Quest Journals
Journal of Research in Mechanical Engineering
Volume 10 ~ Issue 2 (2024) pp: 19-35
ISSN(Online):2321-8185
www.questjournals.org

Research Paper

Discontinuous Galerkin Finite Element Adaptation Method for Biharmonic Eigenvalue Problems with Fixed Boundary Conditions

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Abstract: *This paper proposes an Discontinuous Galerkin finite element method with penalty parameters for biharmonic eigenvalue problems under fixed boundary conditions. A comprehensive error estimation is provided in the study, encompassing both improved a priori error estimation and posteriori error estimation techniques. These estimations highlight the reliability and effectiveness of the posteriori error estimation method for higher-order eigenfunctions as well as the accuracy of eigenvalue estimation. Through adaptive numerical experiments and theoretical analysis, it is shown that the convergence order of this method is effective. The adaptive numerical experiment and theoretical analysis show that the convergence order of the proposed method is effective.*

Key words: *Biharmonic eigenvalue; DGFEM; Prior and posterior error analysis; Adaptive algorithm*
Mathematics Subject Classification: *65N25 · 65N30 · 65N15*

Received 29 Apr., 2024; Revised 06 May, 2024; Accepted 08 May, 2024 © The author(s) 2024.
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I. Introduction

The Biharmonic Eigenvalue problem is one of the most fundamental model problems in mathematics. They have wide applications in various fields such as modeling of thin plate vibrations [1], fluid-structure interactions [2], inverse scattering theory [3], and electronic structures [4]. The biharmonic operator, along with appropriate Dirichlet and Neumann boundary conditions, is widely used to model the isotropic elastic behavior of thin plates and membranes. It has been proven that finite element methods (*FEMs*) are highly effective in numerically solving such fourth-order elliptic problems. Many *FEM* models have been developed, which can be roughly classified into three categories: consistent, non-consistent, and mixed finite element models. Fourth-order compatible finite elements require the finite element space to be a subspace of the *Sobolev* space $H^2(\Omega)$, where Ω represents the com- putational domain. Traditionally, a C^1 consistent space is introduced [5] for this purpose. However, implementing such a finite element space is extremely challenging, especially when it involves high-order basis functions or three-dimensional domains, which is why it is rarely used in practice. Another approach is to reduce the higher-order problem to a lower-order problem and then use mixed finite element methods [6–8]. Of course, for fourth-order el- liptic problems, there are also nonconforming finite element methods and inconsistent finite element methods [5, 7, 8].

In recent years, the *DG* finite element method has emerged as a highly effective dis- cretization technique for solving non-conforming eigenvalue problems. This method has gained widespread popularity due to its remarkable performance in this field. The *DG* method was first used in elliptic problems in 1977 [9]. The *DG* method is capable of op- erating on completely discontinuous finite element spaces, offering significant flexibility in mesh design. This makes it an ideal choice for adaptive algorithms, where the mesh size h can be adjusted based on specific adaptive criteria. This adaptability allows for efficient and accurate simulations in a variety of scenarios.

Reference [10] provides an in-depth discussion on the non-conforming finite elemen-t method applied to linear plate eigenvalue problems. It explores the application of this method and offers valuable insights into its implementation and effectiveness for such prob- lems. References [11, 12] discuss the C^0IPG method for plate vibration redetermination and eigenvalue problems. References [13, 14] discuss lower bounds for vibration redetermination and eigenvalue problems. Reference [15] discusses posteriori error estimation for fourth-order problems using *Morley* elements. Reference [16] discusses the application of plate vibration redetermination and eigenvalue problems in engineering fields such as aerospace and nuclear energy. Reference [17] investigates the relationship between characteristic pairs of plate vi- bration redetermination and eigenvalue problems. Reference [18] provides a detailed analysis of the posteriori error estimation for the biharmonic problem, specifically utilizing quadratic basis functions and the C^0 interior penalty method.

The purpose of this paper is to further analyze the priori and posteriori error analysis of biharmonic eigenvalue problems using the $SIPDG$ method under fixed boundary conditions, building upon the aforementioned studies. Our work is as follows:

1 Refer to reference [19], we extend the DG method originally developed for numerical approximation of second-order elliptic partial differential equations to address biharmonic eigenvalue problems. Notably, we provide a proof demonstrating that $\|u - u_h\|_{0,\Omega} \lesssim h^t\|u - u_h\|_G$, indicating that the DG method achieves convergence order in terms of the mesh size h. Furthermore, our analysis reveals that employing high-order elements enables optimal convergence order for the eigenfunctions in terms of error estimation.

2 We performed both a priori and a posteriori error analyses of the h-type interior penalty DG method in the energy norm, focusing on biharmonic eigenvalue problems. Through the use of the lifting operator, we confirmed the reliability and effectiveness of the posteriori error estimation for eigenfunctions in the DG method. Our analysis offers valuable insights into the accuracy and efficiency of this approach in tackling biharmonic eigenvalue problems.

3 We not only conducted numerical experiments on uniform grids but also on adaptively refined grids. From the numerical results, it can be observed that our method achieves optimal convergence order for eigenvalues.

The structure of this paper is organized as follows. In Section 2, we introduce the model problem, function spaces, and derive the discrete formulation used in our study. In Section 3, we define the energy norm, discuss the boundedness of the lifting operator, and establish the ellipticity and continuity of the equations. Section 4 is devoted to priori error analysis, while Section 5 presents the posteriori error analysis. Finally, in Section 6, we illustrate the theoretical framework with numerical experiments.

II. Preliminaries

In this paper, we utilize $L^p(\omega)$ to denote the standard Lebesgue space, where $1 \le p \le \infty$ and $\omega \subset \mathbb{R}^2$. The corresponding norm is denoted by $\|\cdot\|_{L^p(\omega)}$. However, for simplicity, we represent the norm of $L^p(\omega)$ as $\|\cdot\|_\omega$. Next, we use $H^s(\omega)$ (where $s \ge 0$) to denote the space of real functions on $\Omega \subset \mathbb{R}^2$ in the standard *Hilbertian Sobolev* space. The norm and semi-norm in $H^s(\omega)$ are respectively represented by $\|\cdot\|_{s,\omega}$ and $|\cdot|_{s,\omega}$.

2.1 Model problem, Meshes and finite-element spaces

Let Ω denote a bounded open polygonal region in $\mathbb{R}^2$ with boundary $\partial\Omega$, and let $\mathbf{n}$ denote the unit outward normal. We consider the eigenvalue problem for the following biharmonic equation to find $\lambda \in \mathbb{R}$ and $u \neq 0$ such that

$$\Delta^2 u = \lambda u \quad in\ \Omega, \tag{2.1}$$

together with the boundary conditions of the clamped plate (CP):

$$u = \nabla u \cdot \mathbf{n} = 0 \quad on\ \partial\Omega. \tag{2.2}$$

The weak formulation of problem (2.1) is to seek $(\lambda, u) \in \mathbb{C} \times H_0^2(\Omega)$, such that $u \neq 0$ and

$$A(u, y) = B(u, y), \ \ \forall y \in H_0^2(\Omega) \tag{2.3}$$

where $A(u, y) = \int_\Omega \Delta u \Delta y \mathrm{d}x$, $B(u, y) = \lambda(u, y) = \int_\Omega \lambda u y \mathrm{d}x$.

Let T be a conforming partition of Ω into triangles, where the triangular elements $\kappa \in$ T do not intersect. The length of an edge in element κ is denoted as h_g, and the diameter of element κ is denoted as h_κ, Furthermore, $h = \max_{\kappa \in T} h_\kappa$. Suppose the partition T is shape-regular, constructed via an affine transformation $\Psi_\kappa : \kappa_0 \to \kappa$ with a nonsingular Jacobian, where κ_0 is the reference triangle. The aforementioned affine transformation Ψ_κ ensures that $\bar{\Omega} = \bigcup_{\kappa \in \mathrm{T}} \bar{\kappa}$

For a non-negative integer r, we denote by $P_r(\kappa_0)$ the set of all tensor product polynomials on κ_0 of degree at most r in each coordinate direction. For $r \geq 2$ we consider the finite-element space:

$$S_h^r = \{v \in L^2(\Omega), v|_\kappa \circ \Psi_\kappa \in P_r(\kappa_0), \forall \kappa \in T\}.$$

The piecewise function space defined on the partition T is given by:

$$H^s(T) = \{v \in L^2(\Omega), v|_\kappa \in H^s(\kappa), \forall \kappa \in T\}.$$

The union of all 1D element edges associated with T is denoted as $\mathcal{F}$. Let $\mathcal{F} = \mathcal{F}_b \bigcup \mathcal{F}_i$, where $\mathcal{F}_i = \mathcal{F} \setminus \mathcal{F}_b$, $\mathcal{F}_i$ represents the internal edges, while $\mathcal{F}_b$ represents the edges on the boundary $\partial\Omega$.

Let the function $\mathbf{h} : \Omega \to R$ be represented as a collection of h_κ, where $\mathbf{h}|_\kappa = h_\kappa (\kappa \in T)$ and $\mathbf{h}|_g = \{\mathbf{h}\}(g \subset \mathcal{F})$.

Next, we introduce some trace operators. Let κ^+ and κ^- be two elements in T hat share an edge g, which is denoted by $g = \partial\kappa^+ \cap \partial\kappa^-$. Define the outward normal unit vectors $\mathbf{n}^+$ and $\mathbf{n}^-$ on g corresponding to $\partial\kappa^+$ and $\partial\kappa^-$, respectively. For functions $y : \Omega \to R$ and $\mathbf{q} : \Omega \to R^2$, which may be discontinuous across $\mathcal{F}$, we define the following quantities. For $y^+ = v|_{g \subset \partial\kappa^+}$, $y^- = y|_{g \subset \partial\kappa^-}$, and $\mathbf{q}^+ = \mathbf{q}|_{g \subset \partial\kappa^+}$,$\mathbf{q}^- = \mathbf{q}|_{g \subset \partial\kappa^-}$, we get

$$\{y\} = \frac{1}{2}(y^+ + y^-), \ \ [y] = y^+\mathbf{n}^+ + y^-\mathbf{n}^-, \ \ \{\mathbf{q}\} = \frac{1}{2}(\mathbf{q}^+ + \mathbf{q}^-), \ \ [\mathbf{q}] = \mathbf{q}^+\mathbf{n}^+ + \mathbf{q}^-\mathbf{n}^-.$$

If $g \subset \partial_\kappa \bigcap \mathcal{F}_b$, these definitions are modified as follows:

$$\{y\} = y^+, \ \ [y] = y^+\mathbf{n}^+, \{\mathbf{q}\} = \mathbf{q}^+, \ \ [\mathbf{q}] = \mathbf{q}^+\mathbf{n}^+.$$

With the above definition, we can easily obtain the following identities:

$$\sum_{\kappa \in T} \int_{\partial\kappa} y\mathbf{q} \cdot \mathbf{n} \mathrm{d}s = \int_{\mathcal{F}} [y] \cdot \{\mathbf{q}\} \mathrm{d}s + \int_{\mathcal{F}_i} \{y\}[\mathbf{q}] \mathrm{d}s. \tag{2.4}$$

2.2 SIPDG

For the sake of simplicity in error analysis later on, we introduce the concept of a lifting operator $\mathcal{L} : S_h^r + H_0^2(\Omega) \to S_h^r$

$$\int_\Omega \mathcal{L}(w) y \mathrm{d}x = \int_{\mathcal{F}} ([w]\{\nabla y\} - \{y\}[\nabla w]) \mathrm{d}s \quad \ \ \forall y \in S_h^r. \tag{2.5}$$

Then, we can reformulate the eigenvalue problem of equation (2.1) as a system of first-order equations:

$$\nabla u = \mathbf{t}, \nabla \cdot \mathbf{t} = s, \nabla s = \mathbf{q}, \nabla \cdot \mathbf{q} = \lambda u, \quad in\ \Omega, \lambda \in C \tag{2.6}$$

$$u = 0, \nabla u \cdot \mathbf{n} = 0, \qquad on\ \partial\Omega \tag{2.7}$$

Firstly, define the following numerical fluxes:

$$\hat{u} = \begin{cases} \{u_h\}, & \text{if } g \subset \Gamma_i, \\ 0, & \text{if } g \subset \Gamma_b, \end{cases} \qquad \hat{\mathbf{t}} = \begin{cases} \{\nabla u_h\}, & \text{if } g \subset \Gamma_i, \\ 0, & \text{if } g \subset \Gamma_b, \end{cases}$$

$$\hat{s} = \begin{cases} \{\Delta u_h\} - \xi[\nabla u_h], & \text{if } g \subset \Gamma_i, \\ \Delta u_h - \xi(\nabla u_h \cdot \mathbf{n}), & \text{if } g \subset \Gamma_b, \end{cases} \qquad \hat{\mathbf{q}} = \begin{cases} \{\nabla \triangle u_h\} + \eta[u_h], & \text{if } g \subset \Gamma_i, \\ \nabla \Delta u_h + \eta u_h \cdot \mathbf{n}, & \text{if } g \subset \Gamma_b, \end{cases}$$

Remark 1. *The piecewise constants η and ξ defined on $\Gamma \rightarrow R$ are given by $\eta|_g = C_\eta(\mathbf{h}|_g)^{-3}(C_\eta > 0)$ and $\xi|_g = C_\xi(\mathbf{h}|_g)^{-1}(C_\xi > 0)$, respectively.Of course, in the subsequent IPDG method, in order to ensure stability, we choose η and ξ to be sufficiently large positive constants that depend only on the mesh parameters [20].*

Then, (2.6) can be equivalent to: For any $\mathfrak{w}_h, y_h \in S_h^r$, $\mathbf{z}_h, \mathbf{r}_h \in S_h^r \times S_h^r$, find $u_h, s_h \in S_h^r$, $\mathbf{t}_h, \mathbf{q}_h \in S_h^r \times S_h^r$, such that

$$\int_\kappa \mathbf{t_h} \cdot \mathbf{z_h} \mathrm{d}x = -\int_\kappa u_h \nabla \cdot \mathbf{z_h} \mathrm{d}x + \int_{\partial\kappa} \hat{u}\mathbf{n} \cdot \mathbf{z_h} \mathrm{d}s, \tag{2.8}$$

$$\int_\kappa s_h \mathfrak{w}_h \mathrm{d}x = -\int_\kappa \mathbf{t_h} \cdot \nabla \mathfrak{w}_h \mathrm{d}x + \int_{\partial\kappa} \mathfrak{w}_h \mathbf{n} \cdot \hat{\mathbf{t}} \mathrm{d}s, \tag{2.9}$$

$$\int_\kappa \mathbf{q_h} \cdot \mathbf{r_h} \mathrm{d}x = -\int_\kappa s_h \nabla \cdot \mathbf{r_h} \mathrm{d}x + \int_{\partial\kappa} \hat{s}\mathbf{n} \cdot \mathbf{r_h} \mathrm{d}s, \tag{2.10}$$

$$\int_\kappa \lambda u_h y_h \mathrm{d}x = -\int_\kappa \mathbf{q_h} \cdot \nabla y_h \mathrm{d}x + \int_{\partial\kappa} y_h \mathbf{n} \cdot \hat{\mathbf{q}} \mathrm{d}s, \tag{2.11}$$

where $\mathbf{z}$ and $\mathbf{r}$ are vector test functions, w and y are scalar test functions, and $\mathbf{n}$ is the unit outward normal vector of $\partial\Omega$.

Then for (2.8) – (2.11) using the *Green* formula and the identity (2.4), we get

$$\begin{aligned} &\int_\Omega \Delta_h u_h \Delta_h y_h \mathrm{d}x + \int_{\Gamma_b} \Big(u_h(\nabla\Delta y_h \cdot \mathbf{n}) + y_h(\nabla\Delta \cdot u_h \cdot \mathbf{n}) - \Delta y_h(\nabla u_h \cdot \mathbf{n}) - \Delta u_h(\nabla y_h \cdot \mathbf{n}) \Big) \mathrm{d}s \\ &+ \int_{\Gamma_b} \Big(\eta u_h y_h + \xi(\nabla u_h \cdot \mathbf{n})(\nabla y_h \cdot \mathbf{n}) \Big) \mathrm{d}s + \int_{\Gamma_i} \Big(\eta[u_h] \cdot [y_h] + \xi[\nabla u_h][\nabla y_h] \Big) \mathrm{d}s \\ &+ \int_{\Gamma_i} \Big([u_h] \cdot \{\nabla\Delta y_h\} + [y_h] \cdot \{\nabla\Delta u_h\} - \{\Delta y_h\}[\nabla u_h] - \{\Delta u_h\}[\nabla y_h] \Big) \mathrm{d}s = \int_\Omega \lambda u_h y_h \mathrm{d}x. \end{aligned} \tag{2.12}$$

Recalling the conventions $[y]|_g = y\mathbf{n}$, $[\mathbf{r}]|_g = \mathbf{rn}$, $\{y\}|_g = y$ and $\{\mathbf{r}\}|_g = \mathbf{r}$, equation (2.12) can be expressed in a condensed form

$$\begin{aligned} &\int_\Omega \Delta_h u_h \Delta_h y_h \mathrm{d}x + \int_\Gamma \Big(\eta[u_h] \cdot [y_h] + \xi[\nabla u_h][\nabla y_h] \Big) \mathrm{d}s + \int_\Gamma \Big([u_h] \cdot \{\nabla\Delta y_h\} \\ &+ [y_h] \cdot \{\nabla\Delta u_h\} \Big) \mathrm{d}s - \int_\Gamma \Big(\{\Delta y_h\}[\nabla u_h] + \{\Delta u_h\}[\nabla y_h] \Big) \mathrm{d}s = \int_\Omega \lambda u_h y_h \mathrm{d}x, \end{aligned} \tag{2.13}$$

see that (2.13) gives rise to the $SIP - DG$

$$\exists\, u_h \in S_h^r, \quad st\ A_h(u_h, y_h) = B_h(y_h) \quad \forall\, y_h \in S_h^r, \tag{2.14}$$

where the bilinear form $A_h(\cdot,\cdot)$ and the linear functional $B_h(\cdot)$ can be obtained from (2.13).
We substitute the lifting operator into (2.13) to obtain

$$A_h(u_h,y_h)=\int_\Omega\Big(\Delta_h u_h\Delta_h y_h+\mathcal{L}(u_h)\Delta_h y_h+\Delta_h u_h\mathcal{L}(y_h)\Big)\mathrm{d}x+\int_\Gamma\Big(\eta[u_h]\cdot[y_h]+\xi[\nabla u_h][\nabla y_h]\Big)\mathrm{ds},$$

$$B_h(y_h)=\int_\Omega\lambda_h u_h y_h\mathrm{d}x.$$

3 Improve the stability of operators, continuity and ellipticity of equations

First, define the $DG-$energy norm associated with $A_h(\cdot,\cdot)$, $\forall\, v\in S=S_h^r+H_0^2(\Omega)$, we have

$$||v||_G=(||\Delta_h v||_\Omega^2+||\sqrt{\eta}[v]||_\Gamma^2+||\sqrt{\xi}[\nabla v]||_\Gamma^2)^{\frac{1}{2}}. \tag{3.1}$$

then the h-norm defined on the piecewise function space $H^{3+s}(T)(s>\frac{1}{2})$ is given by:

$$\|v\|_h^2=\|v\|_G^2+h_g\|\{\Delta v\}\|_\Gamma^2+h_g^3\|\{\nabla\Delta v\}\|_\Gamma^2\,. \tag{3.2}$$

Note that on the discontinuous finite element space S_h^r, $\|\cdot\|_G$ and $\|\cdot\|_h$ are equivalent.
Lemma 3.1(see [20] Lemma 5.1) Let $\mathcal{L}$ be the trace lifting defined in (2.5). Then, for $\forall\, v\in S$, the following bound holds:

$$\|\mathcal{L}(v)\|_\Omega^2\leqslant C(\|\sqrt{\eta}[v]\|_\Gamma^2+\|\sqrt{\xi}[\nabla v]\|_\Gamma^2). \tag{3.3}$$

Lemma 3.2 (see [20] Lemma 5.2) Then the bilinear form $A_h(\cdot,\cdot)$ is continuous and coercive in the sense that

$$|A_h(v,y)|\lesssim||v||_G||y||_G\ \ \forall v,y\in S_h^r, \tag{3.4}$$

$$|A_h(v,v)|\gtrsim||v||_G^2\qquad\forall v\in S_h^r. \tag{3.5}$$

4 A Priori Error Estimates for the Eigenvalue Problem

First, we present the source problem associated with equation (2.3) as follows: find $w\in H_0^2(\Omega)$, such that

$$A(w,y)=(f,y),\forall\, y\in H_0^2(\Omega). \tag{4.1}$$

Then, we provide the finite element approximation problem for equation (2.3) as follows: find $(\lambda_h,u_h)\in C\times S_h^r$, such that $u_h\neq 0$ and

$$A_h(u_h,y_h)=\lambda_h(u_h,y_h),\forall\, y_h\in S_h^r. \tag{4.2}$$

The DG approximation of (4.1) is to find $w_h\in S_h^r$, such that

$$A_h(w_h,y_h)=(f,y_h),\forall\, y_h\in S_h^r. \tag{4.3}$$

Definition of a bounded linear operator $\Lambda: L^2(\Omega)\to H_0^2(\Omega)$, such that:

$$A(\Lambda f,y)=(f,y),\ \forall\, f\in L^2(\Omega),\ y\in H_0^2(\Omega). \tag{4.4}$$

Therefore, equation (2.3) can be equivalently expressed in operator form as:

$$\lambda \Lambda u = u, \quad \forall\, u \in H_0^2(\Omega). \tag{4.5}$$

The corresponding finite element approximation problem also has an equivalent form. Firstly, let's define the discrete operator $\Lambda_h : L^2(\Omega) \to S_h^r(\Omega)$, Then, equation (4.2) can be written in the form of an equivalent operator as follows:

$$\lambda_h \Lambda_h u_h = u_h, \quad \forall\, u_h \in S_h^r. \tag{4.6}$$

By the consistency of the $DGFEMs$, we can obtain: Let w be the solution of (4.1), and $f \in L^2(\Omega)$ then:

$$A_h(w, y_h) = (f, y_h), \forall\, y_h \in S_h^r, \tag{4.7}$$

which together with (4.3) and (4.7) we obtain the error formulation:

$$A_h(w - w_h, y_h) = 0, \forall\, y_h \in S_h^r. \tag{4.8}$$

If $f \in L^2(\Omega)$, then the solution $w \in H^{3+t}(\frac{1}{2} > t > 0)$ of (4.1), and the following regularity estimate holds [21]:

$$\|w\|_{3+t} \lesssim \|f\|_{0,\Omega}. \tag{4.9}$$

When w_I represents the quadratic element interpolation of w, and the following interpolation estimate holds:

$$\|w - w_I\|_h \lesssim h^t \|w\|_{3+t}. \tag{4.10}$$

Attention: $[w - w_I] = 0$.

Lemma 4.1(see [22] Lemma 2.1) Let $\kappa \in T$, $g \in \partial\kappa$, and $0 < \epsilon < \frac{1}{2}$, for any $\Theta \in H^{1+\epsilon}(\kappa)$ with $\Delta\Theta \in L^2(\kappa)$, there exists a positive constant C independent of Θ such that:

$$\|\nabla\Theta \cdot \mathbf{n}\|_{\epsilon-\frac{1}{2},g} \leq C\left(\|\nabla\Theta\|_{\epsilon,\kappa} + h_\kappa^{1-\epsilon}\|\Delta\Theta\|_{0,\kappa}\right). \tag{4.11}$$

Theorem 4.1 Let $w \in H^{3+t}$ and $w_h \in S_h^r$ be the solution of (4.1) and (4.3), respectively. Then, the following relations hold:

$$\|w - w_h\|_h \lesssim \inf_{y_h \in S_h^r} \|w - y_h\|_h, \tag{4.12}$$

$$\|w - w_h\|_G \lesssim h^t \|f\|_{0,\Omega}. \tag{4.13}$$

Proof By utilizing (3.4), (3.5) and (4.8), we can obtain

$$\begin{aligned}\|w_h - y_h\|_G^2 &\lesssim |A_h(w_h - y_h, w_h - y_h)| \lesssim A_h(w_h - w, w_h - y_h) + A_h(w - y_h, w_h - y_h)\\ &\lesssim \|w - y_h\|_h \|w_h - y_h\|_G,\end{aligned}$$

then $\|w_h - y_h\|_G \lesssim \|w - y_h\|_h$.

Further applying the triangle inequality, we obtain:

$$\|w - w_h\|_h \lesssim \|w - y_h\|_h + \|w_h - y_h\|_G \lesssim \|w - y_h\|_h. \tag{4.14}$$

Therefore, (4.12) is proven.

To prove (4.13), we can utilize (2.13), (3.4), (3.5), (4.8) and (4.11), we can obtain

$$\begin{aligned}\|w_h - w_I\|_G^2 &\lesssim |A_h(w - w_I, w_h - w_I)| \\ &\lesssim \|w - w_I\|_G \|w_h - w_I\|_G + \sum_g \|\{\Delta(w - w_I)\}\|_{0,g} \|[\nabla(w_h - w_I)]\|_{0,g} \\ &+ \sum_g \|\{\nabla\Delta(w - w_I)\} \cdot \mathbf{n}\|_{t-\frac{1}{2},g} \|(w_h - w_I)_{\kappa^+} - (w_h - w_I)_{\kappa^-}\|_{\frac{1}{2}-t,g}.\end{aligned} \tag{4.15}$$

We can bound the third term in equation (4.15) as follows:

$$\begin{aligned}&\sum_g \|\{\nabla\Delta(w - w_I)\} \cdot \mathbf{n}\|_{t-\frac{1}{2},g} \|(w_h - w_I)_{\kappa^+} - (w_h - w_I)_{\kappa^-}\|_{\frac{1}{2}-t,g} \\ &\lesssim \sum_\kappa \left(\|\nabla\Delta(w - w_I)\|_{t,\kappa^+\cup\kappa^-} + h_\kappa^{1-t}\|\Delta^2(w - w_I)\|_{0,\kappa^+\cup\kappa^-}\right) \left(h^{t-\frac{1}{2}}\|(w_h - w_I)_{\kappa^+} - (w_h - w_I)_{\kappa^-}\|_{0,\partial\kappa}\right) \\ &\lesssim \left((\sum_\kappa \|w\|_{3+t,\kappa^+\cup\kappa^-}^2)^{\frac{1}{2}} + (\sum_k h_\kappa^{2-2t}\|f\|_{0,\kappa^+\cup\kappa^-}^2)^{\frac{1}{2}}\right) h^{t+1}\|w_h - w_I\|_G \\ &\lesssim h^{t+1}\|f\|_{0,\Omega}\|w_h - w_I\|_G.\end{aligned} \tag{4.16}$$

By applying the properties of Cauchy-Schwarz inequality, we estimate this bound for the second term in equation (4.15):

$$\begin{aligned}&\sum_g \|\{\Delta(w - w_I)\}\|_{0,g} \|[\nabla(w_h - w_I)]\|_{0,g} \lesssim (\sum_g \|h_g^{-\frac{1}{2}}[\nabla(w_h - w_I)]\|_{0,g}^2)^{\frac{1}{2}} (\sum_g h_g \|\{\Delta(w - w_I)\}\|_{0,g}^2)^{\frac{1}{2}} \\ &\lesssim h^t \|w_h - w_I\|_G \|w\|_{3+t} \lesssim h^t \|w_h - w_I\|_G \|f\|_{0,\Omega},\end{aligned} \tag{4.17}$$

we obtain:

$$\|w_h - w_I\|_G^2 \lesssim (h^t + h^{1+t})\|f\|_{0,\Omega}\|w_h - w_I\|_G. \tag{4.18}$$

After simplification, we obtain $\|w_h - w_I\|_G \lesssim h^t\|f\|_{0,\Omega}$, further utilizing the triangle inequality, (4.9), and (4.10) yields:

$$\|w - w_h\|_G \lesssim \|w - w_I\|_G + \|w_h - w_I\|_G \lesssim h^t\|f\|_{0,\Omega}. \tag{4.19}$$

Theorem 4.2 Let w and w_h be the solutions of (4.1) and (4.3), respectively. Then, the following relations hold:

$$\|w - w_h\|_{0,\Omega} \lesssim h^t\|w - w_h\|_G, \tag{4.20}$$

$$\|w - w_h\|_{0,\Omega} \lesssim h^{2t}\|f\|_{0,\Omega}. \tag{4.21}$$

Proof By utilizing the symmetry and (4.8), we can derive:

$$\begin{aligned}(w - w_h, f) &= A_h(w - w_h, w) = A_h(w - w_h, w - w_I) + A_h(w - w_h, w_I) \\ &\lesssim \|w - w_h\|_G\|w - w_I\|_G + \sum_g \|\{\Delta(w - w_I)\}\|_{0,g}\|[\nabla(w - w_h)]\|_{0,g} \\ &+ \sum_g \|\{\nabla\Delta(w - w_I)\} \cdot \mathbf{n}\|_{t-\frac{1}{2},g}\|(w - w_h)_{\kappa^+} - (w - w_h)_{\kappa^-}\|_{\frac{1}{2}-t,g}.\end{aligned} \tag{4.22}$$

Similar to the derivation of (4.13), we can obtain for (4.22):

$$|\int_F \{\nabla\Delta(w-w_I)\}[w-w_h]-\{\Delta(w-w_I)\}[\nabla(w-w_h)]\mathrm{d}s| \lesssim h^t\|w-w_h\|_G\|f\|_{0,\Omega}.$$

Therefore, (4.20) is proven.

Substituting (4.13) into (4.20), we obtain:

$$\|w-w_h\|_{0,\Omega} \lesssim h^{2t}\|f\|_{0,\Omega}.$$

Assume λ is the ith eigenvalue of (2.3) with the algebraic multiplicity q and the ascent α, $\lambda_i=\lambda_{i+1}=\cdots=\lambda_{i+q-1}$. When $\|\Lambda_h-\Lambda\|_{0,\Omega}\to 0$, q eigenvalues $\lambda_{i,h}=\lambda_{i+1,h}=\cdots=\lambda_{i+q-1}$ will converge to λ (as stated in Lemma 5 of reference [23]). Let $M(\lambda)$ denote the space of generalized eigenvectors of equation (2.3) associated with the eigenvalue λ, and let $M_h(\lambda)$ be the direct sum of the generalized eigenspace of equation (2.14) associated with the eigenvalue λ_h that converges to λ. And the arithmetic mean of λ can be expressed as: $\bar{\lambda}_h=\frac{1}{q}\sum_{i=j}^{j+q-1}\lambda_{i,h}$.

Theorem 4.3 Assume $M(\lambda)\subset H^{3+m}(m\geqslant t)$, $\nu=min\{m+1,r-1\}$ the following inequalities hold:

$$|\lambda-\lambda_h|\lesssim h^{2\nu}, \tag{4.23}$$

$$\|u-u_h\|_{0,\Omega}\lesssim h^{t+\nu}, \tag{4.24}$$

$$\|u-u_h\|_h\lesssim h^{t+\nu}+h^{\nu}, \tag{4.25}$$

$$\|u-u_h\|_{0,\Omega}\lesssim h^t\|u-u_h\|_h, \tag{4.26}$$

where $u\in M(\lambda)$ and $u_h\in M_h(\lambda)$ are the eigenfunction of (2.3) and (4.2) respectively.

Proof Let's denote $\Lambda f:=w$ and $\Lambda_h f:=w_h$, Combining the operator form and (4.21), we can obtain:

$$\|\Lambda-\Lambda_h\|_{0,\Omega}=\sup_{0\neq f\in L^2(\Omega)}\frac{\|\Lambda f-\Lambda_h f\|_{0,\Omega}}{\|f\|_{0,\Omega}}\lesssim\sup_{0\neq f\in L^2(\Omega)}\frac{h^{2t}\|f\|_{0,\Omega}}{\|f\|_{0,\Omega}}\lesssim h^{2t}\to 0,(h\to 0,).$$

From Theorems 7.1, 7.2, 7.3, and 7.4 in reference [17], we have

$$|\lambda-\lambda_h|\lesssim\sum_{i,l=j}^{j+q-1}|\langle(\Lambda-\Lambda_h)\varphi_i,\varphi_l\rangle|+\left\|(\Lambda-\Lambda_h)|_{M(\lambda)}\right\|_{0,\Omega}^2, \tag{4.27}$$

$$\|u-u_h\|_{0,\Omega}\lesssim\left\|(\Lambda-\Lambda_h)|_{M(\lambda)}\right\|_{0,\Omega}, \tag{4.28}$$

here $\{\varphi_i\}_{i=j}^{j+q-1}$ is any basis for $M(\lambda)$.

From Theorem 4.1 and Theorem 4.2, we can infer that:

$$\left\|(\Lambda-\Lambda_h)|_{M(\lambda)}\right\|_{0,\Omega}=\sup_{f\in M(\lambda),\|f\|_{0,\Omega}=1}\|\Lambda f-\Lambda_h f\|_{0,\Omega}\lesssim\sup_{f\in M(\lambda),\|f\|_{0,\Omega}=1}h^{t+\nu}\|\Lambda f\|_{2+\nu,\Omega}. \tag{4.29}$$

Substituting (4.29) into (4.28), we can obtain (4.24).

By utilizing the properties of operators, (4.8) and (3.5), we can obtain:

$$\begin{aligned}((\Lambda-\Lambda_h)\varphi_i,\varphi_l)&=A_h(\Lambda\varphi_i-\Lambda_h\varphi_i,\Lambda\varphi_l)=A_h(\Lambda\varphi_i-\Lambda_h\varphi_i,\Lambda\varphi_l-(\Lambda\varphi_l)_I)\\&\lesssim\|\Lambda\varphi_i-\Lambda_h\varphi_i\|_h\|\Lambda\varphi_l-(\Lambda\varphi_l)_I\|_h\lesssim h^{\nu}\|\Lambda\varphi_i\|_{3+\nu,\Omega}h^{\nu}\|\Lambda\varphi_l\|_{3+\nu,\Omega}\lesssim h^{2\nu}.\end{aligned} \tag{4.30}$$

By substituting (4.29) and (4.30) into (4.27), we can derive (4.23).

Since $u=\lambda\Lambda u$ and $u_h=\lambda_h\Lambda_h u_h$, by using the triangle inequality, we can derive (4.23) and (4.24) as follows:

$$\left|\|u-u_h\|_h-\|u-\lambda\Lambda_h u\|_h\right|\lesssim\|u_h-\lambda\Lambda_h u\|_h=\|\Lambda_h(\lambda_h u_h-\lambda u)\|_h\lesssim\|\lambda_h u_h-\lambda u\|_{0,\Omega}\lesssim h^{t+\nu}, \tag{4.31}$$

which together with (4.12) yields (4.25).

From (4.12) and (4.20), we obtain:

$$\|u-u_h\|_{0,\Omega} \lesssim \left\|(\Lambda u-\Lambda_h u)|_{M(\lambda)}\right\|_{0,\Omega} \lesssim h^t \|\Lambda u-\Lambda_h u\|_G \lesssim h^t \inf_{v_h\in V_h} \|\Lambda u - v_h\|_h \lesssim h^t \|u-u_h\|_h .$$

Therefore, by obtaining (4.26), the proof is complete.

5 A Posteriori Error Estimates

5.1 Reliability

Lemma 5.1 Let's assume that the mesh T is constructed as in Section 2.1. In the case, there exists an operator $\Pi : S_h^r \to \widetilde{S}_h^{r+2} \bigcap H_0^2(\Omega)$ that satisfies the following bound:

$$\sum_{\kappa\in T} |u_h - \Pi(u_h)|^2_{\alpha,\kappa} \leqslant C(||\mathbf{h}^{\frac{3}{2}-\alpha}[\nabla u_h]||^2_{\Gamma} + C||\mathbf{h}^{\frac{1}{2}-\alpha}[u_h]||^2_{\Gamma}), \tag{5.1}$$

with $=0,1,2$ and $C>0$ is a constant that is independent of $\mathbf{h}$ and u_h.

Proof See Lemma 3.1 in [24].

Theorem 5.1 Let $u \in H_0^2(\Omega)$ be a solution to equations (2.1) and (2.2). Let u_h be the approximate solution obtained using the Discontinuous Galerkin method. Let η and ξ be as defined earlier. Then, there exists a positive constant C that is independent of $\mathbf{h}, u$ and u_h, such that:

$$\begin{aligned} ||u-u_h||^2_G \leqslant & C\Big(||\lambda u - \lambda_h u_h||^2_\Omega + ||\mathbf{h}^2(\lambda_h u_h - \Delta_h^2 u_h)||^2_\Omega + ||\mathbf{h}^{1/2}[\Delta u_h]||^2_{\Gamma_i} \\ & + ||\mathbf{h}^{3/2}[\nabla\Delta u_h]||^2_{\Gamma_i} + C_q(||\mathbf{h}^{-3/2}[u_h]||^2_\Gamma + ||\mathbf{h}^{-1/2}[\nabla u_h]||^2_\Gamma)\Big), \end{aligned} \tag{5.2}$$

where $C_q = max\{1, C_\eta, C_\xi, C_\eta^2, C_\xi^2\}$.

Proof Before the formal proof, we first construct the macro element $\widetilde{S}_h^{r+2}$, and the specific construction method can be found in reference [24]. For each node y of the C^1 conforming finite element space $\widetilde{S}_h^{r+2}$, we define ω_y as the set of elements $\kappa \in T$ that share the node y, i.e., $\omega_y = \{\kappa \in T : y \in \kappa\}$. The cardinality of ω_y is denoted by $|\omega_y|$. It is worth noting that if y lies in the interior of an element, then $|\omega_y| = 1$.

We define the operator $\Pi : S_h^r \to \widetilde{S}_h^{r+2} \bigcap H_0^2(\Omega)$ by

$$G_y(\Pi(u_h)) = \begin{cases} \frac{1}{|\omega_y|}\sum_{\kappa\in\omega_y} G_y(u_h|_\kappa) & \text{if} y \notin \Gamma_{int}, \\ 0 & \text{if} y \in \Gamma_{int}, \end{cases}$$

here, G_y represents any nodal variable at y, which is a nodal point of $\widetilde{S}_h^{r+2}$. It is important to note that if y lies in the interior of an element, then $G_y(\Pi(u_h)) = G_y(u_h)$.

Let $y_h \in S_h^r, y \in H_0^2(\Omega)$ and $\Phi(u_h) = y - y_h$. Thus, the error can be decomposed as:

$$R = u - u_h = (u - \Pi(u_h)) + (\Pi(u_h) - u_h) \equiv R_1 + R_2,$$

thus:

$$A_h(R, y) = A_h(u, y) - A_h(u_h, y) = (\lambda u - \lambda_h u_h, y) + B_h(\Phi) - A_h(u_h, \Phi), \tag{5.3}$$

and therefore:

$$A_h(R_1, y) = (\lambda u - \lambda_h u_h, y) + B_h(\Phi) - A_h(u_h, \Phi) - A_h(R_2, y), \tag{5.4}$$

since $\Pi(u_h) \in \widetilde{S}_h^{r+2}$ and $u \in H_0^2(\Omega)$, we have $R_1 \in H_0^2(\Omega)$.

Therefore,we can deduce that $\pounds(R_1)=0$, and upon setting $y=R_1$ in (5.3), we can also deduce that:

$$\|\Delta R_1\|^2=A_h(R_1,R_1)=(\lambda u-\lambda_h u_h,R_1)+B_h(\Phi)-A_h(u_h,\Phi)-A_h(R_2,R_1), \quad (5.5)$$

First, let's estimate the four terms on the right-hand side of (5.5). Since $\pounds(R_1)=[R_1]=[\nabla R_1]=0$ and using (3.4), we can obtain:

$$\begin{aligned}|A_h(R_2,R_1)|&=\left|\int_\Omega(\Delta_h R_2\Delta_h R_1+\pounds(R_2)\Delta_h R_1)\mathrm{d}x\right|\\&\leqslant\Big(||\Delta_h R_2||_\Omega^2+C(||\sqrt{\eta}[u_h]||_T^2+||\sqrt{\xi}[\nabla u_h]||_T^2)\Big)^{1/2}||\Delta_h R_1||_\Omega.\end{aligned}\quad (5.6)$$

Next, let's estimate the second and third terms on the right-hand side of (5.5). By applying the Green's formula, we obtain:

$$\begin{aligned}B_h(\Phi)-A_h(u_h,\Phi)&=\int_\Omega(\lambda_h u_h-\Delta_h^2 u_h)\Phi\mathrm{d}x-\int_\Omega(\pounds(\Phi)\Delta_h u_h+\pounds(u_h)\Delta_h\Phi)\mathrm{d}x\\&-\sum_{\kappa\in\mathcal{T}}\int_{\partial\kappa}(\Delta u_h\mathbf{n}\cdot\nabla\Phi-\nabla\Delta u_h\cdot\mathbf{n}\Phi)\mathrm{d}s-\int_{\mathcal{F}}(\Phi[u_h]\cdot[\Phi]+\xi[\nabla u_h][\nabla\Phi])\mathrm{d}s,\end{aligned}\quad (5.7)$$

since $u_h,y_h\in S_h^r$ and $y\in H_0^2$, we can use the lifting operator and the identity (2.4) to transform (5.7) into:

$$\begin{aligned}B_h(\Phi)-A_h(u_h,\Phi)&=\int_\Omega((\lambda_h u_h-\Delta^2 u_h)\Phi-\pounds(u_h)\Delta_h\Phi)\mathrm{d}x+\int_{\mathcal{F}_i}\{\Phi\}[\nabla\Delta u_h]\mathrm{d}s\\&-\int_{\mathcal{F}_i}\{\nabla\Phi\}\cdot[\Delta u_h]\mathrm{d}s-\int_{\mathcal{F}}(\eta[u_h]\cdot[\Phi]+\xi[\nabla u_h][\nabla\Phi])\mathrm{d}s.\end{aligned}\quad (5.8)$$

According to [25], for $0\leqslant j\leqslant m\leqslant 2$, there exists a constant $C\geq 0$ independent of T, such that for any $u\in H_0^2(\Omega)$ and $\kappa\in T$, there exists $y_h\in S_h^r$ satisfying:

$$|u-y_h|_{j,\kappa}\leq ch_\kappa^{m-j}|u|_{m,\kappa},0\leqslant j\leqslant m\leqslant 2. \quad (5.9)$$

For the first term on the right-hand side of (5.8), using (5.9) and (3.3), we can obtain:

$$\begin{aligned}\left|\int_\Omega((\lambda_h u_h-\Delta_h^2 u_h)\Phi-\pounds(u_h)\Delta_h\Phi)\mathrm{d}x\right|&\leqslant C\Big(||\mathbf{h}^2(\lambda_h u_h-\Delta_h^2 u_h)||_\Omega^2\\&+||\sqrt{\eta}[u_h]||_{\mathcal{F}}^2+||\sqrt{\xi}[\nabla u_h]||_{\mathcal{F}}^2\Big)^{1/2}|R_1|_{2,\Omega},\end{aligned}\quad (5.10)$$

then, using the shape regularity, mesh regularity, and bounded local variation on the finite element space S_h^r, we can bound the second, third, and fourth terms on the right-hand side of (5.8) as follows:

$$\left|\int_{\mathcal{F}_i}\{\Phi\}[\nabla\Delta u_h]\mathrm{d}s\right|\leqslant C||\mathbf{h}^{3/2}[\nabla\Delta u_h]||_{\mathcal{F}_i}|R_1|_{2,\Omega},\quad\left|\int_{\mathcal{F}_i}\{\nabla\Phi\}\cdot[\Delta u_h]\mathrm{d}s\right|\leqslant C||\mathbf{h}^{1/2}[\Delta u_h]||_{\mathcal{F}_i}|R_1|_{2,\Omega},$$

$$|\int_{\mathcal{F}}(\eta[u_h]\cdot[\Phi]+\xi[\nabla u_h][\nabla\Phi])\mathrm{d}s|\leqslant CC_q^{1/2}(||\mathbf{h}^{-3/2}[u_h]||_{\mathcal{F}}^2+||\mathbf{h}^{-1/2}[\nabla u_h]||_{\mathcal{F}}^2)^{1/2}|R_1|_{2,\Omega}.$$

The first term in (5.4) can be estimated as: $|(\lambda u-\lambda_h u_h,R_1)|\leq C||\lambda u-\lambda_h u_h||_\Omega|R_1|_{2,\Omega}$. Since for $R_1\in H^2(\Omega)$, we have $|R_1|_{2,\Omega}\leq C\|\Delta R_1\|_\Omega$, we can obtain:

$$\begin{aligned}||\Delta R_1||_\Omega^2\leqslant&C\Big(||\lambda u-\lambda_h u_h||_\Omega^2+||\mathbf{h}^2(\lambda_h u_h-\Delta_h^2 u_h)||_\Omega^2+||\mathbf{h}^{1/2}[\Delta u_h]||_{\mathcal{F}}^2\\&+||\mathbf{h}^{3/2}[\nabla\Delta u_h]||_{\mathcal{F}}^2+C_q(||\mathbf{h}^{-3/2}[u_h]||_{\mathcal{F}}^2+||\mathbf{h}^{-1/2}[\nabla u_h]||_{\mathcal{F}}^2)\Big).\end{aligned}\quad (5.11)$$

This provides an estimate for $||\Delta R_1||_\Omega$, Next, we will estimate $||\Delta R_2||_\Omega$.

From lemma 5.1, we can obtain:

$$||\Delta R_2||_\Omega^2 \leqslant \sum_{\kappa\in T} |u_h - \Pi(u_h)|_{2,\kappa}^2 \leqslant C(||\mathbf{h}^{-\frac{3}{2}}[u_h]||_\Gamma^2 + ||\mathbf{h}^{-\frac{1}{2}}[\nabla u_h]||_\Gamma^2). \tag{5.12}$$

Therefore, by using (5.11) and (5.12), as well as the triangle inequality, we can obtain:

$$\begin{aligned}\|u-u_h\|_G^2 \leqslant & C\Big(||\lambda u - \lambda_h u_h||_\Omega^2 + ||\mathbf{h}^2(\lambda_h u_h - \Delta_h^2 u_h)||_\Omega^2 + ||\mathbf{h}^{1/2}[\Delta u_h]||_{\Gamma_i}^2 \\ & + ||\mathbf{h}^{3/2}[\nabla\Delta u_h]||_{\Gamma_i}^2 + C_q(||\mathbf{h}^{-3/2}[u_h]||_\Gamma^2 + ||\mathbf{h}^{-1/2}[\nabla u_h]||_\Gamma^2)\Big).\end{aligned}$$

5.2 Effectiveness

Theorem 5.2 Under the basis of Theorem 5.1, for every element $\kappa \in T$, the following inequality holds true:

$$\|\mathbf{h}^2(\lambda_h u_h - \Delta^2 u_h)\|_\kappa^2 \lesssim \|\Delta(u-u_h)\|_\kappa^2 + \|\mathbf{h}^2(\lambda u - \lambda_h u_h)\|_\kappa^2, \tag{5.13}$$

and for each edge $g \in \Gamma_i$ we have:

$$\|\mathbf{h}^{\frac{1}{2}}[\Delta u_h]\|_g^2 \lesssim \|\Delta_h(u-u_h)\|_{\kappa_1\cup\kappa_2}^2 + \|\mathbf{h}^2(\lambda u - \lambda_h u_h)\|_{\kappa_1\cup\kappa_2}^2, \tag{5.14}$$

$$\|\mathbf{h}^{\frac{3}{2}}[\nabla\Delta u_h]\|_g^2 \lesssim \|\Delta_h(u-u_h)\|_{\kappa_1\cup\kappa_2}^2 + \|\mathbf{h}^2(\lambda u - \lambda_h u_h)\|_{\kappa_1\cup\kappa_2}^2. \tag{5.15}$$

Proof First, determine $\kappa \in T$, then define a polynomial function $y|_\kappa = (\lambda_h u_h - \Delta^2 u_h)b_\kappa^2$ on κ, with $y \in H_0^2(\Omega) \cap H_0^2(\kappa)$, and $y = 0$ when $y \in \Omega \setminus \kappa$. Where $b_\kappa : \kappa \to R$ is a standard bubble function defined as $b_\kappa = b_{\widetilde{\kappa}} \circ F_\kappa$. If κ is a reference triangular element with barycentric coordinates $\lambda_1, \lambda_2, \lambda_3$, then $b_\kappa = 27\lambda_1\lambda_2\lambda_3$.

Take the above y in (5.3) and set $y_h = 0$ to get the result:

$$\int_\kappa \Delta R \Delta y \mathrm{d}x = \int_\kappa (\lambda u - \lambda_h u_h) y \mathrm{d}x + \int_\kappa (\lambda_h u_h - \Delta^2 u_h) y \mathrm{d}s = \int_\kappa (\lambda u - \Delta^2 u_h) y \mathrm{d}s, \tag{5.16}$$

noting that $[y] = [\nabla y] = \{y\} = \{\nabla y\} = 0$ on Γ and that $\mathcal{L}(u) = \mathcal{L}(y) = 0$ on Ω.

Then using the reverse inequality, we can obtain:

$$\|\lambda u - \Delta^2 u_h\|_\kappa \|y\|_\kappa = \|\Delta R\|_\kappa \|\Delta y\|_\kappa \lesssim \mathbf{h}^{-2}\|\Delta R\|_\kappa \|y\|_\kappa, \tag{5.17}$$

and, from the norm equivalence, (5.17) and scaling argument, we obtain:

$$\begin{aligned}&\|\lambda_h u_h - \Delta^2 u_h\|_\kappa^2 \lesssim \int_\kappa (\lambda_h u_h - \Delta^2 u_h)^2 b_\kappa^2 \mathrm{d}x \lesssim \int_\kappa (\lambda_h u_h - \Delta^2 u_h) y \mathrm{d}x \\ &\lesssim \int_\kappa (\lambda_h u_h - \lambda u + \lambda u - \Delta^2 u_h) y \mathrm{d}x \lesssim \mathbf{h}^{-2}\|\Delta(u-u_h)\|_\kappa \|y\|_\kappa + \|\lambda_h u_h - \lambda u\|_\kappa \|y\|_\kappa.\end{aligned} \tag{5.18}$$

For (5.18), using norm scaling and the $Cauchy - Schwarz$ inequality, we can obtain:

$$\|\lambda_h u_h - \Delta^2 u_h\|_\kappa^2 \lesssim \mathbf{h}^{-4}\|\Delta(u-u_h)\|_\kappa^2 + \|\lambda_h u_h - \lambda u\|_\kappa^2, \tag{5.19}$$

multiplying both sides of (5.18) by $\mathbf{h}^4$, then (5.13) obtained.

Next, let's do some preparation work.

Firstly, let the largest rhombus in $\kappa_1 \cup \kappa_2$ be denoted as $\widetilde{\kappa} \subset \kappa_1 \cup \kappa_2$. The diagonal of the rhombus $\widetilde{\kappa}$ is denoted as $g \in \Gamma_i$.

Furthermore, we define a bubble function on the rhombus $\widetilde{\kappa}$, denoted as $b_{\widetilde{\kappa}} : \widetilde{\kappa} \to R$.

Finally, we define an affine function $b_l : \widetilde{\kappa} \to R$, which takes the value of 0 along the edge g and such that $(\nabla b_l \cdot \mathbf{n}|_e = \mathbf{h}^{-2})$.

Using the above preparation work, we consider the function $b_g : \Omega \to R$, with $b_g|_{\widetilde{\kappa}} = b_l b_{\widetilde{\kappa}}^3$ and $b_g = 0$ on $\Omega \setminus \widetilde{\kappa}$, that has the following properties:

$$b_g \in C^2(\Omega) \cap H_0^2(\Omega), \quad [b_g] = [\nabla b_g] = \{b_g\} = 0 \text{ on } \Gamma,$$
$$(\{\nabla b_g\} \cdot \mathbf{n})|_g = (\mathbf{h}^{-1} b_{\widetilde{\kappa}}^3)|_g, \quad \{\nabla b_g\} = 0 \text{ on } \Gamma \backslash g,$$

where $\nabla b_{\widetilde{\kappa}} \cdot \mathbf{n}$ takes the value of 0 along the edge g.

Building upon the aforementioned foundation. Firstly, we define $y = \chi b_g$, where χ is a constant function in the normal direction to $g(i.e., \nabla \chi \cdot \mathbf{n}|_g = 0)$. For this y and for $y_h = 0$ to (5.3), we can obtain:

$$\int_\kappa (\lambda u - \lambda_h u_h)\mathrm{d}x + \int_{\kappa_1 \cup \kappa_2} (\lambda_h u_h - \Delta_h^2 u_h) y \mathrm{d}x - \int_{\kappa_1 \cup \kappa_2} \Delta_h R \Delta_h y \mathrm{d}x = \int_g [\Delta u_h] \cdot \{\nabla y\} \mathrm{d}s, \tag{5.20}$$

in (5.20), let $\chi|_g = (\mathbf{h}^{-1}[\Delta u_h] \cdot \mathbf{n})|_g$, and then use norm equivalence and scaling to obtain:

$$\int_g [\Delta u_h] \cdot \{\nabla y\} \mathrm{d}s = b_{\widetilde{\kappa}}^3 ||\mathbf{h}^{-1}[\Delta u_h]||_g^2 \geqslant C||\mathbf{h}^{-1}[\Delta u_h]||_g^2. \tag{5.21}$$

Furthermore, let $l : g \to R$, where $l(s)$ denotes the length ofthe intersection ofthe line normal to g, crossing g at the point $s \in g$, and $\widetilde{\kappa}$. We can obtain:

$$\|y\|_{\kappa_1 \cup \kappa_2} \leqslant C||\chi||_{\kappa_1 \cup \kappa_2} = C\left(\int_g \chi^2(s) l(s) \mathrm{d}s\right)^{\frac{1}{2}} < C\|\mathbf{h}^{\frac{1}{2}} \chi\|_g = C||\mathbf{h}^{-\frac{1}{2}}[\Delta u_h]||_g, \tag{5.22}$$

from (5.20) and (5.21), we can obtain:

$$||\mathbf{h}^{-1}[\Delta u_h]||_g^2 \leqslant (\|\mathbf{h}^{\frac{1}{2}}(\lambda u - \lambda_h u_h)\|_{\kappa_1 \cup \kappa_2} + ||\mathbf{h}^{\frac{1}{2}}(\lambda_h u_h - \Delta_h^2 u_h)||_{\kappa_1 \cup \kappa_2} + \mathbf{h}^{-\frac{3}{2}}||\Delta_h R||_{\kappa_1 \cup \kappa_2})||\mathbf{h}^{-\frac{1}{2}} y||_{\kappa_1 \cup \kappa_2}, \tag{5.23}$$

by utilizing (5.13) and (5.22) in (5.23) and then multiplying both sides of (5.23) by $\mathbf{h}^3$, the (5.14) is proven.

To estimate $[\nabla \Delta u_h]$ we first observe that we have

$$b_{\widetilde{\kappa}}^3 \in C^2(\Omega) \cap H_0^2(\Omega), \quad [b_{\widetilde{\kappa}}^3] = [\nabla b_{\widetilde{\kappa}}^3] = \{\nabla b_{\widetilde{\kappa}}^3\} \cdot \mathbf{n} = 0 \text{ on } \Gamma \quad \text{and} \quad [b_{\widetilde{\kappa}}^3] = 0 \text{ on } \Gamma \backslash g. \tag{5.24}$$

We set $y = \psi b_{\widetilde{\kappa}}^3$, where ψ is a constant function in the normal direction to g. For this y and for $y_u = 0$ (5.3) yields

$$\int_\kappa (\lambda u - \lambda_h u_h)\mathrm{d}x + \int_{\kappa_1 \cup \kappa_2} (\lambda_h u_h - \Delta_h^2 u_h) y \mathrm{d}x - \int_{\kappa_1 \cup \kappa_2} \Delta_h R \Delta_h y \mathrm{d}x = \int_g [\nabla \Delta u_h]\{y\} \mathrm{d}s, \tag{5.25}$$

in (5.25), let $\psi|_g = [\nabla \Delta u_h]|_g$. By following a similar proof as (5.14), we can obtain (5.15).

5.2 Reliability of eigenvalue estimation

Lemma 5.2 Let (λ, u) and (λ_h, u_h) be the eigenpairs of (2.3) and (4.2), respectively, with $(u_h, u_h) \neq 0$. In this case, we can deduce:

$$\lambda - \lambda_h = \lambda \frac{(u - u_h, u - u_h)}{(u_h, u_h)} - \frac{A_h(u - u_h, u - u_h)}{(u_h, u_h)}. \tag{5.26}$$

Theorem 5.3 Under the conditions of Lemma (5.1), let $M(\lambda) \subset H^{3+t}(\Omega)(\frac{1}{2} \geqslant t > 0)$, then

$$|\lambda - \lambda_h| \lesssim \aleph(u_h)^2 + \epsilon\aleph(u_h)^2 + \epsilon \sum_\kappa |\Delta(u - u_{I'})|_{0,\kappa}\aleph(u_h) + \epsilon \sum_\kappa h_\kappa^1 |\Delta(u - u_{I'})|_{1,\kappa}\aleph(u_h)$$
$$+ \epsilon \sum_\kappa \big((h^{-\frac{1}{2}} + h^{-\frac{3}{2}})\|f\|_{0,\Omega} + \aleph(u_h) + h_\kappa^{1-t}(\|\lambda u - \lambda_h u_h\|_{0,\kappa} + \|\lambda_h u_h - \Delta^2 u_h\|_{0,\kappa})\big)h^{t+1}\|u - u_h\|_G, \tag{5.27}$$

where $u_{I'}$ is the first-order interpolation of u,

$$\aleph(u_h)^2 = ||\mathbf{h}^2(\lambda_h u_h - \Delta_h^2 u_h)||_\Omega^2 + ||\mathbf{h}^{1/2}[\Delta u_h]||_{\digamma_i}^2 + ||\mathbf{h}^{3/2}[\nabla\Delta u_h]||_{\digamma_i}^2$$
$$+ C_q(||\mathbf{h}^{-3/2}[u_h]||_\digamma^2 + ||\mathbf{h}^{-1/2}[\nabla u_h]||_\digamma^2).$$

Proof According to Theorem 4.3, we know that $\|u - u_h\|_{0,\Omega}$ is a higher-order term compared to $\|u - u_h\|_G$. Then, by using (5.26) and the estimate for u_h in (5.2), we can derive:

$$\begin{aligned}
|\lambda - \lambda_h| &\lesssim \|u - u_h\|_G\|u - u_h\|_G \\
&+ |\sum_g \int_g \Big(\{\nabla\Delta(u - u_h)\} \cdot [u - u_h] + \{\nabla\Delta(u - u_h)\} \cdot [u - u_h]\Big)ds| \\
&+ |\sum_g \int_g \Big(\{\Delta(u - u_h)\} \cdot [\nabla(u - u_h)] + \{\Delta(u - u_h)\} \cdot [\nabla(u - u_h)]\Big)ds| \\
&\lesssim \aleph(u_h)^2 + 2\epsilon \sum_g \|\{\nabla\Delta(u - u_h)\} \cdot \mathbf{n}\|_{t-\frac{1}{2},g}\|(u - u_h)_{\kappa^+} - (u - u_h)_{\kappa^-}\|_{\frac{1}{2}-t,g} \\
&+ 2\epsilon \sum_g \|\{\Delta(u - u_h)\}\|_{0,g}\|[\nabla(u - u_h)]\|_{0,g}.
\end{aligned} \tag{5.28}$$

From the estimate in (4.11), along with the interpolation estimate and inverse estimate, we can deduce

$$\begin{aligned}
&\epsilon \sum_g \|\{\nabla\Delta(u - u_h)\} \cdot \mathbf{n}\|_{t-\frac{1}{2},g} \lesssim \epsilon \sum_\kappa \big(\|\nabla\Delta(u - u_h)\|_{t,\kappa^+\cup\kappa^-} + h_\kappa^{1-t}\|\Delta^2(u - u_h)\|_{0,\kappa^+\cup\kappa^-}\big) \\
&\lesssim \epsilon \sum_\kappa \big(\|\Delta(u - u_h)\|_{1+t,\kappa} + h_\kappa^{1-t}(\|\lambda u - \lambda_h u_h\|_{0,\kappa} + \|\lambda_h u_h - \Delta^2 u_h\|_{0,\kappa})\big) \\
&\lesssim \epsilon \sum_\kappa \big(\|u - u_{I'}\|_{3+t,\kappa} + h^{-t-1}(\|\Delta(u - u_{I'})\|_{0,\kappa} + \|\Delta(u - u_h)\|_{0,\kappa}) \\
&\quad + h_\kappa^{1-t}(\|\lambda u - \lambda_h u_h\|_{0,\kappa} + \|\lambda_h u_h - \Delta^2 u_h\|_{0,\kappa})\big) \\
&\lesssim \epsilon \sum_\kappa \big(h^{-\frac{1}{2}}\|f\|_{0,\Omega} + h^{-\frac{3}{2}}\|f\|_{0,\Omega} + \|u - u_h\|_G + h_\kappa^{1-t}(\|\lambda u - \lambda_h u_h\|_{0,\kappa} + \|\lambda_h u_h - \Delta^2 u_h\|_{0,\kappa})\big).
\end{aligned}$$

From the inverse estimates, the trace inequality and the interpolation estimate, we get

$$\begin{aligned}
&\sum_g \|(u - u_h)_{\kappa^+} - (u - u_h)_{\kappa^-}\|_{\frac{1}{2}-t,g} \lesssim \sum_g h^{t-\frac{1}{2}}\|(u - u_h)_{\kappa^+} - (u - u_h)_{\kappa^-}\|_{0,g} \\
&\lesssim h^{t+1}(\sum_g \|h^{-\frac{3}{2}}[u - u_h]\|_{0,g}^2)^{\frac{1}{2}} \lesssim h^{t+1}\|u - u_h\|_G,
\end{aligned}$$

then, combining the above two inequalities, we obtain

$$\begin{aligned}
&\epsilon \sum_g \|\{\nabla\Delta(u - u_h)\} \cdot \mathbf{n}\|_{t-\frac{1}{2},g}\|(u - u_h)_{\kappa^+} - (u - u_h)_{\kappa^-}\|_{\frac{1}{2}-t,g} \\
&\lesssim \epsilon \sum_\kappa \big((h^{-\frac{1}{2}} + h^{-\frac{3}{2}})\|f\|_{0,\Omega} + \aleph(u_h) + h_\kappa^{1-t}(\|\lambda u - \lambda_h u_h\|_{0,\kappa} + \|\lambda_h u_h - \Delta^2 u_h\|_{0,\kappa})\big)h^{t+1}\|u - u_h\|_G.
\end{aligned}$$

The second term in (5.28) can be estimated as follows

$$
\begin{aligned}
&\epsilon \sum_{g\in\Gamma} \|\{\Delta(u-u_h)\}\|_{0,g}\|[\nabla(u-u_h)]\|_{0,g} \\
&\lesssim \epsilon\big(h_\kappa^{-\frac{1}{2}}\|u-u_h\|_G + \sum_\kappa h_\kappa^{-\frac{1}{2}}|\Delta(u-u_{I'})|_{0,\kappa} + \sum_\kappa h_\kappa^{\frac{1}{2}}|\Delta(u-u_{I'})|_{1,\kappa}\big)h^{\frac{1}{2}}\aleph(u_h) \\
&\lesssim \epsilon\aleph(u_h)^2 + \epsilon\sum_\kappa |\Delta(u-u_{I'})|_{0,\kappa}\aleph(u_h) + \epsilon\sum_\kappa h_\kappa^1|\Delta(u-u_{I'})|_{1,\kappa}\aleph(u_h).
\end{aligned}
$$

Remark 2. *From Theorems 5.1 and 5.2, we understand that the estimator $\aleph(u_h)^2$ for the eigenfunction error $\|u-u_h\|_G^2$ is both reliable and efficient up to data oscillation. Therefore, the adaptive algorithm based on this estimator can generate a well-graded mesh, enabling the approximate eigenfunctions to achieve the optimal convergence rate of $O(dof^{-m})$ in $\|\cdot\|_G^2$. As a result, we can anticipate that $\epsilon\sum_\kappa\left(|u-u_{I'}|_{2,\kappa}+h^{-1}|u-u_{I'}|_{3,\kappa}\right)\lesssim dof^{-m}$, which leads to $\|\lambda-\lambda_h\|\leqslant dof^{-m}$ from (5.27). The numerical experiments in Sect. 6 demonstrate the reliability and efficiency of $\aleph(u_h)^2$ as the error estimator for λ_h.*

6 Numerical experiment

In this section, we will demonstrate the practical performance of the a posteriori error estimators proposed in the adaptive refinement process through numerical results. We use $MATLAB2022a$ to solve the discrete eigenvalue problem, and our program was implemented using the $iFEM$ package. When $\Omega\in R^2$, we solve (2.14) on the unit square $[0,1]^2$ and the L-shaped domain $[\frac{-1}{2},\frac{1}{2}]^2/(0,\frac{1}{2})\times(\frac{-1}{2},0)$. For ease of reference, let S represent the square region and L represent the L-shaped region mentioned in the table. In the upcoming example, we set the internal penalty parameters $\eta=\xi=70$, and set the constant C generated in Theorem 5.2 to be 1 for calculating the DG solution of equation (2.14). As the exact eigenvalues are not known, we consider reference eigenvalues $\lambda\approx 1294.9339795796$ in the square region S and $\lambda\approx 6702.97945136574$ in the L-shaped region L. These reference eigenvalues have been obtained through adaptive computation to ensure the highest level of accuracy.

Domain	*Die*	*dof*	λ_1	*Error*
	1	9216	7.010810385566216e+03	5.734186072308789e+03;
	2	10788	6.887887391633877e+03	3.963003904029943e+03;
	3	13272	6.852830595264825e+03	2.964057770679027e+03;
	4	17376	6.808612049380864e+03;	2.123049786633385e+03;
	5	24480	6.779988924160483e+03	1.541112814381431e+03;
Ω_L	6	35988	6.748579866377175e+03	1.055818370150666e+03;
	7	49992	6.738503615381788e+03	0.762981947681214e+03;
	8	68904	6.728894529748604e+03	0.544936518766507e+03;
	9	103308	6.721107342835468e+03	0.384637390535364e+03
	10	151932	6.714791825102930e+03	0.266363127506993e+03
	11	212796	6.712125540666304e+03	0.189232893908050e+03
	12	298524	6.709691518402161e+03	0.132823141315932e+03

Table 1: Numerical solution of eigenvalues using 12 iterations of p2 adaptive method with an initial mesh size of 1/16 on the domain Ω_L.

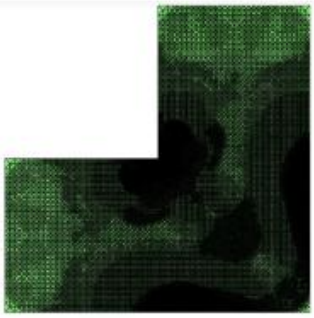

Figure 1: Iterating 12 times with p2 adaptive mesh refinement on the initial mesh size of 1/16 for the domain Ω_L.

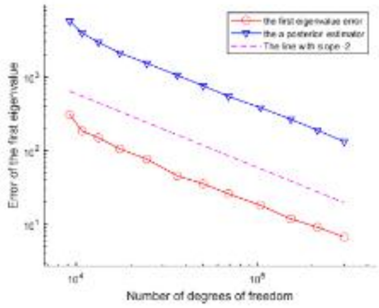

Figure 2: Error and indicator subcurve plot for 12 iterations of p2 adaptive eigenvalue computation on the initial mesh size of 1/16 for the domainΩ_L.

Domain	*l*	*dof*	λ_1	*Error*
	1	3072	1.449289880289645e+03	1.814671886456622e+03
	2	4416	1.330502520545450e+03	0.819472333018605e+03
	3	6240	1.310010557754699e+03	0.474600744323542e+03
	4	9840	1.305071417843494e+03	0.333887289987416e+03
	5	15384	1.300626257832301e+03	0.215319004620546e+03
	6	22500	1.299737112716709e+03	0.138692890520540e+03
Ω_S	7	36300	1.298051282728464e+03	0.092951340263445e+03
	8	59496	1.296561336690429e+03	0.058016006336130e+03
	9	87192	1.296142965517671e+03	0.036914388859048e+03
	10	140562	1.295725508290966e+03	0.024363914971054e+03
	11	230352	1.295362102788036e+03	0.015128348635248e+03
	12	338616	1.295239426283698e+03	0.009661505225656e+03

Table 2: Numerical solution of eigenvalues using 12 iterations of p2 adaptive method with an initial mesh size of 1/16 on the domain Ω_S.

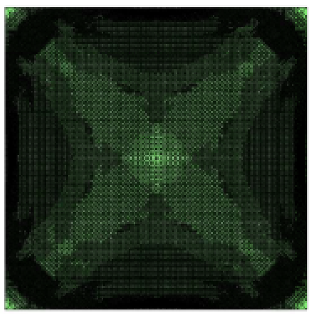

Figure 3: Iterating 12 times with p2 adaptive mesh refinement on the initial mesh size of 1/16 for the domain Ω_S.

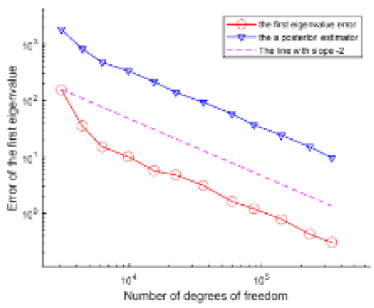

Figure 4: Error and indicator subcurve plot for 12 iterations of p2 adaptive eigenvalue computation on the initial mesh size of 1/16 for the domain Ω_S.

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