamma}, & \frac{j-1}{2^{k}}\leq t\leq\frac{j}{2^{k}},\\ 0, & \text{otherwise,}\end{cases}$$

Figure 1. shows graphs of the GFSMLF for n = 4 with various values of MLF parameter α.

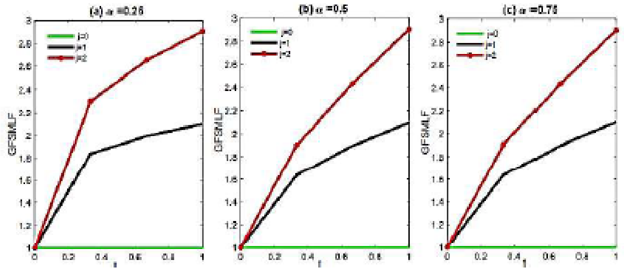

Figure 1. Graph of GFSMLF given in Eq (5) for n = 4 with various values of MLF parameter α, where β = 0.7 and γ = 1.

Obviously, most authors in this field use the following approximation:

$$O_{nm}(t)=\sum_{j=1}^{n}\sum_{k=1}^{m}c_{jk}Q_{j}^{\alpha\beta,\gamma}(t)=C\overline{Q}(t), \qquad (6)$$

where $C = \{c_{jk}\}_{j=1,k=1}^{n,m}$ is $n * m$ unknown constants. However, in this section, we suggest the following approximation of O(t) in terms of GSFMLF as

$$O_{nm}(t)=\sum_{j=1}^{n}\sum_{k=1}^{m}a_{j}b_{k}Q_{j}^{\alpha\beta,\gamma}(t)=\overline{A}\,\overline{Q}(t), \qquad (7)$$

and its fractional derivative is defined as

$$D^{F}O_{nm}(t)=\overline{A}\,D^{F}\overline{Q}(t), \qquad (8)$$

where $\overline{A} = [b_1, b_2, ..., b_m](\Sigma_{j=1}^{n}a_j)$; $A = [a_1, a_2, ..., a_n]$, $B = [b_1, b_2, ..., b_m]$ and $\overline{Q}(t) = [Q_j^{\alpha\beta,\gamma}]_{j=1}^{n}$. Here, A and B represent n + m unknowns and this reduces the effort to implement the present approach. This family of functions is not normalized or orthogonal, in contrast to most of the wavelet functions.

4. Operational integral fractional Mittag matrix

As we may have a nonlinear term in the integrand, it may be difficult to treat this situation making use of the GSFMLF method. So, in this subsection, we shall construct the OIFMM method for this object. Let $\chi(t)$ be any function, and it can be approximated via GSFMLF as:[5]

$$\chi(t) = \sum_{k=0}^{n} r_k W_k^{\alpha,\beta,\gamma}(t) = R^T \Phi(t), \tag{9}$$

with $\Phi(t) = [W_k^{\alpha,\beta,\gamma}(t), W_k^{\alpha,\beta,\gamma}(t), \ldots, W_k^{\alpha,\beta,\gamma}(t)]^T$, as defined by Eq. (٢).

The unknown coefficients $R = [r_k]_{k=0}^{n}$ can be written as

where $\Theta = [\theta_{lk}]_{l,k=0}^{n}$ is to be obtained. By combining Eqs (٨) and (6.1.12), we conclude that

$$\chi(t) = \chi\, \Theta^T \Phi(t), \qquad (١٠)$$

or

$$\Theta^T \Phi(x) = I \Rightarrow \Theta^T = [\Phi(x)]^{-1}. \tag{11}$$

It is clear that Θ in Eq. (١١) can be easily calculated. For an approximation of fractional integrals, we can integrate Eq. (١٠) with fractional order F to obtain

$$I^F \chi(t) = \chi[\Theta^T I^F \Phi(t)] = I^F{}_W(t)\chi^T ,$$

where $I^F{}_W(t) = \Theta^T I^F \Phi(t)$ is the OIFMM.

٥. Gauss-Jacobi quadrature for Riemann-Liouville integral operator

Let $v(t) \in C[0, 1]$ and $r \in N$. By applying change of variables

$$t = x^r,\ s = y^r, \qquad 0 \le y \le x \le 1, \tag{12}$$

and defining $w(x) := v(x^r)$, the Riemann-Liouville integral operator will be as follows:[6]

$$\left(J^\alpha v\right)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} v(s)\, ds$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^x \left(x^r - y^r\right)^{\alpha-1} w(y) r y^{r-1}\, dy = \left(J^\alpha_{x,|x^r,1|} w\right)(x) := \left(J_r^\alpha w\right)(x). \tag{13}$$

In this equation, $J^\alpha_{x,|x^r,1|}$ is the modified Erdelyi-Kober integration operator with base function x^r and weight function 1 that we simply represent it by J_r^α. Since we approximate $\left(J_r^\alpha w\right)(x)$ by Gauss-Jacobi quadrature, then we define

$$\omega_{1-\alpha,r}(x,y) = \frac{r y^{r-1}}{\Gamma(\alpha)} \begin{cases} \left(\frac{x^r - y^r}{x-y}\right)^{\alpha-1} & x \ne y, \\ \left(r x^{r-1}\right)^{\alpha-1} & x = y, \end{cases} \quad 0 \le y \le y \le x \le 1, \tag{14}$$

and then try to convert the interval of integration in (١٣) into the interval [−1, 1] by setting the following two other mappings:

$$x = \frac{\xi+1}{2}, \qquad y = \frac{\eta+1}{2}, \quad -1 \le \xi,\ \eta \le 1, \tag{15}$$

$$\eta = \frac{\xi+1}{2}\theta + \frac{\xi-1}{2} := \eta_\xi(\theta), \qquad -1 \le \theta \le 1. \qquad (١٦)$$

By applying change of variables (6.2.9) to Eq. (6.2.7) we will get

$$\left(J_r^\alpha w\left(\frac{\xi+1}{2}\right)\right) = \frac{1}{2^\alpha}\int_0^x (\xi-\eta)^{\alpha-1} \omega_{1-\alpha,r}\left(\frac{\xi+1}{2}, \frac{\eta+1}{2}\right) w\left(\frac{\eta+1}{2}\right) d\eta. \qquad (١٧)$$

and by defining $z(\xi) := w\left(\frac{\xi+1}{2}\right)$ and then applying the second change of variable (16) to Eq. (17) we reach the final integral as follows:

$$\left(J_r^\alpha z\right)(z) := \left(J_r^\alpha w\left(\frac{\xi+1}{2}\right)\right)$$

$$= \left(\frac{\xi+1}{2}\right)^\alpha \frac{1}{2^\alpha}\int_{-1}^{1} (1-\theta)^{\alpha-1} \omega_{1-\alpha,r}\left(\frac{\xi+1}{2}, \frac{\eta_\xi(\theta)+1}{2}\right) z\left(\eta_\xi(\theta)\right) d\theta$$

$$= \int_{-1}^{1} (1-\theta)^{\alpha-1} \overline{\omega}_{1-\alpha,r}\left(\xi, \eta_\xi(\theta) z\left(\eta_\xi(\theta)\right)\right) d\theta, \tag{18}$$

Here we note that for $\xi > -1$ and fixed, $\overline{\omega}_{1-\alpha,r}(\xi,\eta_\xi(\theta))$ as a function of θ is smooth on $[-1,1]$. Now, assume that $w^{a,b}(\theta) = (1-\theta)^a(1+\theta)^b$, $a, b > -1$ is the weight function of the orthogonal Jacobi polynomials $\{p_N^{a,b}(\theta)\}_{N=0}^{\infty}$ on the interval $[-1, 1]$. Then we approximate the integral in (١٧) by the Gauss-Jacobi quadrature formula in the following form:

$$\begin{aligned}&(J_r^\alpha z)(\xi)\\ &\approx \sum_{k=\infty}^{N} \overline{\omega}_{1-\alpha,r}(\xi,\eta_\xi(\overline{\theta}_k))z(\eta_\xi(\overline{\theta}_k))\overline{w}_k := (J_r^{\alpha,N} z)(\xi),\ -1 \le \xi \le 1,\end{aligned} \tag{١٩}$$

in which $\overline{\theta}_k$ and $\overline{w}_k$ for $k = 0, \ldots, N$, are the nodes and weights of the quadrature corresponding to the weight function $w^{\alpha-1,0}(\theta)$.

6. Jacobi collocation method with smoothing transformation

Consider integro-differential equation (6.2.1) and denote $v(t) := (D_*^\alpha u)(t)$, then $u(t) = u_0 + (J^\alpha v)(t)$. We obtain the following integral equation: [7]

$$\begin{aligned}v(t) &= f(t, u_0 + (J^\alpha v)(t))\\ &+ \sum_{q=1}^{Q} \frac{1}{\Gamma(1-\mu_q)} \int_0^t (t-s)^{\alpha-1} R_q\left(t, s, u_0(s), (J^\alpha v)(s)\right), \left(J^{\alpha-\beta_q} v\right)(s) ds,\end{aligned} \tag{20}$$

After approximating the Riemann-Liouville integral operator using the Gauss-Jacobi quadrature, we try to apply the same scheme to the integral parts of integral equation (20) together with the collocation method. In each step of computations we will do the same operations on the terms $(J^\alpha v)(s)$ and $(J^{\alpha-\beta_q} v)(s)$ and we will show these operations just by employing the same symbols that we used we will not discuss the details.

Here we note that $v(t)$, the solution of Eq. (20), behaves like $t^{1-\mu Q}$ when $t \to 0^+$, thus to reach high order precision in the numerical solution we derive

$$\begin{aligned}v(t) &= f(t, u_0 + (J_r^\alpha w)(t))\\ &+ \sum_{q=1}^{Q} \int_0^t (x-y)^{-\mu_q} \omega_{1-\alpha,r}(x,y) R_{q,r}\left(x, y, u_0(s) + (J_r^\alpha w)(y), (J_r^{\alpha-\beta_q} w)(y)\right) dy,\end{aligned} \tag{21}$$

where we use the symbols introduced in Eqs. (١٣) and define

$$\begin{aligned}&g(x, u_0 + (J_r^\alpha w)(x)) = f(x_r, u_0 + (J^\alpha v)(x^r)),\\ &R_{q,r}\left(x, y, u_0(s) + (J_r^\alpha w)(y), (J_r^{\alpha-\beta_q} w)(y)\right)\\ &\quad = R_{q,r}\left(x^r, y^r, u_0(s) + (J_r^\alpha w)(y^r), (J_r^{\alpha-\beta_q} w)(y^r)\right)\end{aligned} \tag{22}$$

Employing linear transformation (6.2.9) gives

$$\begin{aligned}z(\xi) &= g\left(\frac{\xi+1}{2}, u_0 + (J_r^\alpha)(\xi)\right)\\ &+ \sum_{q=1}^{Q} \frac{1}{2^{1-\mu_k}} \int_{-1}^{1} (\xi-\mu)^{-\mu_k} \omega_{1-\alpha,r}\left(\frac{\xi+1}{2}, \frac{\eta+1}{2}\right)\\ &\times R_{q,r}\left(\frac{\xi+1}{2}, \frac{\eta+1}{2}, u_0 + (J_r^\alpha z)(\eta), (J_r^{\alpha-\beta_q} z)(\eta)\right) d\eta.\end{aligned} \tag{23}$$

and finally by applying change of variable (١٥), Eq. (٢٣) will be in the following form:

$$\begin{aligned}z(\xi) &= g\left(\frac{\xi+1}{2}, u_0 + (J_r^\alpha)(\xi)\right)\\ &+ \sum_{q=1}^{Q} \int_{-1}^{1} (1-\theta)^{-\mu_q} \overline{\omega}_{\mu_q,r}(\xi,\eta_\xi(\theta))\\ &\times R_{q,r}\left(\xi, \eta_\xi(\theta) + (J_r^\alpha z)(\eta_\xi(\theta)), (J_r^{\alpha-\beta_q} z)\eta_\xi(\theta)\right) d\theta\end{aligned}$$

$$:= g\left(\frac{\xi+1}{2}, u_0 + \left(J_r^{\alpha}\right)(\xi)\right) + \sum_{q=1}^{Q}\left(J_r^{\alpha}z, J_r^{\alpha-\beta_q}z\right)(\xi) \qquad (24)$$

$$R_{q,r}\left(\xi, \eta_\xi(\theta) + u_0 + \left(J_r^{\alpha}z\right)(\eta_\xi(\theta)), \left(J_r^{\alpha-\beta_q}z\right)\eta_\xi(\theta)\right)$$

$$= R_{q,r}\left(\frac{\xi+1}{2}, \frac{\eta_\xi(\theta)+1}{2}, u_0 + \left(J_r^{\alpha}z\right)(\eta_\xi(\theta)), \left(J_r^{\alpha-\beta_q}z\right)(\eta_\xi(\theta))\right). \qquad (25)$$

Let P_N is the space of polynomials of degree at most N. Also, suppose that $\{\xi_i\}_{i=0}^{N}$ are the roots of the Jacobi polynomial $p_{N+1}^{a,b}(\xi)$ of degree N +1. We choose Lagrange fundamental polynomials $\{L_i(\xi)\}_{i=0}^{N}$ constructed on the points $\{\xi_i\}_{i=0}^{N}$ as a basis for P_N. A collocation solution for Eq. (23) is a polynomial $z_N(\xi) \in P_N$ with a representation $z_N(\xi) = \sum_{j=0}^{N} z_j L_j(\xi)$ that satisfies the following collocation conditions:

$$z_N(\xi_i) = g\left(\frac{\xi_i+1}{2}, u_0 + \left(J_r^{\alpha}\right)(\xi_i)\right)$$

$$+ \sum_{q=1}^{Q}\int_{-1}^{1}(1-\theta)^{-\mu_q}\,\overline{\varpi}_{\mu_q,r}\left(\xi_i, \eta_{\xi_i}(\theta)\right)$$

$$\times K_{q,r}\left(\xi_i, \eta_{\xi_i}(\theta) + \left(J_r^{\alpha}z\right)(\eta_{\xi_i}(\theta)), \left(J_r^{\alpha-\beta_q}z\right)\eta_{\xi_i}(\theta)\right)d\theta, \qquad (26)$$

for i = 0, ...,N. We look for a discrete collocation solution $\overline{z}_N(\xi) \in P_N$, $\overline{z}_N(\xi) = \sum_{j=0}^{N}\overline{z}_j L_j(\xi)$, then we approximate the integral terms in Eq. (6.2.21) using Gauss-Jacobi quadrature in two steps as follows:

$$\int_{-1}^{1}(1-\theta)^{-\mu_q}\,\overline{\varpi}_{\mu_q,r}\left(\xi,\eta_\xi(\theta)\right)K_{q,r}\left(\xi,\eta, u_0 + \left(J_r^{\alpha}z\right)(\eta_\xi(\theta), \left(J_r^{\alpha-\beta_q}z\right)\left(\eta_\xi\left(\overline{\theta}_k^q\right)\right)\right)d\theta$$

$$\approx \sum_{k=0}^{N}\overline{\varpi}_{\mu_q,r}\left(\xi,\eta_\xi\left(\overline{\theta}_k^q\right)\right)K_{q,r}\left(\xi,\eta_\xi\left(\overline{\theta}_k^q\right), u_0 + \left(J_r^{\alpha}z\right)\left(\eta_\xi\left(\overline{\theta}_k^q\right)\right), \left(J_r^{\alpha-\beta_q}z\right)\eta_\xi\left(\overline{\theta}_k^q\right)\right)\overline{w}_k^q$$

$$\approx \sum_{k=0}^{N}\overline{\varpi}_{\mu_q,r}\left(\xi,\eta_\xi\left(\overline{\theta}_k^q\right)\right)K_{q,r}\left(\xi,\eta_\xi\left(\overline{\theta}_k^q\right), u_0 + \left(J_r^{\alpha}z\right)\left(\eta_\xi\left(\overline{\theta}_k^q\right)\right), \left(J_r^{\alpha-\beta_q}z\right)\eta_\xi\left(\overline{\theta}_k^q\right)\right)\overline{w}_k^q$$

$$:= \left(\overline{J}_r^{\alpha-\beta_q,N}K_{q,r}\overline{J}_r^{\alpha-N}z, \overline{J}_r^{\alpha-\beta_q,N}z\right)(\xi), \quad q = 1,\dots,Q, \qquad (27)$$

where $\overline{\theta}_k^q$ and $\overline{w}_k^q$ are the nodes and weights with respect to the weight function $w^{-\mu_q,0}(\theta)$. Now, discrete collocation solution $\overline{z}_N(\xi)$ is obtained by satisfying the following discrete collocation equation: [7]

$$\overline{z}_N(\xi_i) = g\left(\frac{\xi_i+1}{2}, u_0 + \left(\overline{J}_r^{\alpha,N}\overline{z}_N\right)(\xi_i)\right)$$

$$+ \sum_{q=1}^{Q}\sum_{k=0}^{N}K_{q,r}\left(\xi_i, \eta_{\xi_i}\left(\overline{\theta}_k^q\right) + \left(\overline{J}_r^{\alpha,N}\overline{z}_N\right)\left(\eta_{\xi_i}\left(\overline{\theta}_k^q\right)\right), \left(\overline{J}_r^{\alpha-\beta_q}\overline{z}_N\right)\left(\eta_{\xi_i}\left(\overline{\theta}_k^q\right)\right)\right)\overline{w}_k^q$$

$$= g\left(\frac{\xi_i+1}{2}, u_0 + \left(\overline{J}_r^{\alpha,N}\overline{z}_N\right)(\xi_i)\right) + \sum_{q=1}^{Q}\left(\overline{J}_r^{\alpha-\beta_q,N}K_{q,r}\left(\overline{J}_r^{\alpha-N}\overline{z}_N, \overline{J}_r^{\alpha-\beta_q,N}\overline{z}_N\right)\right)(\xi_i) \qquad (28)$$

for i = 0, ...,N. By replacing the representation $\overline{z}_N(\xi) = \sum_{j=0}^{N}\overline{z}_j L_j(\xi)$, we derive

$$\overline{z}_i = g\left(\frac{\xi_i+1}{2}, u_0 + \left(\sum_{j=0}^{N}\overline{z}_j\overline{J}_r^{\alpha,N}L_j\right)(\xi_i)\right)$$

$$+ \sum_{q=1}^{Q}\sum_{k=0}^{N}K_{q,r}\left(\xi_i, \eta_{\xi_i}\left(\overline{\theta}_k^q\right), u_0 + \left(\sum_{j=0}^{N}\overline{z}_j\overline{J}_r^{\alpha,N}L_j\right)\left(\eta_{\xi_i}\left(\overline{\theta}_k^q\right)\right)\right.$$

$$\left.\left(\sum_{j=0}^{N}z_j\overline{J}_r^{\alpha-\beta_m,N}L_j\right)\left(\eta_{\xi_i}\left(\overline{\theta}_k^q\right)\right)\right)\overline{w}_k^q,$$

$$= g\left(\frac{\xi_i+1}{2}, u_0 + \left(\sum_{j=0}^{N}\overline{z}_j\overline{J}_r^{\alpha,N}L_j\right)(\xi_i)\right)$$

$$+ \sum_{q=1}^{Q}\left(\overline{J}_r^{\alpha-\mu_k,N}K_{q,r}\left(\sum_{j=0}^{N}\overline{z}_j\overline{J}_r^{\alpha,N}L_j, \sum_{j=0}^{N}\overline{z}_j\overline{J}_r^{\alpha-\beta_m,N}L_j\right)\right)(\xi_i). \qquad (29)$$

for i = 0, ...,N. Eq. (29) is a nonlinear system with unknowns $\overline{z}_i$, i = 0, ...,N, and by solving it we obtain the discrete collocation solution $\overline{z}_N(\xi) = \sum_{j=0}^{N}\overline{z}_j L_j(\xi)$. According to the change of variables, the relation between the solution v(t) of Eq. (18), the solution w(x) of Eq. (19) and the solution $z(\xi)$ of Eq. (32) is as $v(t) = w(\sqrt[\gamma]{t}) = z(2\sqrt[\gamma]{t} - 1)$, $0 \leq t \leq 1$, then the numerical solution for these equations will be as

$\overline{v}_N(t) = \overline{w}_N(\sqrt[\gamma]{t}) = \overline{z}_N(2\sqrt[\gamma]{t} - 1)$.

Moreover, the numerical solution $\overline{u}_N(t)$ of (١١) can be computed as follows:

$$\begin{aligned} \overline{u}_N(t) &= u_0 + \left(J^{\alpha}\overline{v}_N\right)(t) \\ &= u_0 + \left(J_r^{\alpha}\overline{w}_N\right)(x) \\ &= u_0 + \left(J_r^{\alpha}\overline{z}_N\right)(\xi) \\ &\approx u_0 + \left(\overline{J}_r^{\alpha,N}\overline{z}_N\right)(\xi) \\ &= u_0 + \left(\overline{J}_r^{\alpha,N}\overline{z}_N\right)\left(2\sqrt[\gamma]{t} - 1\right), \quad 0 \leq t \leq 1. \end{aligned} \qquad (٣٠)$$

7. Numerical experiments and results

To show the accuracy and the efficiency of the proposed Jacobi collocation method, we apply the proposed method on different numerical examples of d (11). The algorithm of the suggested method has been performed in Mathematica software. In each example, we report the root mean square of the absolute errors, ERMS,

at equidistant points, $t_i = \frac{i}{10}$, $i = 1, 2, \ldots, 10$.[8]

Example(7.1)[8] Consider the following nonlinear fractional Volterra integro-differential equation:

$$\left(D_*^{\alpha}u\right)(t) = f(t, u(t)) + \int_0^t (t-s)^{-\mu} t^{2-\mu} s^{1-2\alpha+\beta} u(s)\left(D_*^{\beta}u\right)(s)\,ds, \; t \in [0,1], \quad (٣١)$$

$$u(0) = 0,$$

where

$$f(t,u) = \frac{\Gamma(2+\alpha-\mu)}{\Gamma(2-\mu)} t^{1-\mu} - \frac{\Gamma(4+4\alpha-2\mu)\Gamma(1-\mu)\Gamma(2+\alpha-\mu)}{\Gamma(5+4\alpha-3\mu)\Gamma(1-\mu)\Gamma(2+\alpha-\beta-\mu)} t^2 u^4.$$

The exact solution of this problem is $u(t) = t^{1+\alpha-\mu}$. For $\alpha = \frac{\pi}{6}$, $\beta = \frac{\pi}{8}$ and $\mu = \frac{2}{\pi}$, we have $u(t) \in A^1[0, 1]$. The absolute error ERMS of the proposed method for different values of N and r = 1, 5, 9, 13 are listed in Table 1. When N is big enough, increasing r and then $d = [r(1-\frac{2}{\pi})] + 1$ decreases the error as we expect from the error bound (4.1). In Figure 1. we represent the errors obtained in Table 1. for different values of N and r. From this figure we observe the accuracy of the proposed method which verifies the theoretical results.

Table 1: The global ERMS errors for a range of increasing values of N and r for Example (7.1).

N	r = 1	r = 5	r = 9	r = 13
8	1.55e − 04	8.38e − 07	2.42e − 05	4.20e − 04
16	2.88e − 05	1.34e − 08	2.13e − 10	3.19e − 11
24	6.60e − 06	1.28e − 09	6.35e − 12	5.81e − 14
32	3.07e − 06	2.44e − 10	4.66e − 13	2.02e − 15

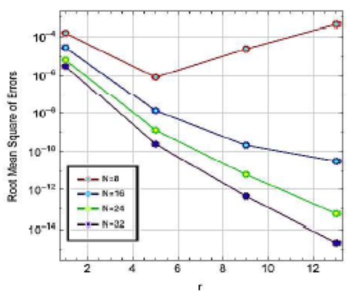

Figure 2. : Graph of the root mean square of the absolute errors for Example (7.2).

Example(7.2)[8] We examine the following fractional integro-differential equation:

$$\left(D_*^{\frac{\sqrt{2}}{3}} u\right)(t) = f(t) + \int_0^t (t-s)^{-\frac{\sqrt{2}}{2}} \frac{\left(D_*^{\frac{\sqrt{2}}{5}} u\right)(s)}{1+u^2(s)} ds, \qquad t \in [0,1], \qquad (32)$$

$u(0) = 1,$

In this equation, the function f(t) is chosen such that the exact solution of this problem to be the Mittag-Leffler function $u(t)=E_{\frac{\sqrt{2}}{2}}\left(t^{\frac{\sqrt{2}}{2}}\right) \in A^1[0, 1]$

defined by : $E_b(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(bk+1)}$.

We derive errors ERMS by applying the proposed Jacobi collocation method on Eq. (32) for several values of N and r and list these errors in Table 2and Figure 2.

Table 2: The global ERMS errors for a range of increasing values of N and r for Example (7.2)[8].

N	r = 1	r = 5	r = 9	r = 13
8	4.82e − 04	2.11e − 04	3.06e − 03	1.03e − 02
16	7.57e − 05	1.72e − 07	2.76e − 06	3.88e − 05
32	1.71e − 05	8.10e − 09	4.50e − 11	1.67e − 10
40	1.10e − 05	2.40e − 09	9.00e − 12	1.19e − 13

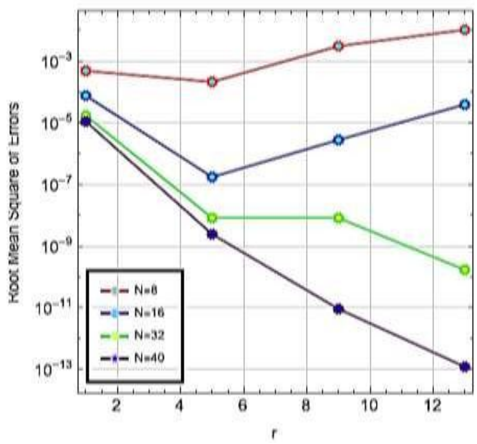

Figure 2: Graph of the root mean square of the absolute errors for Example (7.2).

Conclusion:

This paper introduces an innovative spectral approach for solving nonlinear weakly singular fractional integro-differential equations using an advanced Jacobi collocation technique. Our method systematically addresses three fundamental challenges: the nonlinearity of the equations, the fractional derivative operators, and the weakly singular kernel. Through a carefully designed algorithm, we transform the original problem into a well-conditioned system of algebraic equations while preserving the spectral accuracy characteristic of Jacobi polynomials. The theoretical analysis establishes exponential convergence rates under appropriate regularity conditions, supported by comprehensive error estimates. Numerical simulations demonstrate the superior performance of our approach compared to existing methods in terms of both accuracy and computational efficiency, particularly for long-time integration and problems with strong nonlinearities.

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