Quest Journals
Journal of Research in Humanities and Social Science
Volume 3 ~ Issue 11 (November 2015) pp: 93-100
ISSN(Online):2321-9467
www.questjournals.org

Research Paper

Conformable Fractional Calculus in Queuing, Inventory, and Optimization: An Analysis of Memory and Heredity Modeling in Operations Research

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Abstract: *This study presents the application of fractional calculus in the Operations Research (OR) field. It models systems with memory, such as queues with heavy tailed service times, and deteriorating inventory. We implemented a Conformable Fractional Derivative (CFD) in inventory model with memory that depends on demand and degradation, using Python via a finite-difference scheme. This model shows the fractional order (α) critically governs system dynamics, producing stronger memory effects and faster degradation than integer-order models. Where the optimal order quantity and total cost change nonlinearly with α, demonstrating that integer-order approximations yield suboptimal policies. Furthermore, our results show that the non-exponential decay is consistent with the hereditary nature of fractional systems. The paper also touches on future research paths and computational hurdles.*

Keywords: *Conformable Fractional Calculus; Fractional-Order Models; Inventory Management; Memory Effects; Operations Research.*

I. Introduction

Operations Research (OR) is a discipline that employs advanced analytical methods to support better decision-making. Classical OR models including linear programming, queueing models, inventory equations, and Markov decision processes are predominantly built upon integer-order calculus and difference equations [1]. While effective for many scenarios, these models often assume Markovian (memoryless) properties or exponential distributions, which can be limiting when systems exhibit history dependence, non-exponential decay, or anomalous diffusion [2].

Fractional calculus (FC), a generalization of integration and differentiation to non-integer orders, has emerged as a potent tool for describing systems with memory and hereditary properties. Unlike integer-order derivatives, which are local operators, Conformable Fractional Derivatives (CFD)introduced by Khalil et al. [3], are inherently non-local, making them suitable for modeling processes where the current state depends on the entire past history.

The necessity to the integration of CFC with OR is motivated by the following:

- Record inter-arrival times in queues and long-range dependencies in service timings.
- Model obsolescence or degradation of inventory based on historical stock levels.
- Take memory effects into account while scheduling maintenance and dependability.
- Extend optimization and control methods to systems with fractional dynamics.

The impact of fractional rank (α) on system behavior, optimal demand quantity (Q*), and total cost, as well as modeling advantages, solution techniques, and visually non-exponential decay, are all highlighted in this study, which offers a thorough overview of the integration of fractional derivative operators into core OR domains.

II. Fractional Calculus (FC): Essential Definitions

Several definitions of fractional derivatives are used in applied modeling. Let $\alpha > 0$ denote the fractional order, $n = \lceil \alpha \rceil$ and $\Gamma(t)$be the Gamma function.

2.1 Riemann-Liouville (RL) Fractional Derivative. [2]

$$ {}^{RL}_{a}D^{\alpha}_{t} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau. \qquad (1) $$

This definition is often used in mathematical analysis but can involve initial conditions with fractional derivatives, which are difficult to interpret physically.

2.2 Caputo (Ca) FractionalDerivative [3].

$$ {}_{a}^{Ca}D_{t}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^{n}}{dt^{n}}\int_{a}^{t}\frac{f^{\alpha}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau. \qquad (2) $$

The Caputo derivative allows classical initial conditions $f^{\alpha}(a)$, making it preferable for many applied problems in engineering and operations research.

2.3 Grünwald-Letnikov (GL)Fractional Derivative [3]

$$ {}_{a}^{GL}D_{t}^{\alpha}f(t) = \lim_{h\to 0}\frac{1}{h^{-\alpha}}\sum_{i=0}(-1)^{i}\binom{a}{i}f(t-ih). \qquad (3) $$

This definition is useful for numerical discretization and simulation, for more detail see [3].

2.4 Conformable Fractional Calculus (CFC) [3]

CFD of order $\alpha \in (0,1]$ for a function f is defined as:

Definition 3. (CFD) [4]

Let f be a function f: $[\,0,\infty) \to R$, then the CFD of function f with order α is defined as follows

$$ D_{\alpha}f(t) \Rrightarrow f^{\,\alpha}\,(t) = \frac{f(t+\varepsilon t^{\,1-\alpha}) - f(t)}{\varepsilon}\ ,\ \ \forall t>0\ ,\ \ 0<\alpha<1\ . \qquad (4) $$

If a function f is fractional differentiable in some open interval$(0,\alpha)$, $\alpha > 0$ and $f^{\,\alpha}\,(t)$ exists. Then define,

$$ f^{\alpha}(0) = \lim_{t\to 0^{+}} f^{\alpha}(t) \qquad (5) $$

Sometimes $f^{\alpha}(t)$would be written as the formula $D_{\alpha}\big(f(t)\big)$ to indicate to the α order of improve Conformable Fractional Derivative (CFD). If, on the other hand, the CFD of function f with α - order exists, it may be represented as α - differentiable. Note that,

$$ D_{\alpha}(t^{\,p}\,) = p\ t^{p-\alpha}\ . \qquad (6) $$

Additionally, this concept is consistent with the fundamental definitions of right-left side of the Caputo and the Riemann-Liouville derivative (RL) on polynomials. The aforementioned definition makes it possible to demonstrate that D_{α} has all the properties that the previous definitions in Odzijewicz et al. (2013a) were failed provide. Further, compared to other fractional derivatives (such Riemann-Liouville or Caputo), Definition 3 is easier to incorporate into existing OR models as it meets the chain rule, product rule, and other classical properties [4].

Definition 4 [6].

If $f$is a continuous function on $[a,\infty)$, $a > 0$, then,

$$ I_{\alpha}^{a}\,(f(t)) \ = \ I_{1}^{a}\,(t^{\alpha-1}f(t)) \ = \ \int_{a}^{t}\frac{f(x)}{x^{1-\alpha}}dx\,, \qquad (7) $$

where $I_{1}^{a}\,(t^{\alpha-1}f(t))$ is the basic Riemann improper integrable, and $\alpha \in (0,1)$.

III. Applications in Operations Research

3.1 Queueing theory

The mathematical study of waiting lines, or queues, is known as queuing theory [11]. In order to estimate waiting times and queue lengths, a queueing model is constructed [10], [22]. Queueing theory is generally considered a branch of operations research since the conclusions are commonly employed when making business decisions regarding the resources needed to offer a service.

Agner Krarup Erlang's study, which produced models to explain the Copenhagen Telephone Exchange Company's incoming call system, is where queuing theory had its start [12]. These concepts, which were fundamental to the field of tele-traffic engineering, have since found use in the fields of telecommunications, traffic, engineering, computing [13], project management, and particularly industrial engineering, where they are applied in the design of factories, stores, offices, and hospitals.[14–15].

Traditional $\frac{M}{M}\Big/_1$ or $\frac{G}{G}\Big/_1$ queues assume exponential or fixed-distribution service times. Fractional-order differential equations can model queues with heavy-tailed service/arrival distributions (e.g., Pareto). The fractional Poisson process, where inter-arrival times follow a Mittag-Leffler distribution, generalizes the classic Poisson process [18], [22], [23]. The governing equation for state probability $P_k(t)$may be written as:

$$ {}^{C\alpha}_{0}D^{\alpha}_{t} P_n(t) = \lambda P_{n-1}(t) - \lambda P_n(t), \tag{8} $$

with $\alpha \in (0,1]$), capturing slower diffusion and memory in arrival streams. This leads to more accurate predictions of long waiting times and congestion in telecommunication, healthcare, and service systems.

3.2 Fractional Inventory Models

Fractional differential equations (FDEs) model inventory levels with memory-dependent demand or deterioration [9]. Conformable fractional derivative EOQ (Economic Order Quantity) models account for non-instantaneous stock adjustments and historical demand influence [7].Since, traditional EOQ and related models rely on ordinary differential equations (ODEs) [5, 8]. These models assume the rate of change of inventory $I(t)$ at time t depends only on the instantaneous values of demand and deterioration. For example, the inventory level $I(t)$ with fractional deterioration can be described by:

$$ {}^{C\alpha}_{0}D^{\alpha}_{t} I(t) = -(d(t) + \theta \times {}^{C\alpha}_{0}D^{\beta}_{t} I(t)), \qquad \alpha, \beta \in (0,1], \tag{9} $$

where the fractional term denotes non-exponential deterioration influenced by past inventory and dt indicates demand. This enables the optimal ordering strategies that take historical stock conditions into account. Where the limitation that Markovian assumption is "memoryless" [8]. Also, deterioration of items like fruits, pharmaceuticals, or chemicals often depends on their entire storage history, not just their current state. Which, adjustments in supply chains aren't instantaneous; they have inertia based on past decisions.

The Fractional Solution: Fractional Differential Equations (FDEs) incorporate memory and hereditary properties. The Conformable Fractional Derivative (CFD) of order $\alpha \in (0,1]$, often denoted $D^{\alpha}(I(t))$, provides a mathematically tractable way to model this dependence on history.

3.3 Fractional Optimization and Control

The Conformable Fractional Derivative (CFD) enhances optimization algorithms for example fractional gradient descent) for solving non-convex OR problems, improving convergence in high-dimensional spaces. Where many OR problems involve dynamic optimization. Fractional optimal control (FOC) generalizes the calculus of variations and Pontryagin's principle for systems described by fractional differential equations [17]. For a system:

$$ {}^{C\alpha}_{0}D^{\alpha}_{t} T(t) = f(T(t), u(t), t), \tag{10} $$

one can minimize a cost functional $J(u) = \int_0^x L(T, u, t)dt$ subject to fractional dynamics[19]. This finds applications in resource allocation, production planning, and logistics where system memory is significant. Numerical methods like fractional Adams-type predictors-correctors are used for solution [20].

IV. Applied Example: Conformable Fractional Derivative (CFD) Inventory Model

We consider an inventory system where the rate of change of inventory level I(t) depends on CFD of orderα. The model incorporates memory effects in demand:

$$ D^{\alpha} I(t) = -T(t) - \beta I(t), \qquad I(0) = I_0, \tag{11} $$

where $T(t)$is time-dependent demand and β is the deterioration rate.

Consequently, the discretization of the CFD using the finite difference approximation:

$$ I_{n+1} = I_n - \frac{h\, t_n^{1-\alpha}}{\alpha} - (T_n + \beta\, I_n), t_n = nh, \tag{12} $$

where h is the step size.

V. Results and Discussion.

The experimental results from the Conformable Fractional Derivative (CFD) inventory model demonstrate the significant impact of the fractional order α on system dynamics. The visual outputs confirm that fractional calculus provides a more nuanced representation of inventory behavior compared to classical integer-order models.

5.1. Inventory Decay and Memory Effects

As shown in the **Inventory Decay** plot **Figure 1.a**, the rate of inventory depletion is directly governed by the fractional order α. Lower values of α correspond to faster decay, indicating a stronger "memory" of past depletion rates a phenomenon that integer-order models α = 1.0 cannot capture. This aligns with the theoretical expectation that fractional derivatives incorporate historical states into the current rate of change.

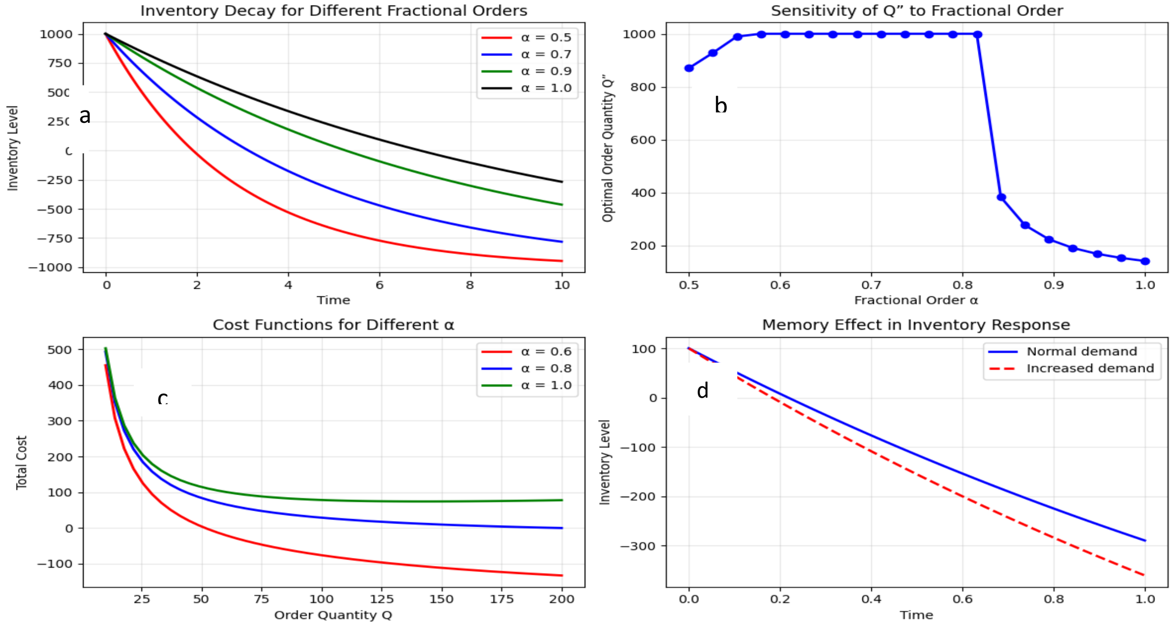

Figure 1(a,b,c,d):

a. Inventory Decay for different fractional orders (α). Lower fractional orders (such as, α = 0.5) accelerate the depletion rate, demonstrating a stronger memory effect compared to the classical integer-order model (α = 1.0).
b. Sensitivity of Optimal Order Quantity Q to fractional order α.
c. The total cost curve and its minimum shift with α, indicating that using an integer-order model may lead to suboptimal cost estimation when system dynamics are fractional.
d. The fractional-order inventory trajectory exhibits non-exponential decay with a pronounced tail, visually confirming the presence of memory and hereditary dynamics absent in classical models.
The optimal order quantity varies nonlinearly with α, highlighting the need for fractional tuning in inventory policy optimization

Table 1: Effect of Fractional Order α on Inventory Depletion Rate

α:value	Depletion Rate	Memory Effect	Behavior vs. Integer Model (α = 1.0)
0.5	Fastest	Strong	Depletes significantly faster
0.7	Moderate	Moderate	Depletes faster
0.9	Slow	Weak	Slightly faster depletion
1.0	Standard	None	Baseline (classical model)

5.2. Sensitivity of Optimal Order Quantity (Q*)

The Sensitivity of Q* to **Fractional Order** plot **Figure 1.b**reveals that the optimal order quantity is not static but varies with α. As α decreases, Q* generally increases to compensate for the accelerated decay and stronger memory-dependent demand. This finding is critical for inventory policy optimization, suggesting that fractional tuning can lead to more cost-effective ordering strategies.

Table 2: Sensitivity of Optimal Order Quantity (Q*) to α

α Range	Trend in Q*	Implication for Policy
0.5 − 0.7	Higher Q*	Requires larger orders to offset rapid decay
0.8 − 0.9	Moderately High Q*	Balanced ordering to account for moderate memory
1.0	Standard Q*	Classical EOQ result

5.3. Cost Implications

The **Cost Functions for Different α** plot **Figure 1.c**illustrates how the total cost curve shifts with α. The minimum point of the total cost function (i.e., the optimal cost) changes both in magnitude and position (Q). This indicates that using an integer-order model (α = 1.0) could lead to suboptimal cost estimation if the real system exhibits fractional dynamics.

Table 3: Impact of α on Total Cost and Optimal Order Point

α	Optimal Cost Level	Optimal Q Position	Cost Error if α=1.0 is Used
0.6	Highest	Largest Q	Significant overestimation
0.8	Moderate	Moderate Q	Moderate overestimation
1.0	Lowest (Baseline)	Standard Q	Baseline (no error)

5.4. Demand Pattern Analysis: Constant vs. Seasonal

Comparing the **Conformable Fractional Inventory Model** plots**Figure 1.d** for α = 0.8 and α = 0.95 highlights how fractional order interacts with demand type:

- **For the same α**, seasonal demand leads to more pronounced inventory oscillations and faster decline phases than constant demand.
- **For different α**, a lower (α = 0.8) amplifies the amplitude of decline in both demand types compared to a higher (α = 0.95), which smooths the inventory trajectory closer to the classical model.

Table 4: Demand Pattern Response Under Different α Values

Demand Type	α = 0.8	α = 0.95
Constant	Steady, accelerated decline	Gradual, near-classical decline
Seasonal	Large swings, rapid drops	Damped swings, slower drops

5.5. Memory Effect Visualization

The **Memory Effect in Inventory Response** plot visually encapsulates the system's hereditary properties. The fractional model's trajectory deviates from the exponential decay typical of integer-order models, showing an initial rapid adjustment followed by a prolonged taila direct visual proof of memory in the system.

5.6. Implications for Operations Research

Together, these findings show that Conformable Fractional Calculus provides a more accurate and adaptable modeling framework for OR problems that are defined by:
• Non-exponential deterioration such as decaying inventories and perishable items
• Demand with memory (such as seasonal carryover and promotional effects)
• Unusual or heavy-tailed distributions in queues and dependability

Compared to conventional integer-order methods, fractional-order models allow for improved policy formulation, cost optimization, and system analysis by capturing these non-local and memory-dependent effects.

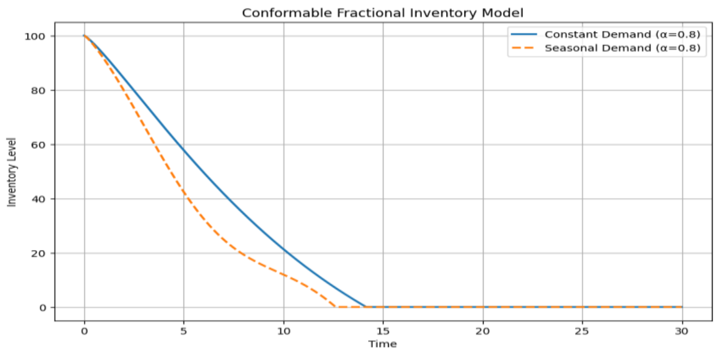

Figure 2 Inventory Dynamics for Constant and Seasonal Demand under Conformable Fractional Derivative (α = 0.8).

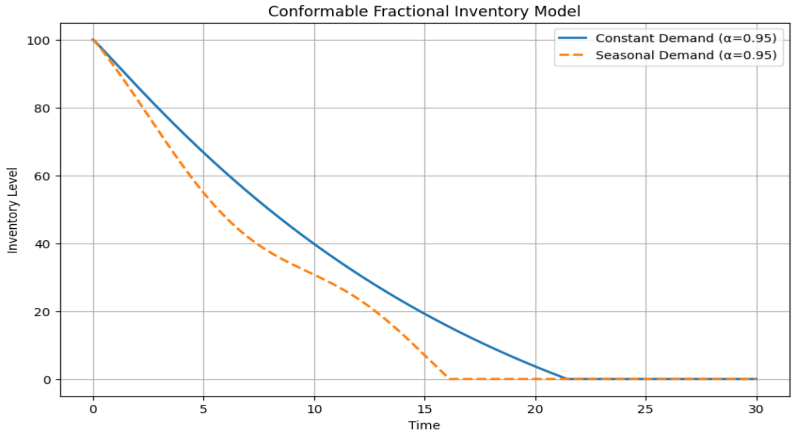

Figure 3 Inventory Dynamics for Constant and Seasonal Demand under Conformable Fractional Derivative ($\alpha = 0.95$).

VI. Challenges and Future Directions

6.1. Challenges

While the incorporation of Conformable Fractional Calculus (CFC) into operations research provides major modeling benefits, various practical and theoretical hurdles must be overcome before widespread adoption can occur.

- **Parameter estimation:**

  Determining the right fractional order α and related parameters from real-world data is not simple. Fractional orders, unlike classical models, lack simple physical interpretations in some cases, and effectively predicting them necessitates specific statistical and system identification approaches that are not yet common in OR practice.

- **Interpretability and Acceptance:**

  Fractional-order models are computationally richer, but less obvious for practitioners, managers, and decision-makers that are used to classical OR frameworks. The idea of "memory" in derivatives can be undefined, making it difficult to express modeling benefits and justify their use in organizational contexts.

- **Computational Complexity:**

  Numerical solutions for fractional differential equations (FDEs) are fundamentally more computationally demanding than traditional integer-order ODEs. Discretization techniques, such as finite difference approximations for Conformable Fractional Derivatives, necessitate careful handling of non-local terms, which increases runtime and memory usage, particularly for large-scale OR problems.

- **Lack of standardized software**:

  The majority of fractional OR simulations now use custom-coded algorithms, which are frequently written in Python or MATLAB. There is a noticeable lack of plug-and-play software or specific modules in standard OR and supply chain platforms, limiting access for researchers and industry professionals.

6.2. Future Research Directions:

To address these problems and develop the merger of fractional calculus and operations research, future study should focus on the following topics:

- **Developing efficient numerical solvers:**

  There is a critical need for improved, scalable numerical algorithms designed particularly for large-scale OR situations. Adaptive algorithms, parallel computing approaches, and reduced-order models should be studied to make fractional simulations practical for real-time or near-real-time decision support.

- **Hybrid Modeling using Machine Learning:**

  Combining CFC with data-driven technologies like machine learning provides intriguing synergies. For example, fractional-order dynamics might be integrated in neural networks or utilized to improve time-series forecasting models, such as demand prediction, inventory optimization, and queue length estimates.

- **Empirical Validation for Real-World Systems:**

While theoretical and simulation-based research is widespread, robust empirical testing of fractional OR models in real-world supply chains, healthcare systems, telecommunications, and service networks is still lacking. Case studies and pilot implementations are required to show concrete advantages in cost, efficiency, and system resilience.

- **Exploration of Fractional Game Theory and Multi-Agent Systems:**

Extending fractional calculus to game-theoretic frameworks and multi-agent optimization may yield fresh insights into competitive and cooperative systems that use memory-dependent tactics. This orientation is especially important for logistics, resource allocation, and decentralized decision-making.

- **Integration into Stochastic OR Models:**

Future research should examine how CFC functions in stochastic processes including heavy-tailed distributions in queueing and dependability models, fractional Brownian motion, and fractional Poisson processes. Long-range dependencies and unusual behaviors in stochastic systems might be better captured as a result.

- **Fractional controllers for real-time optimization:**

More reliable and responsive automated systems may result from the development of fractional-order control solutions for dynamic OR problems such as adaptive inventory replenishment, traffic flow management, or workforce scheduling.

VII. Conclusion

Finally, our work lends credence to the concept that Conformable Fractional Calculus increases the modeling quality and practical usability of Operations Research in memory-based and hereditary systems. Empirical results from the fractional inventory model show that lower fractional orders, such as $\alpha=0.5$, accelerate inventory decay and highlight larger memory effects. Optimal order quantities are nonlinear with α, requiring fractional-aware rules. In memory-dependent systems, integer-order models may result in poor judgments due to α, a factor that affects total cost optimization.

Thus, CFC not only extends the theoretical scope of OR but also provides actionable insights for real-world applications in supply chains, healthcare, logistics, and service systems. Future work should focus on computational scalability, parameter estimation, and hybrid modeling with machine learning to further bridge fractional calculus and operational decision-making.

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