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Research Paper

On the Inequalities of the Block Power Projection Operators

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Abstract
Originally studied by Gohberg and Krein, the block projection operators admit a natural extension to the setting of quasi-normed ideals and noncommutative integration. A. Bikchentaev and F. Sukochev [30] establish several uniform submajorisation inequalities for block projection operators. We do an application on their study and show that in the quasi-normed setting, for $\mathcal{L}^{1-\epsilon}$-spaces with $0 \le \epsilon < 1$, the reverse inequality holds and valid.

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I. Introduction

Gohberg and Krein in their book, ([15], Ch. 3, p. 82, Theorem 4.2) asserts that for any sequence with small change $\{P_j^r\}_{j=1}^{\omega}(\omega \le \infty)$ of mutually orthogonal power projections and for any symmetrically-normed ideal $\mathfrak{S}_{\Phi_\mathrm{m}}$ in the algebra $B(H)$ of all bounded operators on the infinite-dimensional Hilbert space H we have

$$\left\| \sum_{m,j=1}^{\omega} P_j^r A_m P_j^r \right\|_{\Phi_\mathrm{m}} \le \sum_m \| A_m \|_{\Phi_\mathrm{m}} \qquad (1)$$

for every $A_m \in \mathfrak{S}_{\Phi_\mathrm{m}}$. Here, Φ_m is symmetrically-norming function in the sense of ([15], Ch. 3, p. 71]) and $\mathfrak{S}_{\Phi_\mathrm{m}}$ is the symmetrically-normed ideal generated by Φ_m. The extension of this fundamental inequality was presented in ([10]. Corollary 3.4) in the form of Hardy-Littlewood-Pólya submajorization inequality (denoted below by $\prec\prec$)

$$\sum_{m,j=1}^{\omega} P_j^r (A_m) P_j^r \prec\prec \sum_m A_m \qquad (2)$$

Equivalently, if $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is a Banach ideal in $B(H)$ equipped with fully symmetric norm, then

$$\left\| \sum_{m,j=1}^{\omega} P_j^r (A_m) P_j^r \right\|_{\mathcal{E}} \le \sum_m \| A_m \|_{\mathcal{E}},\ \forall A_m \in \mathcal{E} \qquad (3)$$

The estimate (3) properly extends (1) and suggests the following natural question, which we address here:

Does (3) hold for an arbitrary symmetrically (quasi-)normed ideal in $B(H)$?

Recall, that an ideal $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ of $B(H)$ is said to be a symmetrically (quasi)normed ideal if its (quasi)-norm $\|\cdot\|_{\mathcal{E}}$ satisfies the following estimates

$\| X(X+\epsilon) \|_{\mathcal{E}} \leq \| X \|_{\mathcal{E}} \| X+\epsilon \|_{\infty}$
$\| (X+\epsilon)X \|_{\mathcal{E}} \leq \| X \|_{\mathcal{E}} \| X+\epsilon \|_{\infty}$ for all $X \in \mathcal{E}, \epsilon > 0$ and $(X+\epsilon) \in B(H)$. The ideals $\mathfrak{S}_{\Phi_m}$ featured in [15] and fully symmetric ideals mentioned above are special subclasses of general symmetrically (quasi)normed ideals in $B(H)$. The results concern two main subclasses of quasi-normed ideals. For convenience, we denote the classical trace-class ideal equipped with its natural norm as $(\mathfrak{S}_1, \|\cdot\|_1)$.

(i) The class of symmetrically normed ideals (this class is properly larger than the class of all fully symmetric ideals). Every such ideal is an intermediate ideal between $(\mathfrak{S}_1, \|\cdot\|_1)$ and $B(H)$.

(ii) The class of quasi-normed ideals which do not admit a symmetric norm and which are proper subsets of the ideal $\mathfrak{S}_1$.

The following short paragraphs described the finishing by [30]. We recall that the class of all fully symmetric ideals coincides with the class of all Banach ideals $\mathcal{E}$ which are exact interpolation spaces for the Banach pair $(\mathfrak{S}_1, B(H))$. For examples of symmetrically normed ideals which fail to be interpolation spaces for the latter pair see [20] and [27]. The classical examples of quasi-normed ideals from (ii) are given by Schatten-von Neumann ideals $\mathfrak{S}_{1-\epsilon}$, $0 < \epsilon < 1$.

The results here concern the question stated above and show the difference between (i) and (ii). We present the main results in the more general setting of (semifinite) noncommutative integration theory and Δ-normed symmetric spaces.

For the class (i) our methods are based on the concept of uniform majorization introduced in [20] (see also [25]). This concept is a generalization of Hardy-Littlewood-Pólya submajorization and is an important tool in the study of symmetric norms rather than merely fully symmetric norms. The first main result establishes a uniform submajorization version of inequality (2). As a consequence, inequality (3) holds for any symmetric norm.

Surprisingly, for the class (ii) the inequality (3) is reversed (and this is the second main result). Even in the setting of ideals of $B(H)$, this is a completely new result, as before only normed ideals and Banach spaces have been considered.

Perhaps even more surprising, if we consider infinite sequences of projections, then there are examples where (2) and (3) completely fail for the class (i) of symmetric operator spaces. Furthermore, the reverse inequality in the quasi-normed setting also does not extends to infinite sums (see Section 5 where we present such a counterexample for the Δ-normed space of all τ-measurable operators).

We observing that (3) plays an important role in noncommutative analysis and has significant applications in the study of extreme points [10], sets of uniformly absolutely continuous norm ([13], Section 6), derivation problem [4, 5, 21], isometries [17, 29] and other topics.

2. Preliminaries

2.1. Singular value functions. Let (I, m) denote the measure space $I = (0, \infty)$ (resp. $I = \mathbb{Z}_+$), where $(0, \infty)$ (resp. $\mathbb{Z}_+$) is the set of positive real (resp. of nonnegative integer) numbers, equipped with Lebesgue measure (resp. counting measure) m. Let $L(I, m)$ be the space of all measurable real-valued functions (resp. sequences) on I equipped with Lebesgue measure (resp. counting measure) m i.e. functions which coincide almost everywhere are considered identical. Define $S_m(I, m)$ to be the subset of $L(I, m)$ which consists of all functions (resp. sequences) $(y+\epsilon)$ such that $m(\{1+\epsilon: |(y+\epsilon)(1+\epsilon)| > 1+\epsilon\})$ is finite for some $\epsilon \geq 0$.

For $(y+\epsilon) \in S_m(I, m)$ (where $I = (0, \infty)$), we denote by $\mu_m(y+\epsilon)$ the decreasing rearrangement of the function $|y+\epsilon|$. That is,

$$\mu_m(1+\epsilon, y+\epsilon) = \inf\{\epsilon \geq 0: m(\{|y+\epsilon| > 1+\epsilon\}) \leq 1+\epsilon\},\ \epsilon \geq 0$$

On the other hand, if $I=\mathbb{Z}_+$, and m is the counting measure, then $S_m(I)=\ell_\infty(I)$, where $\ell_\infty(I)$ denotes the space of all bounded sequences on I. In this case, for a sequence $(y+\epsilon)=\{(y+\epsilon)_n\}_{n\geq 0}$ in $\ell_\infty(\mathbb{Z}_+)$, we denote by $\mu_m(y+\epsilon)$ the decreasing rearrangement of the sequence $|y+\epsilon|=\{|(y+\epsilon)_n|\}_{n\geq 0}$.

For $(y+\epsilon), y\in S_m(I,m)$, we say that y is submajorized by $(y+\epsilon)$ in the sense of Hardy-Littlewood-Pólya (written $y \prec\prec_{\text{head}} (y+\epsilon)$) if

$$\int_0^{1+\epsilon}\sum_m \mu_m(1+\epsilon,y)d(1+\epsilon)\leq \int_0^{1+\epsilon}\sum_m \mu_m(1+\epsilon,y+\epsilon)d(1+\epsilon),\ \epsilon>0$$
$$\left(\text{ or } \sum_{m,k=0}^{n}\mu_m(k,y)\leq \sum_{m,k=0}^{n}\mu_m(k,y+\epsilon),\ n\geq 0\right)$$

A more standard notation for Hardy-Littlewood-Pólya submajorization (or weak submajorization) is $y\prec_{\text{w}}(y+\epsilon)$. We have chosen to use the notation $y\prec\prec_{\text{head}}(y+\epsilon)$ to distinguish this submajorization from its reverse version introduced below in subsection 4.2 (see [30]).

Let $\mathcal{M}$ be a semifinite von Neumann algebra on a separable Hilbert space H equipped with a faithful normal semifinite trace τ.

Let $\text{Proj}(\mathcal{M})$ denote the lattice of all projections in $\mathcal{M}$, $\mathbf{1}$ be the unit of $\mathcal{M}$. A linear operator $X:\mathfrak{D}(X)\to H$, where the domain $\mathfrak{D}(X)$ of X is a linear subspace of H, is said to be affiliated with $\mathcal{M}$ if $(X+\epsilon)X\subseteq X(X+\epsilon)$ for every $(X+\epsilon)\in\mathcal{M}'$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$ (notation: $X\eta\mathcal{M}$). For any self-adjoint operator A_m on H, its spectral measure is denoted by E_{A_m}. A self-adjoint operator A_m is affiliated with $\mathcal{M}$ if and only if $E_{A_m}(B)\in\text{Proj}(\mathcal{M})$ for any Borel set $B\subseteq\mathbb{R}$. A closed and densely defined operator $A_m\eta\mathcal{M}$ is called τ-measurable if $\tau\big(E_{|A_m|}(1+\epsilon,\infty)\big)<\infty$ for sufficiently large $(1+\epsilon)$, where $|A_m|=\sqrt{(A_m)^*(A_m)}$. We denote the set of all τ-measurable operators by $S_m(\mathcal{M},\tau)$. For every $A_m\in S_m(\mathcal{M},\tau)$, we define its singular value function $\mu_m(A_m)$ by setting

$$\mu_m(1+\epsilon,A_m)=\inf\{\|\,(A_m)(\mathbf{1}-P^r)\,\|_{L_\infty(\mathcal{M})}: P^r\in\text{Proj}(\mathcal{M}),\ \tau(P^r)\leq 1+\epsilon\},\ \epsilon\geq 0$$

Equivalently, for positive self-adjoint operators $A_m\in S_m(\mathcal{M},\tau)$, we have

$$n_{A_m}(1+\epsilon)=\tau\left(E_{A_m}(1+\epsilon,\infty)\right),\ \mu_m(1+\epsilon,A_m)=\inf\{1+\epsilon: n_{A_m}(1+\epsilon)<1+\epsilon\},\ \epsilon\geq 0$$

For more details on generalised singular value functions, see [14] and 25.

If $(B+\epsilon), B\in S_m(\mathcal{M},\tau)$, then we say that B is submajorized by $(B+\epsilon)$ (in the sense of Hardy-Littlewood-Pólya), denoted by $\mu_m(B)\prec\prec_{\text{head}}\mu_m(B+\epsilon)$, if

$$\int_0^{1+\epsilon}\sum_m \mu_m(1+\epsilon,B)d(1+\epsilon)\leq\int_0^{1+\epsilon}\sum_m \mu_m(1+\epsilon,B+\epsilon)d(1+\epsilon),\ \epsilon>0$$

If $\mathcal{M}=B(H)$ and τ is the standard trace Tr, then it is not difficult to see that $S_m(\mathcal{M})=S_m(\mathcal{M},\tau)=\mathcal{M}$ (see [25]). In this case, for $(B+\epsilon)\in S_m(\mathcal{M},\tau)$, we have

$$\mu_m(n,B+\epsilon)=\mu_m(1+\epsilon,B+\epsilon),\ \epsilon\in[n-1,n), n\in\mathbb{Z}_+$$

The sequence $\{\mu_m(n,B+\epsilon)\}_{n\in\mathbb{Z}_+}$is just the sequence of singular values of the operator $(B+\epsilon)\in B(H)$. If we consider $L^\infty(I,m)$ as an Abelian von Neumann algebra acting via multiplication on the Hilbert space $L^2(I,m)$, with the trace given by integration with respect to m, then $S_m(I,m)$ consists of all measurable functions on I which are bounded except on a set of finite measure. In this case for $(g+\epsilon) \in S_m(I,m)$, the generalized singular value function $\mu_m(g+\epsilon)$ is precisely the classical decreasing rearrangement of the function $|g+\epsilon|$ defined above (see [30]).

2.2. Symmetric (Quasi-)Banach Function and Operator Spaces. For the general theory of symmetric spaces, see [3, 24, 25].

Definition 2.1. Let $\mathcal{E}$ be a linear subspace in $S_m(\mathcal{M},\tau)$ equipped with a complete (quasi-)norm $\|\cdot\|_{\mathcal{E}}$. We say that $\mathcal{E}$ is a symmetric operator space (on $\mathcal{M}$, or in $S_m(\mathcal{M},\tau)$) if for $(B+\epsilon) \in \mathcal{E}$ and for every $B \in S_m(\mathcal{M},\tau)$ with $\mu_m(B) \leq \mu_m(B+\epsilon)$, we have $B \in \mathcal{E}$ and $\| B \|_{\mathcal{E}} \leq \| B+\epsilon \|_{\mathcal{E}}$.

A symmetric function (or sequence) space is the term reserved for a symmetric operator space when $\mathcal{M} = L_\infty(I,m)$, where $I = (0,\infty)$ (or $\mathcal{M} = \ell_\infty(I)$ with counting measure, where $I = \mathbb{Z}_+$).

Recall the construction of a symmetric (quasi-)Banach operator space (or noncommutative symmetric (quasi-)Banach space) $E(\mathcal{M},\tau)$. The following fundamental theorem was proved in [20] (see also [25], Question 2.5.5, p. 58) and [28].

Theorem 2.2. Let $(E,\|\cdot\|_E)$ be a symmetric function (or sequence) space on $(0,\infty)$ (or $\mathbb{Z}_+$) and let $\mathcal{M}$ be a semifinite von Neumann algebra. Set

$$E(\mathcal{M},\tau) = \{(B+\epsilon) \in S_m(\mathcal{M},\tau) : \mu_m(B+\epsilon) \in E\}, \quad \| B+\epsilon \|_{E(\mathcal{M},\tau)} := \| \mu_m(B+\epsilon) \|_E$$

So defined $\left(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M},\tau)}\right)$ is a symmetric operator space.

The main result of [20] (see also [25], Section 3) shows that the correspondence

$$(E,\|\cdot\|_E) \leftrightarrow \left(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M},\tau)}\right)$$

is a one-to-one correspondence between the set of all symmetric operator space in $S_m(\mathcal{M},\tau)$ and the set of all symmetric function spaces in $S_m(I,m)$ whenever $(\mathcal{M},\tau)$ does not contain any minimal projections or is atomic and all minimal projections have equal trace. Of course, depending on $(\mathcal{M},\tau)$ the symmetric function space $E \subset S_m(I,m)$ is considered either on (0,1), or on $(0,\infty)$ or on $\mathbb{Z}_+$.

3. Inequalities for Uniform Submajorizations

Throughout the sequel, let $\mathcal{M}$ be an arbitrary semifinite von Neumann algebra, with some distinguished faithful normal semifinite trace τ. Let $(B+\epsilon), B \in S_m(\mathcal{M},\tau)$. We say that B is uniformly submajorized by $(B+\epsilon)$ (written $B \triangleleft (B+\epsilon)$) if there exists $\lambda \in \mathbb{N}$ such that

$$\int_{\lambda a}^{b} \sum_m \mu_m(1+\epsilon, B)\, d(1+\epsilon) \leq \int_a^b \sum_m \mu_m(1+\epsilon, B+\epsilon)\, d(1+\epsilon),\ \lambda a < b$$

The notion of uniform submajorization originally introduced in [20] (see also [25]). It has a wider area of applicability than Hardy-Littlewood-Pólya submajorization (in particular, it makes sense for arbitrary elements $(B+\epsilon), B \in S_m(\mathcal{M},\tau)$, whereas the latter submajorization is meaningful only for $(B+\epsilon), B \in L_r(\mathcal{M},\tau) + \mathcal{M}$. On the other hand, uniform submajorization imposes stricter conditions on the behavior of singular numbers of operators $(B+\epsilon)$ and B than their classical counterpart. The next theorem, the first main result, extends (2) to uniform submajorization (see [30]).

Theorem 3.1. If $e_1, e_2, \dots, e_n \in \mathcal{M}$ are projections with $e_i e_j = 0, i \neq j$, and if $(y+\epsilon) \in S_m(\mathcal{M},\tau)$, then

$$e_1(y+\epsilon)e_1 + e_2(y+\epsilon)e_2 + \cdots + e_n(y+\epsilon)e_n \triangleleft (y+\epsilon)$$

Proof. Firstly, we note that for any $(B+\epsilon), B \in S_m(\mathcal{M},\tau)$, the following inequality holds

$$(2B+\epsilon) \triangleleft \mu_m(B+\epsilon)+\mu_m(B)$$

Indeed, the case when $(B+\epsilon), B \geq 0$ is established in [20, Lemma 8.4]. For arbitrary operators $(B+\epsilon)$ and B, it follows from the triangle inequality observed in [23] (see also [11] or [25], Lemma 2.3.15) that

$$|2(B+\epsilon)| \leq U|B+\epsilon|U^* + V|B|V^*$$

where U and V are partial isometries in $\mathcal{M}$. Again appealing to ([20], Lemma 8.4), we obtain

$$\begin{aligned}\mu_m(2(B+\epsilon)) &\leq \mu_m(U|B+\epsilon|U^*+V|B|V^*) \triangleleft \mu_m(U|B+\epsilon|U^*)+\mu_m(V|B|V^*)\\ &\leq \mu_m(B+\epsilon)+\mu_m(B)\end{aligned}$$

By induction, we have

$$\sum_{k=1}^{n} (B+\epsilon)_k \triangleleft \sum_{m,k=1}^{n} \mu_m((B+\epsilon)_k), \forall (B+\epsilon)_1,\dots,(B+\epsilon)_k \in S_m(\mathcal{M},\tau) \qquad (4)$$

For every subset $\mathcal{A}_m \subset \{1,\dots,n\}$, define a partial isometry $u_{\mathcal{A}_m} \in \mathcal{M}$ by setting

$$u_{\mathcal{A}_m} := \sum_{m,k=1}^{n} \left(2\chi_{\mathcal{A}_m}(k)-1\right)e_k$$

Indeed, since $e_i e_j = 0, i \neq j$, it immediately follows that $u_{\mathcal{A}_m}u^*_{\mathcal{A}_m} = u^*_{\mathcal{A}_m}u_{\mathcal{A}_m} = e_1+e_2+\cdots+e_n$. We have

$$\begin{aligned}\sum_{\mathcal{A}_m} u_{\mathcal{A}_m}(y+\epsilon)u^*_{\mathcal{A}_m} &= \sum_{m}\sum_{\mathcal{A}_m}\sum_{k_1,k_2=1}^{n} \left(2\chi_{\mathcal{A}_m}(k_1)-1\right)e_{k_1}(y+\epsilon)\left(2\chi_{\mathcal{A}_m}(k_2)-1\right)e_{k_2}\\ &= \sum_{k_1,k_2=1}^{n} e_{k_1}(y+\epsilon)e_{k_2}\sum_{m}\sum_{\mathcal{A}_m}\left(2\chi_{\mathcal{A}_m}(k_1)-1\right)\left(2\chi_{\mathcal{A}_m}(k_2)-1\right)\end{aligned}$$

A direct computation yields that

$$\sum_{m}\sum_{\mathcal{A}_m}\left(2\chi_{\mathcal{A}_m}(k_1)-1\right)\left(2\chi_{\mathcal{A}_m}(k_2)-1\right) = \begin{cases}0, & k_1 \neq k_2\\ 2^n, & k_1 = k_2\end{cases}$$

Therefore, we have

$$e_1(y+\epsilon)e_1+e_2(y+\epsilon)e_2+\cdots+e_n(y+\epsilon)e_n = \frac{1}{2^n}\sum_{m}\sum_{\mathcal{A}_m} u_{\mathcal{A}_m}(y+\epsilon)u^*_{\mathcal{A}_m}$$

Hence, by (4), we have

$$e_1(y+\epsilon)e_1+e_2(y+\epsilon)e_2+\cdots+e_n(y+\epsilon)e_n \triangleleft \frac{1}{2^n}\sum_{m}\sum_{\mathcal{A}_m}\mu_m\left(u_{\mathcal{A}_m}(y+\epsilon)u^*_{\mathcal{A}_m}\right)$$

$$\leq \frac{1}{2^n}\sum_{m}\sum_{\mathcal{A}_m}\mu_m(y+\epsilon) = \mu_m(y+\epsilon)$$

The following corollary extends (3) to arbitrary symmetric operator spaces (see [30]).

Corollary 3.2. Let $\left(E(\mathcal{M},\tau), \|\cdot\|_{E(\mathcal{M},\tau)}\right)$ be a symmetric operator space on $(\mathcal{M},\tau)$ defined in Theorem 2.2. If $e_1, e_2, \dots, e_n \in \mathcal{M}$ are projections with $e_i e_j = 0, i \neq j$, and if $(y+\epsilon) \in E(\mathcal{M},\tau)$, then

$$\|e_1(y+\epsilon)e_1 + e_2(y+\epsilon)e_2 + \cdots + e_n(y+\epsilon)e_n\|_{E(\mathcal{M},\tau)} \leq \| y+\epsilon \|_{E(\mathcal{M},\tau)}$$

Proof. The assertion of Corollary 3.2 follows from that of Theorem 3.1 combined with ([25], Corollary 3.4.3) (see also [20], p. 84).

Now, we are ready to present the strengthened version of a triangle inequality for uniform submajorizations. This improves the main result in [26] and complements the result of ([9], Lemma A.1).

Theorem 3.3 (see [30]). Let $T_m, S_m \in S_m(\mathcal{M},\tau), {S_m}^* = S_m, T_m \geq 0$. If $-T_m \leq S_m \leq T_m$, then $S_m \triangleleft T_m$.

Proof. Set $(1-\epsilon) = E_{S_m}(0,\infty)$. We have $(S_m)_+ = (1-\epsilon)S_m(1-\epsilon) \leq (1-\epsilon)T_m(1-\epsilon)$ and $(S_m)_- = -(\epsilon)S_m(\epsilon) \leq (\epsilon)T_m(\epsilon)$. Thus, by Theorem 3.1

$$|S_m| = (S_m)_+ + (S_m)_- \leq (1-\epsilon)T_m(1-\epsilon) + (\epsilon)T_m(\epsilon) \triangleleft T_m$$

i.e., $S_m \triangleleft T_m$.

Corollary 3.4 (see [30]). If $(T_m)_i \in S_m(\mathcal{M},\tau), (T_m)_i^* = (T_m)_i, i = 1,2$, then

$$|(T_m)_1 + (T_m)_2| \triangleleft |(T_m)_1| + |(T_m)_2|$$

Proof. We need only to observe that

$$-(|(T_m)_1| + |(T_m)_2|) \leq (T_m)_1 + (T_m)_2 \leq |(T_m)_1| + |(T_m)_2|$$

and apply Theorem 3.3.

It is quite remarkable that the result of Theorem 3.1 fails for infinite sequences of pairwise orthogonal projections. That is, generally speaking, the implication

$$\sum_{j=1}^{\infty} e_j(y+\epsilon)e_j \triangleleft (y+\epsilon),\ \ (y+\epsilon) \in S_m(\mathcal{M},\tau)$$

fails for the situation when $e_1, e_2, \dots \in \mathcal{M}$ are projections with $e_i e_j = 0, i \neq j$, and the series on the left hand side is understood convergent in measure topology (see e.g. [14] or [25]). We demonstrate this failure in Theorem 3.5 below. To make the presentation smoother, we recall a few notions and intorduce some notations.

Following [20], a symmetric (function or sequence) space E is called relatively fully symmetric if and only if

$$(g+\epsilon), g \in E, g \prec\prec_{\text{head}} (g+\epsilon) \Rightarrow \| g \|_E \leq \| g+\epsilon \|_E$$

The space E is relatively fully symmetric if and only if E is a closed subspace of a fully symmetric space [20]. For clarity, we shall also address to those spaces as those whose norm is monotone with respect to Hardy-Littlewood-Pólya submajorization. There exist symmetric sequence spaces E which do not admit an equivalent relatively fully symmetric norm (in particular, any such space is not a closed subspace of any fully symmetric sequence space). We refer to [27] for such examples.

Let $E = E(\mathbb{Z}_+)$be a symmetric sequence space, let $H = \ell_2$ be a space of all square summable sequences with standard basis (e_n) and let $\mathcal{E}$ be the corresponding symmetrically normed ideal (see Theorem 2.2). We shall use a standard notation for elements from $\mathcal{E}$. The matrix $\left((y+\epsilon)_{i,j}\right)$ representing the elements $(y+\epsilon) \in \mathcal{E}$ is defined by $(y+\epsilon)_{i,j} = \left((y+\epsilon)e_j, e_i\right), 1 \le i,j < \infty$. We shall use the matrix elements $e_{ij} \in \mathcal{E}$ defined by

$$e_{ij}(k,l) = \delta_i^k \delta_j^l, 1 \le i,j,k,l < \infty$$

Clearly, the sequence $(e_{kk})_{k\ge 0}$ is a sequence of pairwise orthogonal one-dimensional projections in $B(\ell_2)$.

Theorem 3.5 (see [30]). Let $(E, \|\cdot\|_E)$ be a symmetric Banach sequence space whose norm is not monotone with respect to the Hardy-Littlewood submajorization. There exists a positive operator $A_m \in \mathcal{E}$ such that

$$\left\| \sum_{k\ge 0} \sum_m e_{kk} A_m e_{kk} \right\|_{\mathcal{E}} > \sum_m \| A_m \|_{\mathcal{E}}$$

Proof. Let us consider a closed subspace F of E generated by the closure in E of all finitely supported sequences from E. It is well-known that $(F, \|\cdot\|_E)$ is a separable symmetric sequence space and hence it is fully symmetric (see e.g. [24], Theorem II.4.10). Taking into account that the space ℓ_1 is a subset of any symmetric sequence space (see e.g. [25], Example 2.6.7(c)), in particular $\ell_1 \subset F$, we infer that for any elements $(y+\epsilon), y \in \ell_1$, the assumption $y \prec\prec_{\text{head}} (y+\epsilon)$ implies $\| y \|_E \le \| y+\epsilon \|_E$.

By the assumption there exist elements $(y+\epsilon) = \mu_m(y+\epsilon), y = \mu_m(y) \in E$ such that $y \prec\prec_{\text{head}} (y+\epsilon)$ and $\| y \|_E > \| y+\epsilon \|_E$. The preceding argument shows, that it is not possible that both elements $(y+\epsilon)$ and y belong to ℓ_1; in particular, taking into account that ℓ_1 is fully symmetric, we must have $(y+\epsilon) \notin \ell_1$. Let us show that, there exists $z = \mu_m(z) \notin \ell_1$ such that $z \prec\prec_{\text{head}} (y+\epsilon)$ and $\| z \|_E > \| y+\epsilon \|_E$. To this end, assume that $y \in \ell_1, y \prec\prec_{\text{head}} (y+\epsilon)$ and $\| y \|_E > \| y+\epsilon \|_E$ and set

$$f(\lambda) = \lambda(y+\epsilon) + (1-\lambda)y,\ \lambda \in (0,1)$$

Obviously, the mapping $\lambda \to f(\lambda)$ is continuous from $(0,1)$ into $(E, \|\cdot\|_E)$ and therefore, there exists $\lambda_0 \in (0,1)$ such that

$$\|f(\lambda_0)\|_E > \| y+\epsilon \|_E$$

Let us set $z = f(\lambda_0)$ and observe that, by the definition, $z = \mu_m(z)$ and that for every $n \ge 0$, we have

$$\sum_{m,k=0}^{n} \mu_m(k,z) = \lambda \sum_{m,k=0}^{n} \mu_m(k,y+\epsilon) + (1-\lambda) \sum_{m,k=0}^{n} \mu_m(k,y) \le \sum_{m,k=0}^{n} \mu_m(k,y+\epsilon)$$

that is $z \prec\prec_{\text{head}} (y+\epsilon)$. Since $(y+\epsilon) \notin \ell_1$, we conclude that $z \notin \ell_1$. Thus, until the end of the proof, we may assume that $(y+\epsilon) = \mu_m(y+\epsilon), y = \mu_m(y) \in E$ such that $y \prec\prec_{\text{head}} (y+\epsilon)$, that $\| y \|_E > \| y+\epsilon \|_E$ and that $(y+\epsilon), y \notin \ell_1$.

By the fundamental Kaftal-Weiss theorem (see [22] and also [25], Theorem 7.5.2), there exists a positive compact operator $A_m \in B(\ell_2)$ such that $\mu_m A_m = (y+\epsilon)$ and such that

$$e_{kk}A_m e_{kk} = y(k)e_{kk},\ k \geq 0$$

In particular,

$$\sum_{m} \mu_m \left(\sum_{k\geq 0} e_{kk}A_m e_{kk} \right) = y$$

Since $\| y \|_E > \| y + \epsilon \|_E$, we obtain $\left\|\sum_{m,k\geq 0} e_{kk}A_m e_{kk}\right\|_{\mathcal{E}} > \sum_m \| A_m \|_{\mathcal{E}}$.

4. Reverse Inequality for the Block Projection Operator

4.1. Δ-normed spaces. We recall the definition of Δ-norm, which extends and generalizes the notion of quasi-norm. Let Ω be a linear space over the field $\mathbb{C}$. A function $\|\cdot\|$ from Ω to $\mathbb{R}$ is a Δ-norm [19], if for all $(y + \epsilon), y \in \Omega$ the following properties hold:

(1) $\| y + \epsilon \| \geqslant 0, \| y + \epsilon \| = 0 \Leftrightarrow y = -\epsilon$;

(2) $\| \alpha(y + \epsilon) \| \leqslant \| y + \epsilon \|$ for all $|\alpha| \leq 1$;

(3) $\lim_{\alpha\to 0} \| \alpha(y + \epsilon) \| = 0$;

(4) $\| 2(y + \epsilon) \| \leq C_\Omega \cdot (\| y + \epsilon \| + \| y \|)$ for a constant $C_\Omega \geq 1$ independent of $(y + \epsilon), y$.

The couple $(\Omega, \|\cdot\|)$ is called a Δ-normed space.

Definition 4.1. [16, 18] Let a semifinite von Neumann algebra $\mathcal{M}$ be equipped with a faithful normal semifinite trace τ. Let $\mathcal{E}$ be a linear subspace in $S_m(\mathcal{M}, \tau)$ equipped with a Δ-norm $\|\cdot\|_{\mathcal{E}}$. We say that $\mathcal{E}$ is a symmetrically Δ-normed operator space if $X \in \mathcal{E}$ and every $X + \epsilon \in S_m(\mathcal{M}, \tau)$ the assumption $\mu_m(X + \epsilon) \leq \mu_m(X)$ implies that $(X + \epsilon) \in \mathcal{E}$ and $\| X + \epsilon \|_{\mathcal{E}} \leq \| X \|_{\mathcal{E}}$.

More information concerning symmetrically Δ-normed operator spaces may be found in ([7], pp. 1427-1429).

Remark 4.2. By defining that

$$\| y + \epsilon \|_{S_m} = \inf_{\epsilon\geq 0}\{1 + \epsilon + \mu_m(1 + \epsilon; y + \epsilon)\}, \qquad (y + \epsilon) \in S_m(\mathcal{M})$$

we obtain a symmetric Δ-norm $\|\cdot\|_{S_m}$ on $S_m(\mathcal{M}, \tau)$, ([18], Remark 3.4). Moreover, the topology induced by $\|\cdot\|_{S_m}$ is equivalent to the measure topology ([18], Proposition 4.1).

We end this subsection (see [30]) by discussing interpolation between $L_r(\mathcal{M}, \tau)$ and $L_{r-1}(\mathcal{M}, \tau)$, where $L_r(\mathcal{M}, \tau)$ consists of elements in $S_m(\mathcal{M}, \tau)$ whose supports have finite trace [8, 18]. We denote $\| y + \epsilon \|_{L_{r-1}} = \tau(\mathrm{supp}(y + \epsilon)), (y + \epsilon) \in L_{r-1}(\mathcal{M}, \tau)$. For $T_m : L_{r-1}(\mathcal{M}, \tau) \to L_{r-1}(\mathcal{M}, \tau)$, we write

$$\| T_m \|_{L_{r-1}\to L_{r-1}} = \sup_{f\in L_{r-1}} \frac{\tau(\mathrm{supp}(T_m f))}{\tau(\mathrm{supp}(f))}$$

For the interpolation couple $(L_{r-1}(\mathcal{M}, \tau), L_r(\mathcal{M}, \tau))$ of Δ-normed spaces, the space $(L_{r-1} \cap L_r)(\mathcal{M}, \tau)$ is equipped with a group-norm by setting

$$\| y + \epsilon \|_{L_{r-1}\cap L_r} = \max\{\| y + \epsilon \|_{L_{r-1}}, \| y + \epsilon \|_{L_r}\},\ (y + \epsilon) \in L_{r-1}(\mathcal{M}, \tau) \cap L_r(\mathcal{M}, \tau)$$

and the space $(L_{r-1} + L_r)(\mathcal{M}, \tau)$ is equipped with a Δ-norm by setting

$$\| y+\epsilon \|_{L_{\mathrm{r}-1}+L_{\mathrm{r}}} = \inf_{(y+\epsilon)=x_{\mathrm{r}-1}+x_{\mathrm{r}}, x_{\mathrm{r}-1}\in L_{\mathrm{r}-1}, x_{\mathrm{r}}\in L_{\mathrm{r}}} \|x_{\mathrm{r}-1}\|_{L_{\mathrm{r}-1}} + \|x_{\mathrm{r}}\|_{L_{\mathrm{r}}},\ (y+\epsilon)\in (L_{\mathrm{r}-1}+L_{\mathrm{r}})(\mathcal{M},\tau)$$

A space $E(\mathcal{M},\tau)$ is said to be intermediate for $L_{\mathrm{r}-1}(\mathcal{M},\tau)$ and $L_{\mathrm{r}}(\mathcal{M},\tau)$ if the continuous embeddings

$$L_{\mathrm{r}-1}(\mathcal{M},\tau)\cap L_{\mathrm{r}}(\mathcal{M},\tau)\subset E(\mathcal{M},\tau)\subset L_{\mathrm{r}-1}(\mathcal{M},\tau)+L_{\mathrm{r}}(\mathcal{M},\tau)$$

hold. Let $E(\mathcal{M},\tau)$ be a symmetrically Δ-normed space intermediate between $L_{\mathrm{r}-1}(\mathcal{M},\tau)$ and $L_{\mathrm{r}}(\mathcal{M},\tau)$.

Definition 4.3. If every linear operator on $L_{\mathrm{r}-1}(\mathcal{M},\tau)+L_{\mathrm{r}}(\mathcal{M},\tau)$ whose reductions on $L_{\mathrm{r}-1}(\mathcal{M},\tau)$ and $L_{\mathrm{r}}(\mathcal{M},\tau)$ are both contractions is also a bounded operator from $E(\mathcal{M},\tau)$ to $E(\mathcal{M},\tau)$ and if

$$\| T_m \|_{E\to E}\le \tilde{C}_E$$

for some positive constant $\tilde{C}_E$, which depends only on E, then $E(\mathcal{M},\tau)$ is called an interpolation space between the spaces $L_{\mathrm{r}-1}(\mathcal{M},\tau)$ and $L_{\mathrm{r}}(\mathcal{M},\tau)$.

4.2. Reverse submajorization. We need below another partial orderings, which is defined for functions from $(L_{\mathrm{r}-1}+L_{\mathrm{r}})(0,\infty)$, where $L_{\mathrm{r}-1}\subset S_m(0,\infty)$ is the collection of all functions whose support has finite measure. For $(g+\epsilon), g\in (L_{\mathrm{r}-1}+L_{\mathrm{r}})(0,\infty)$, we write $g \prec\prec_{\text{tail}} (g+\epsilon)$ if and only if

$$\int_{1+\epsilon}^{\infty}\sum_m \mu_m(1+\epsilon,g)d(1+\epsilon)\le \int_{1+\epsilon}^{\infty}\sum_m \mu_m(1+\epsilon,g+\epsilon)d(1+\epsilon),\ \epsilon\ge 0$$

For the case $(g+\epsilon), g\in L_{\mathrm{r}}(0,\infty), \| g+\epsilon \|_1 = \| g \|_1$, the notion of reverse submajorization is equivalent to that of supermajorization. In other words, we have $g\prec\prec_{\text{tail}} (g+\epsilon)$ is equivalent (under the above conditions) to $(g+\epsilon)\prec\prec_{\text{head}} g$. A classical notation for supermajorization is $g\prec^w (g+\epsilon)$.

Proposition 4.4 (see [30]). Let E be a symmetrically Δ-normed space which is an interpolation space between $L_{\mathrm{r}-1}$ and L_{r}. If $(y+\epsilon)\in E$ and $y\prec\prec_{\text{tail}} (y+\epsilon)$, then $y\in E$ and $\| y \|_E\le c_E \| y+\epsilon \|_E$.

Proof. Let us fix such $(y+\epsilon)$ and y. Setting $q=1$ in ([8], Lemma 3.9) yields an operator $T_m: L_{\mathrm{r}-1}+L_{\mathrm{r}}\to L_{\mathrm{r}-1}+L_{\mathrm{r}}$ such that $T_m(y+\epsilon)=y$ and

$$\| T_m \|_{L_{\mathrm{r}-1}\to L_{\mathrm{r}-1}}\le 4,\ \ \| T_m \|_{L_{\mathrm{r}}\to L_{\mathrm{r}}}\le 6$$

Let $1+\epsilon\to\sigma_{1+\epsilon}, \epsilon\ge 0$, be the action of multiplicative group $\mathbb{R}_+$by dilations, that is,

$$(\sigma_{1+\epsilon}f)(1+\epsilon)=f(1),\ \epsilon>0$$

Define an operator $S_m: L_{\mathrm{r}-1}+L_{\mathrm{r}}\to L_{\mathrm{r}-1}+L_{\mathrm{r}}$ by setting

$$S_m f=\frac{2}{3}\sigma_{\frac{1}{4}}f,\ f\in L_{\mathrm{r}-1}+L_{\mathrm{r}}$$

so that

$$(S_m)^{-1}f=\frac{3}{2}\sigma_4 f,\ f\in L_{\mathrm{r}-1}+L_{\mathrm{r}}$$

It is immediate that

$$\| S_m\circ T_m \|_{L_{\mathrm{r}-1}\to L_{\mathrm{r}-1}}\le 1,\ \ \| S_m\circ T_m \|_{L_{\mathrm{r}}\to L_{\mathrm{r}}}\le 1$$

Since E is an interpolation space between L_{r-1} and L_r, it follows that $\| S_m \circ T_m \|_{E\to E} \le \tilde{C}_E$, where $\tilde{C}_E$ is an interpolation constant from Definition 4.3. Since $(S_m)^{-1}: E \to E$ is a bounded mapping, it follows that

$$\| y \|_E = \|((S_m)^{-1} \circ S_m \circ T_m)(y+\epsilon)\|_E \le \|(S_m)^{-1}\|_{E\to E} \| S_m \circ T_m \|_{E\to E} \cdot \| y+\epsilon \|_E$$

Let I be either finite or infinite interval equipped with Lebesgue measure. If $f, g \in L_r(I)$, then we say $g \prec_{\text{head}} f$ if $g \prec\prec_{\text{head}} f$ and also $\| g \|_1 = \| f \|_1$.

Example 4.5 (see [30]). If $0 \le (y+\epsilon), y \in L_{1-\epsilon}(I)$ and $y \prec\prec_{\text{tail}} (y+\epsilon), 0 \le \epsilon < 1$, then $y \in L_{1-\epsilon}(I)$ and $\| y \|_{1-\epsilon} \le \| y+\epsilon \|_{1-\epsilon}$.

Proof. Step 1: Suppose first that $0 \le (y+\epsilon), y \in (L_r \cap L_{1-\epsilon})(I)$ are such that $(y+\epsilon) \prec_{\text{head}} y$, or equivalently, that $y \prec_{\text{tail}} (y+\epsilon)$.

If $I = (0,1)$, then the inequality $\| y \|_{1-\epsilon} \le \| y+\epsilon \|_{1-\epsilon}$ is established in Lemma 25 in [2]. Alternatively, one can infer this inequality from Theorem 2.5 in [11] (applied to the convex function $1+\epsilon \to -(1+\epsilon)^{1-\epsilon}$).

Suppose that $I = (0,\alpha), 0 < \alpha < \infty$. Let $\sigma_{\alpha^{-1}}$ be the dilation action on $S_m(\mathbb{R}_+, m)$ given by $(\sigma_{\alpha^{-1}} f)(1+\epsilon) = f(\alpha(1+\epsilon))$. It is immediate that $\sigma_{\alpha^{-1}}(y+\epsilon) \prec_{\text{head}} \sigma_{\alpha^{-1}} y$. Functions $\sigma_{\alpha^{-1}}(y+\epsilon)$ and $\sigma_{\alpha^{-1}} y$ live on the interval $(0,1)$. By the preceding paragraph, we have

$$\| y \|_{1-\epsilon} = \alpha^{\frac{1}{1-\epsilon}} \|\sigma_{\alpha^{-1}} y\|_{1-\epsilon} \le \alpha^{\frac{1}{1-\epsilon}} \|\sigma_{\alpha^{-1}}(y+\epsilon)\|_{1-\epsilon} = \| y+\epsilon \|_{1-\epsilon}$$

This proves the inequality $\| y \|_{1-\epsilon} \le \| y+\epsilon \|_{1-\epsilon}$ for the functions on finite interval.

Consider now the case of the semiaxis. We may assume without loss of generality that $(y+\epsilon) = \mu_m(y+\epsilon)$ and $y = \mu_m(y)$.

Let n be a positive integer. We have $(y+\epsilon)\chi_{(0,n)} \prec\prec_{\text{head}} y\chi_{(0,n)}$. Let $t(n) \le n$ be selected so that

$$\int_0^n (y+\epsilon)dm = \int_0^{t(n)} y dm$$

By the preceding paragraph, we have

$$\|(y+\epsilon)\chi_{(0,n)}\|_{1-\epsilon} \ge \|y\chi_{(0,t(n))}\|_{1-\epsilon}$$

Let us denote

$$1+\epsilon = \liminf_{n\to\infty} t(n)$$

We have

$$\| y+\epsilon \|_{1-\epsilon} = \lim_{n\to\infty} \|(y+\epsilon)\chi_{(0,n)}\|_{1-\epsilon} \ge \lim_{n\to\infty} \|y\chi_{(0,t(n))}\|_{1-\epsilon} = \|y\chi_{(0,1+\epsilon)}\|_{1-\epsilon}$$

Choosing a sequence $n_k \uparrow \infty$ such that $t(n_k) \to (1+\epsilon)$ as $k \to \infty$. We have

$$\int_0^{1+\epsilon} y dm = \lim_{k\to\infty} \int_0^{t(n_k)} y dm = \lim_{k\to\infty} \int_0^{n_k} (y+\epsilon)dm = \int_0^{\infty} (y+\epsilon)dm$$

However, by the assumption, we have $(y+\epsilon) \prec_{\text{head}} y$ and, in particular,

$$\int_0^\infty y dm = \int_0^\infty (y+\epsilon) dm$$

This implies, $y|_{(1+\epsilon,\infty)} = 0$. So, the inequality above yields the assertion.

Step 2: Consider now the general case. Without loss of generality, $(y+\epsilon) = \mu_m(y+\epsilon)$ and $y = \mu_m(y)$. By Lemma 3.5 in [8], there exists a collection $(\Delta_k)_{k\geq 0}$ of pairwise disjoint sets such that

(1) $(y+\epsilon)|_{\Delta_k} \prec_{\text{head}} y\big|_{\Delta_k}$ for all $k \geq 0$;

(2) $\epsilon \geq 0$ on the complement of $\cup_{k\geq 0} \Delta_k$.

It follows from Step 1 that

$$\left\|y\chi_{\Delta_k}\right\|_{1-\epsilon} \leq \left\|(y+\epsilon)\chi_{\Delta_k}\right\|_{1-\epsilon}$$

Obviously,

$$\left\|y\chi_{(\cup_{k\geq 0}\Delta_k)^c}\right\|_{1-\epsilon} \leq \left\|(y+\epsilon)\chi_{(\cup_{k\geq 0}\Delta_k)^c}\right\|_{1-\epsilon}$$

Thus,

$$\begin{aligned} \| y \|_{1-\epsilon}^{1-\epsilon} &= \sum_{k\geq 0} \left\|y\chi_{\Delta_k}\right\|_{1-\epsilon}^{1-\epsilon} + \left\|y\chi_{(\cup_{k\geq 0}\Delta_k)^c}\right\|_{1-\epsilon}^{1-\epsilon} \\ &\leq \sum_{k\geq 0} \left\|(y+\epsilon)\chi_{\Delta_k}\right\|_{1-\epsilon}^{1-\epsilon} + \left\|(y+\epsilon)\chi_{(\cup_{k\geq 0}\Delta_k)^c}\right\|_{1-\epsilon}^{1-\epsilon} = \| y+\epsilon \|_{1-\epsilon}^{1-\epsilon} \end{aligned}$$

4.3. Reverse inequality for Δ-normed spaces.

Lemma 4.6 (see [30]). Let $e_1, e_2, \dots, e_n \in \mathcal{M}$ be projections with $e_i e_j = 0, i \neq j$, and such that $\vee_{i=1}^n e_i = 1$. If $0 \leq x^2 \in (L_{r-1} + L_r)(\mathcal{M}, \tau)$, then

$$x^2 \prec\prec_{\text{tail}} e_1 x^2 e_1 + e_2 x^2 e_2 + \cdots + e_n x^2 e_n$$

Proof. For $x^2 \in L_r(\mathcal{M}, \tau)$, it is proved in ([10], Corollary 3.4) (see also [13], Lemma 6.1) and [6] that

$$e_1 x^2 e_1 + e_2 x^2 e_2 + \cdots + e_n x^2 e_n \prec\prec_{\text{head}} x^2$$

In fact, for a positive x^2, we obviously have

$$e_1 x^2 e_1 + e_2 x^2 e_2 + \cdots + e_n x^2 e_n \prec_{\text{head}} x^2$$

Thus,

$$x^2 \prec_{\text{tail}} e_1 x^2 e_1 + e_2 x^2 e_2 + \cdots + e_n x^2 e_n$$

This proves the assertion for $x^2 \in L_r(\mathcal{M}, \tau)$.

Consider now the general case. We have $\min\{x^2, m\} \in L_r(\mathcal{M}, \tau)$ for all $m \in \mathbb{Z}_+$. By the preceding paragraph, we have

$$\begin{aligned} \min\{x^2, m\} &\prec_{\text{tail}} e_1 \min\{x^2, m\} e_1 + e_2 \min\{x^2, m\} e_2 + \cdots + e_n \min\{x^2, m\} e_n \\ &\leq e_1 x^2 e_1 + e_2 x^2 e_2 + \cdots + e_n x^2 e_n \end{aligned}$$

Thus,

$$\int_{1+\epsilon}^{\infty}\sum_{m}\ \mu_m(1+\epsilon,x^2)d(1+\epsilon)=\lim_{m\to\infty}\int_{1+\epsilon}^{\infty}\sum_{m}\ \mu_m(1+\epsilon,\min\{m,x^2\})d(1+\epsilon)$$
$$\leq\lim_{m\to\infty}\int_{1+\epsilon}^{\infty}\sum_{m}\ \mu_m(1+\epsilon,e_1x^2e_1+e_2x^2e_2+\cdots+e_nx^2e_n)d(1+\epsilon)$$
$$=\int_{1+\epsilon}^{\infty}\sum_{m}\ \mu_m(1+\epsilon,e_1x^2e_1+e_2x^2e_2+\cdots+e_nx^2e_n)d(1+\epsilon)$$

Theorem 4.7 (see [30]). Let $e_1,e_2,\dots,e_n\in\mathcal{M}$ be projections with $e_ie_j=0,1\le i\neq j\le n$, and such that $\vee_{i=1}^{n}e_i=1$. Let E be an interpolation space between $L_{\mathrm{r}-1}$ and L_{r}. If $0\le x^2\in E(\mathcal{M},\tau)$, then

$$\left\|\sum_{i=1}^{n}\ e_ix^2e_i\right\|_E\ \geq c_E\ \|\ x^2\ \|_E$$

Proof. The assertion follows from Lemma 4.6 and Proposition 4.4.

Example 4.8 (see [30]). Let $e_1,e_2,\dots,e_n\in\mathcal{M}$ be projections with $e_ie_j=0,1\le i\neq j\le n$, and such that $\vee_{i=1}^{n}e_i=\mathbf{1}$. If $0\le x^2\in L_{1-\epsilon}(\mathcal{M},\tau),0\le\epsilon<1$, then

$$\left\|\sum_{i=1}^{n}\ e_ix^2e_i\right\|_{1-\epsilon}\ \geq\|\ x^2\ \|_{1-\epsilon}$$

Proof. The assertion follows from Lemma 4.6 and Example 4.5.

5. Appendix (see [30])

Let $\{\mathcal{H}_n=\ell_2^n\}_{n=1}^{\infty}$ be a sequence of finite dimensional Hilbert spaces and consider their Hilbertian direct sum

$$\mathcal{H}=\overset{2}{\bigoplus}_{n\geq1}\mathcal{H}_n$$

Let $\{(A_m)_n\}_{n=1}^{\infty}$ be a sequence of self-adjoint operators, with $(A_m)_n\in B_m(\mathcal{H}_n)$. Let A_m denote their direct sum (notation $A_m=\bigoplus_{n=1}^{\infty}(A_m)_n$). Namely A_m is defined on the domain

$$\mathfrak{D}(A_m)=\left\{\{\xi_n\}_{n=1}^{\infty}\in\mathcal{H}:\sum_{m,n=1}^{\infty}\ \|(A_m)_n(\xi_n)\|^2<\infty\right\}$$

by setting $A_m(\xi)=\{(A_m)_n(\xi_n)\}_{n=1}^{\infty}$ for any $\xi=\{\xi_n\}_{n=1}^{\infty}$ in $\mathfrak{D}(A_m)$. Then A_m is a selfadjoint (possibly unbounded) operator on $\mathcal{H}$.

Consider the von Neumann algebra

$$\mathcal{N}=\bigoplus_{n\geq1}B_m(\mathcal{H}_n)$$

equipped with the trace

$$\tau=\bigoplus_{n\geq1}\alpha_n^m\mathrm{Tr}_n,\ \ \text{where}\ \ \alpha_n^m=\frac{1}{n\log^2(1+n)}$$

and where Tr_n is the standard trace on the algebra $B_m(\mathcal{H}_n)$, which we shall below view as the algebra of all complex $n \times n$ matrices $\left((y+\epsilon)_{ij}\right)_{i,j=1}^{n}$. Observe that $\sum_{n=1}^{\infty} \alpha_n^m < \infty$ whereas $\sum_{n=1}^{\infty} n\alpha_n^m$ diverges. We shall define an unbounded operator

$$A_m = \bigoplus_{n=1}^{\infty} (A_m)_n,\ (A_m)_n = \left(n(y+\epsilon)_{ij}\right)_{i,j=1}^{n}, \text{ where } (y+\epsilon)_{ij} = 1, 1 \leq i,j \leq n$$

Observe that we may also view the operator A_m as $\bigoplus_{n=1}^{\infty} n^2 q_n$ where q_n is a selfadjoint one dimensional projection from $B_m(\mathcal{H}_n)$ given by the matrix $\left(q_{ij}^{(n)}\right)_{i,j=1}^{n}$ where $q_{ij}^{(n)} = \frac{1}{n}$ for all $1 \leq i,j \leq n, n \geq 1$. Obviously, we have that A_m is a selfadjoint positive operator such that $(A_m)\eta\mathcal{N}$. Let us show that $A_m \in S_m(\mathcal{N},\tau)$. Indeed, let $\lambda = 1$. Estimating the value of distribution function $n_{A_m}(1)$, we have

$$n_{A_m}(1) = \tau\left(E_{A_m}(1,\infty)\right) = \sum_{m,k\geq 2} \alpha_k^m \mathrm{Tr}_k(q_k) = \sum_{m,k\geq 2} \alpha_k^m < \infty$$

Now, let us consider the element

$$B_m = \bigoplus_{n=1}^{\infty} (B_m)_n\ (B_m)_n = \left(y_{ij}\right)_{i,j=1}^{n}, \text{ where } y_{ij} = n, 1 \leq i = j \leq n, \text{ and } y_{ij} = 0, i \neq j.$$

Again, we obviously have that B_m is a self-adjoint positive operator such that $B_m\eta\mathcal{N}$. Let us show that $B_m \notin S_m(\mathcal{N},\tau)$. Indeed, take λ equal to an arbitrary positive integer say N. Denoting $\mathbf{1}_n$ the unit element of $B_m(\mathcal{H}_n)$, we have

$$n_{B_m}(N) = \tau\left(E_{B_m}(N,\infty)\right) = \sum_{m,k\geq 2} \alpha_k^m \mathrm{Tr}_k(\mathbf{1}_k) = \sum_{m,k\geq N} k\alpha_k^m = \infty$$

Observing that for every $n \geq 1$, we have $(B_m)_n = \sum_{k=1}^{n} \sum_m e_k^n (A_m)_n e_k^n$, where the sequence $(e_k^n)_{k=1}^{n}$ is a sequence of one dimensional projections from $B_m(\mathcal{H}_n)$ given by

$$e_k^n = \left(\delta_{ij}^k\right)_{i,j=1}^{n}, \text{ where } \delta_{ij}^k = 1, \text{ when } i = j = k, \text{ and } \delta_{ij}^k = 0, \text{ otherwise},$$

we arrive at the situation when for a τ-measurable operator A_m there exists a sequence of pairwise orthogonal projections $(e_n)_{n\geq 1} \subset \mathcal{N}$ such that $\sum_{n\geq 1} \sum_m e_n A_m e_n$ is not τ-measurable.

An argument above shows that the inequality established in Theorem 4.7 makes no sense for infinite sequences of pairwise orthogonal projections. Indeed, in the setting of that theorem, if for example $E = L_{1-\epsilon}, 0 \leq \epsilon < 1$, and $x^2 \in L_{1-\epsilon}(\mathcal{M},\tau)$, one simply cannot speak about $\|\sum_{n\geq 1} e_n x^2 e_n\|_{1-\epsilon}$ when the operator $\sum_{n\geq 1} e_n x^2 e_n$ fails to be τ-measurable.

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