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Research Paper

Stretching of an Elastic Membrane

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An elastic membrane in the x_1x_2 -plane with boundary circle $x_1^2 + x_2^2 = 1$ is stretched so that a point P:(x_1,x_2) goes over into the point Q:(y_1,y_2) given by

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Ax = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \text{ in components, } y_1 = 5x_1+3x_2;\ y_2 = 3x_1+5x_2$$

Find the **principal directions**, that is, the directions of the position vector **x** of P for which the direction of the position vector **y** of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?

Solution. We are looking for vectors **x** such that y=λx. Since y=Ax, this gives Ax=λx , the equation of an eigenvalue problem. In components,Ax=λx is

$5x_1+3x_2 = \lambda x_1 \mid (5\text{-}\lambda)x_1 + 3x_2 = 0;\ \ 3x_1+5x_2 = \lambda x_2 \mid 3x_1+(5\text{-}\lambda)x_2 = 0$

The characteristic equation is

$$\begin{bmatrix} (5-\lambda) & 3 \\ 3 & (5-\lambda) \end{bmatrix} = (5-\lambda)^2 - 9 = 0$$

Its solutions are$\lambda_1 = 8$ and$\lambda_2 = 2$Theseare the eigenvalues of our problem. For $\lambda = \lambda_1 = 8$ our system becomes

$-3x_1 + 3x_2 = 0;\ 3x_1 - 3x_2 = 0 \mid x_1 = x_2 = 1$

For $\lambda_2 = 2$, our system becomes

$3x_1 + 3x_2 = 0;\ 3x_1 + 3x_2 = 0 \mid x_1 = 1;\ x_2 = -1$

We thus obtain as eigenvectors of **A**, for instance,$[1 \quad 1]^T$ corresponding to λ_1 and $[1 \quad -1]^T$corresponding to λ_2 (or a nonzero scalar multiple of these). These vectors make 45° and 135° angles with the positive x_1 direction. They give the principal directions, the answer to our problem. The eigenvalues show that in the principal directions the membrane is stretched by factors 8 and 2, respectively; see Fig.

Accordingly, if we choose the principal directions as directions of a new Cartesian u_1u_2 coordinate system, say, with the positive u_1 -semi-axis in the first quadrant and the positive $u_2$semi-axis in the second quadrant of the x_1x_2 -system, and if we set u_1=r.cosϕ, u_2=r.sinϕthen a boundary point of the unstretched circular membrane has coordinatescosϕ,sinϕ Hence, after the stretch we have

$Z_1 = 8\cos\phi;\ Z_2 = 2\sin\phi$

Since, $\cos^2\phi + \sin^2\phi = 1$ this shows that the deformed boundary is an ellipse

$$\frac{Z_1^2}{8^2} + \frac{Z_2^2}{2^2} = 1$$