Quest Journals
Journal of Software Engineering and Simulation
Volume 11 ~ Issue 1 (2025) pp: 09-29
ISSN(Online) :2321-3795 ISSN (Print):2321-3809
www.questjournals.org

Research Paper

Best Sharp Estimates of Unimodular Multipliers on Certain Frequency Decomposition Spaces

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Abstract
The pioneers' authors in [27] give a general method for the study of unimodular multipliers on certain function spaces defined by decomposition on the frequency plane. These basic spaces include modulation spaces, $(1-\epsilon)$-modulation spaces and (homogeneous and inhomogeneous) Besov spaces. We give a complete characterization of the Fourier multipliers on these valid function spaces, and a characterization of unimodular multipliers under some various assumptions. As applications, we obtain some sharp boundedness properties of unimodular Fourier multipliers between these function spaces. We also obtain the asymptotic estimates for certain free dispersive semigroups.
Keywords: Fourier multipliers, Modulation space, $(1-\epsilon)$-modulation space, Besov space, Characterization, Dispersive semigroup, Asymptotic estimates.

***R**eceived 19 Jan., 2024; **R**evised 31 Jan., 2025; **A**ccepted 03 Feb., 2025* *© The author(s) 2025.*
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I. Introduction

From the basic studies in harmonic analysis is to study the boundedness of certain linear operators on various function spaces or distribution spaces. These operators are usually raised from partial differential equation (PDE), mathematical physics, probability theory and other fields. We are interested with the Fourier multiplier operator $e^{i(1+\epsilon)|\Delta|^{\frac{\beta}{2}}}$, with the Laplace operator Δ, is the fundamental semi-group of the Schrödinger equation when $\beta = 2$ and is the fundamental semi-group of the wave equation when $\beta = 1$. On the other hand, choosing a right function space (or distribution space) as a ground frame is a crucial step in order to obtain the well-posedness of certain Cauchy or boundary value problems of partial differential equations. Thus, an important criteria or a compact features is to study various function spaces and distribution spaces to fit different PDE problems. So, one of the most efficient methods of defining function spaces is to use the frequency decomposition of $\mathbb{R}^n$. In the Euclidean space $\mathbb{R}^n$, a family of countable subsets $Q = \{Q_i\}_{i\in I}$ is called an admissible covering (see [10]) of $\mathbb{R}^n$ if
(1) $\mathbb{R}^n = \bigcup_{i\in I} Q_i$;
(2) $\sup_{i\in I} \#(\{j \in I : Q_i \cap Q_j \neq \emptyset\}) < \infty$.
For χ_{Q_i} be the characterization function of Q_i and $\tilde{\chi}_{Q_i}$ be a smooth modification of χ_{Q_i} so that it is a smooth bump functions related to Q_i for each $i \in I$. For sequences of distributions f_σ, consider the frequency projection

$$P_{Q_i}(f_\sigma) = \tilde{\chi}_{Q_i}\mathcal{F}(f_\sigma).$$

Here $\mathcal{F}$ is the Fourier transform with the inverse Fourier transform $\mathcal{F}^{-1}$. Choose an appropriate sequence of positive numbers $\{\lambda_j\}$. For $s \in \mathbb{R}$ and $1 \leq \epsilon \leq \infty$, we can define the norm

$$\| f_\sigma \|_{\Gamma^s_{1+\epsilon,1+2\epsilon}} = \left\| \left\{ \sum_{j\in I} \sum_{\sigma} \left(\lambda_j^s \left|\mathcal{F}^{-1}P_{Q_j}(f_\sigma)\right|\right)^{1+2\epsilon} \right\}^{\frac{1}{1+2\epsilon}} \right\|_{L^{1+\epsilon}}.$$

The space $\Gamma^s_{1+\epsilon,1+2\epsilon}$ then is the set of all f_σ satisfying $\| f_\sigma \|_{\Gamma^s_{1+\epsilon,1+2\epsilon}} < \infty$. When we let Q_j be the dyadic regions $\{\xi \in \mathbb{R}^n : 2^j \le |\xi| < 2^{j+1}\}$ and $\{\lambda_j\} = \{2^j\}$, the space $\Gamma^s_{1+\epsilon,1+2\epsilon}$ becomes the homogeneous Triebel–Lizorkin space that includes the classical Lebesgue space $L^{1+\epsilon}$ and the Sobolev space $W^{1+\epsilon,s}$ if one chooses appropriate parameters s and $(1+2\epsilon)$. Related to the Triebel–Lizorkin space is the famous Littlewood–Paley theory that plays a remarkable role in the study of harmonic analysis and PDE.
Also, we can define the frequency decomposition space

$$\Phi^s_{1+\epsilon,1+2\epsilon} = \left\{ f_\sigma : \| f_\sigma \|_{\Phi^s_{1+\epsilon,1+2\epsilon}} = \left\{ \sum_{j\in I} \sum_{\sigma} \left(\lambda_j^s \left\| \mathcal{F}^{-1} P_{Q_j}(f_\sigma) \right\| \right)^{1+2\epsilon} \right\}^{\frac{1}{1+2\epsilon}} < \infty \right\}.$$

$\Phi^s_{1+\epsilon,1+2\epsilon}$ becomes the modulation space $M^s_{1+\epsilon,1+2\epsilon}$ if we choose the uniform unit cubes $\{Q_j : j \in \mathbb{Z}^n\}$ in the frequency space and $\lambda_j = (1 + |j|^2)^{\frac{1}{2}}$ and $\Phi^s_{1+\epsilon,1+2\epsilon}$ becomes the Besov space $B^s_{1+\epsilon,1+2\epsilon}$ if we choose a dyadic decomposition on the frequency space. Besides $M^s_{1+\epsilon,1+2\epsilon}$ and $B^s_{1+\epsilon,1+2\epsilon}$, the $(1-\epsilon)$-modulation spaces $M^{s,1-\epsilon}_{1+\epsilon,1+2\epsilon}$, $0 < \epsilon \le 1$, can be derived from $\Phi^s_{1+\epsilon,1+2\epsilon}$ by choosing a sequence of $(1-\epsilon)$ coverings (see Section 2).

First of all we give the precise definitions of the spaces $M^s_{1+\epsilon,1+2\epsilon}$, $B^s_{1+\epsilon,1+2\epsilon}$ and $M^{s,1-\epsilon}_{1+\epsilon,1+2\epsilon}$. The Besov space is a well known function space. We leave it to the reader. Below, we keep the statements of the historical developments of the modulation spaces $M^s_{1+\epsilon,1+2\epsilon}$ and $(1-\epsilon)$-modulation spaces $M^{s,1-\epsilon}_{1+\epsilon,1+2\epsilon}$, with a bit change if possible.

The modulation space $M^s_{1+\epsilon,1+2\epsilon}$ was originally introduced by Feichtinger [8] in 1983 by the short-time Fourier transform. It can be used to measure the size and smoothness of a function in a way different from the $L^{1+\epsilon}$ space and the Sobolev space. It have been discover that this space has a discrete version based on the uniform unit decomposition on the frequency space. Based on this alternative definition, many notable performances were showed on the modulation space when deal with PDE and pseudo-differential operators. For more knowledge on the modulation space, see [8, 12, 22] for many elementary properties of modulation space, [2, 1, 3] for the study of boundedness on modulation spaces for certain operators and [4, 22, 21, 23] for the study of nonlinear evolution equations related to modulation space.

As we mentioned above, the inhomogeneous Besov space $B^s_{1+\epsilon,1+2\epsilon}$ is another frequency-decomposition function space based on the dyadic decomposition. Thus, it is interesting to build a bridge connecting the modulation space and the Besov space. To this end, the general framework of a decomposition method considered by Feichtinger and Gröbner in [9, 7] to construct the $(1-\epsilon)$-modulation spaces, which is an intermediate space between the modulation space and the Besov space with respect to the parameters $0 \le \epsilon \le 1$. We also see [14], which contains a comprehensive study of $(1-\epsilon)$-modulation spaces. Modulation spaces is a special $(1-\epsilon)$-modulation space in the case $\epsilon = 1$, and the Besov space $B^s_{1+\epsilon,1+2\epsilon}$ can be regarded as the limit case of $M^{s,1-\epsilon}_{1+\epsilon,1+2\epsilon}$ as $\epsilon \to 2$ (see [11]). So, for the sake of convenience, we can view the Besov space as a special $(1-\epsilon)$-modulation space and use $M^{s,1}_{1+\epsilon,1+2\epsilon}$ to denote the inhomogeneous Besov space $B^s_{1+\epsilon,1+2\epsilon}$. In [13], the conclusion on $(1-\epsilon)$-modulation spaces was proved to be unable to be obtained by a simple interpolation between modulations and Besov spaces.

In addition to the above function spaces, the homogeneous Besov space $\dot{B}^s_{1+\epsilon,1+2\epsilon}$ is another important function space, which is also a popular working frame in the study of partial differential equations. Now, we refer the $(1-\epsilon)$-modulation spaces and (inhomogeneous and homogeneous) Besov spaces collectively as frequency decomposition spaces.

We study certain unimodular Fourier multipliers on the frequency decomposition spaces (see [27]). Suppose X and $X+\epsilon$ are two function spaces. We call a tempered distribution m a Fourier multiplier from X to $X+\epsilon$, if there exists a constant $\epsilon \ge 0$ such that

$$\| \sum_{\sigma} T_m(f_\sigma) \|_{X+\epsilon} \le (1+\epsilon) \sum_{\sigma} \| f_\sigma \|_X,$$

for all sequences f_σ in the Schwartz space $S(\mathbb{R}^n)$, where

$$T_m f_\sigma = m(D) f_\sigma = \mathcal{F}^{-1}(m \mathcal{F} f_\sigma)$$

is the Fourier multiplier operator associated with m, and m is called the symbol or multiplier of T_m. Let $\mathcal{M}_{\mathcal{F}}(X, X+\epsilon)$ denote the set of all symbols such that the corresponding Fourier multipliers are bounded from X to $X+\epsilon$. We set the operator norm of T_m in the following:

$$\| m | \mathcal{M}_{\mathcal{F}}(X, X+\epsilon) \| = \| T_m \|_{X \to X+\epsilon} = \sup \{ \| T_m f_\sigma \|_{X+\epsilon} : f_\sigma \in S(\mathbb{R}^n), \| f_\sigma \| X = 1 \}.$$

Fourier multipliers arise naturally in formal solutions of linear partial differential equations with constant coefficients and in the summabilities of Fourier series. So as before, the linear operator $e^{i(1+\epsilon)|\Delta|^{\frac{\beta}{2}}}$ is the fundamental solution of the Schrödinger equation if $\beta = 2$, and the fundamental solution of the wave equation if $\beta = 1$. This operator is a Fourier multiplier with symbol $e^{i(1+\epsilon)|\xi|^\beta}$. Notice that a Fourier multiplier T_m is a convolution operator, and the frequency decomposition space $\Phi^s_{1+\epsilon,1+2\epsilon}$ is defined also using the convolution on each piece of decomposition. By the commutative property of the convolution operators, we have

$$T_m \mathcal{F}^{-1} P_{Q_j}(f_\sigma) = \mathcal{F}^{-1} P_{Q_j}(T_m f_\sigma).$$

Thus, a Fourier multiplier bounded on the Lebesgue space $L^{1+\epsilon}$ is automatically bounded on any frequency decomposition space $\Phi^s_{1+\epsilon,1+2\epsilon}$. But it is not true vice versa. We know that the Fourier multiplier $e^{i|\Delta|}$ is not bounded on any $L^{1+\epsilon}$ except in the case $\epsilon = 1$, while this operator is bounded on any modulation space $M^s_{1+\epsilon,1+2\epsilon}$. Thus, establishing the boundedness of Fourier multipliers on the modulation spaces, if they are unbounded on the Lebesgue spaces. Particularly, if such a Fourier multiplier was raised from some PDE problems, we naturally expect that the modulation spaces can serve a good substitution of the Lebesgue spaces and that it together with the Fourier multiplier play a notable role to study certain well-posedness of the corresponding PDE problem. Based on the above, we extend one of the main results in [10] to establish characterization of the boundedness for the Fourier multiplier on the $(1-\epsilon)$-modulation space $M^{s,1-\epsilon}_{1+\epsilon,1+2\epsilon}$ and on the Besov spaces. We use a quite different method from [10] to prove the theorem by means of the corresponding Wiener amalgam spaces, before we present some preliminary knowledge in Section 2. We give two applications for this boundedness criterion. The first application is a simple proof for the sharpness of embedding between different $(1-\epsilon)$-modulation spaces. In the second application, under some mild assumptions on the function μ_σ, for the unimodular multipliers $e^{i\mu_\sigma(D)}$ we will establish its sufficient and necessary conditions of the boundedness on the $(1-\epsilon)$-modulation spaces. We also obtain the asymptotic estimates for the operator norms of $e^{i(1+\epsilon)\mu_\sigma(D)}$. These results are substantial extensions to all the previous known results. We can see the details in Theorems 4.2, 4.8, 4.9, 4.10 and some comments related to these theorems (see [27]).

2. Preliminaries

Now let C be a positive constant that may depend on $n, 1+\epsilon, 1+2\epsilon, s_i, 1-\epsilon, \beta$. The notation $X \lesssim X+\epsilon$ denotes the statement that $X \le C(X+\epsilon)$, the notation $X \sim X+\epsilon$ means the statement $X \lesssim X+\epsilon \lesssim X$, and the notation $X \simeq X+\epsilon$ denotes the statement $X = C(X+\epsilon)$. For a multi-index $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$, we denote

$$|k|_\infty := \max_{i=1,2\dots n} |k_i|, \qquad \langle k \rangle := (1 + |k|^2)^{\frac{1}{2}}.$$

Let $S := S(\mathbb{R}^n)$ be the Schwartz space and $S' := S'(\mathbb{R}^n)$ be the space of all tempered distributions. The Fourier transform $\mathcal{F} f_\sigma$ and the inverse Fourier $\mathcal{F}^{-1} f_\sigma$ of $f_\sigma \in S(\mathbb{R}^n)$ is defined by

$$\mathcal{F} f_\sigma(\xi) = \hat{f_\sigma}(\xi) = \int_{\mathbb{R}^n} \sum_\sigma f_\sigma(x) e^{-2\pi i x\cdot\xi} dx, \qquad \mathcal{F}^{-1} f_\sigma(x) = \hat{f_\sigma}(-x) = \int_{\mathbb{R}^n} \sum_\sigma f_\sigma(\xi) e^{2\pi i x\cdot\xi} d\xi.$$

We now bring the definitions of some function spaces that will be discussed here. First, we need to give the partition of unity on frequency space for $0 < \epsilon \le 1$. Suppose that $c > 0$ and $C > 0$ are two appropriate constants, and choose a Schwartz function sequence $\{\eta_k^{1-\epsilon}\}_{k\in\mathbb{Z}^n}$ satisfying

$$\begin{cases} |\eta_k^{1-\epsilon}(\xi)| \ge 1, \quad \text{if } \left|\xi - \langle k\rangle^{\frac{1-\epsilon}{\epsilon}} k\right| < c\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}; \\ \operatorname{supp}\eta_k^{1-\epsilon} \subset \left\{\xi : \left|\xi - \langle k\rangle^{\frac{1-\epsilon}{\epsilon}} k\right| < C\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}\right\}; \\ \sum_{k\in\mathbb{Z}^n} \eta_k^{1-\epsilon}(\xi) \equiv 1, \quad \forall \xi \in \mathbb{R}^n; \\ |\partial^\gamma \eta_k^{1-\epsilon}(\xi)| \le C_{1-\epsilon} \langle k\rangle^{-\frac{(1-\epsilon)|\gamma|}{\epsilon}}, \quad \forall\xi \in \mathbb{R}^n, \quad \gamma \in (\mathbb{Z}^+ \cup \{0\})^n, \end{cases}$$

where $C_{1-\epsilon}$ is a positive constant depending only on n and $(1-\epsilon)$. This $\{\eta_k^{1-\epsilon}(\xi)\}_{k\in\mathbb{Z}^n}$ constitutes a smooth decomposition of $\mathbb{R}^n$. The frequency decomposition operators associated with above function sequence can be defined by

$$\Box_k^{1-\epsilon} := \mathcal{F}^{-1} \eta_k^{1-\epsilon} \mathcal{F}$$

for $k \in \mathbb{Z}^n$. Let $1 \le \epsilon \le \infty, s \in \mathbb{R}, 0 < \epsilon \le 1$. The $(1-\epsilon)$-modulation space associated with above decomposition is defined by

$$M^{s,1-\epsilon}_{1+\epsilon,1+2\epsilon}(\mathbb{R}^n) = \left\{ f_\sigma \in S'(\mathbb{R}^n) : \| f_\sigma \|_{M^{s,1-\epsilon}_{1+\epsilon,1+2\epsilon}(\mathbb{R}^n)} = \left(\sum_{k\in\mathbb{Z}^n} \sum_\sigma \langle k\rangle^{\frac{s(1+2\epsilon)}{\epsilon}} \| \Box_k^{1-\epsilon} f_\sigma \|_{L^{1+\epsilon}}^{1+2\epsilon} \right)^{\frac{1}{1+2\epsilon}} < \infty \right\}$$

with the usual modification when $\epsilon = \infty$. For simplicity, we denote $M^s_{1+\epsilon,1+2\epsilon} = M^{s,0}_{1+\epsilon,1+2\epsilon}$ and $\eta_k(\xi) = \eta^0_k(\xi)$. For $0 < \epsilon \leq 1$ and each $k \in \mathbb{Z}^n$, define two subsets of $\mathbb{Z}^n$ by

$$\Lambda^{1-\epsilon}_k = \{l \in \mathbb{Z}^n : \square^{1-\epsilon}_l \circ \square^{1-\epsilon}_k \neq 0\} \tag{2.1}$$

And

$$\Lambda^{1-\epsilon,*}_k = \left\{l \in \mathbb{Z}^n : \square^{1-\epsilon}_l \circ \square^{1-\epsilon}_m \neq 0 \text{ for some } m \in \Lambda^{1-\epsilon}_k\right\}. \tag{2.2}$$

We denote

$$\square^{1-\epsilon,*}_k = \sum_{l\in\Lambda^{1-\epsilon}_k} \square^{1-\epsilon}_l, \qquad \eta^{1-\epsilon,*}_k = \sum_{l\in\Lambda^{1-\epsilon}_k} \eta^{1-\epsilon}_l. \tag{2.3}$$

Next, we recall a standard dyadic decomposition of $\mathbb{R}^n$. Let $\varphi(\xi)$ be a smooth bump function supported in the ball $\left\{\xi : |\xi| < \frac{3}{2}\right\}$ and be equal to 1 on the ball $\left\{\xi : |\xi| \leq \frac{4}{3}\right\}$. Denote

$$\psi(\xi) = \varphi(\xi) - \varphi(2\xi), \tag{2.4}$$

and a function sequence

$$\psi_j(\xi) = \psi(2^{-j}\xi), \qquad j \in \mathbb{Z}. \tag{2.5}$$

For integers $j \in \mathbb{Z}^+$, we define the Littlewood–Paley operators

$$\Delta_j = \mathcal{F}^{-1}\psi_j(\xi)\mathcal{F},$$
$$\Delta_0 = \mathcal{F}^{-1}\varphi(\xi)\mathcal{F}.$$

Let $1 \leq \epsilon \leq \infty$ and $s \in \mathbb{R}$. For $f_\sigma \in S'$ we set

$$\| f_\sigma \|_{B^s_{1+\epsilon,1+2\epsilon}} = \left(\sum_{j=0}^{\infty} \sum_{\sigma} 2^{js(1+2\epsilon)} \| \Delta_j f_\sigma \|^{1+2\epsilon}_{L^{1+\epsilon}}\right)^{\frac{1}{1+2\epsilon}}, \tag{2.6}$$

with the usual modification when $\epsilon = \infty$. The (inhomogeneous) Besov space is the space of all tempered distributions f_σ satisfying $\| f_\sigma \|_{B^s_{1+\epsilon,1+2\epsilon}} < \infty$.

To introduce the homogeneous Besov spaces, we define the Littlewood–Paley projections

$$\dot{\Delta}_j = \mathcal{F}^{-1}\psi j(\xi)\mathcal{F}$$

for $j \in \mathbb{Z}$. We have $I = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j$ in the sense of $S'/\mathcal{P}$. Let $1 \leq \epsilon \leq \infty$ and $s \in \mathbb{R}$. For $f_\sigma \in S'/\mathcal{P}$, we set

$$\| f_\sigma \|_{\dot{B}^s_{1+\epsilon,1+2\epsilon}} = \left(\sum_{j\in\mathbb{Z}} \sum_{\sigma} 2^{js(1+2\epsilon)} \| \dot{\Delta}_j f_\sigma \|^{1+2\epsilon}_{L^{1+\epsilon}}\right)^{\frac{1}{1+2\epsilon}}, \tag{2.7}$$

with usual modification if $\epsilon = \infty$. The homogeneous Besov space is the space of all $f_\sigma \in S'/\mathcal{P}$ satisfying $\| f_\sigma \|_{\dot{B}^s_{1+\epsilon,1+2\epsilon}} < \infty$.

Now, we introduce the definitions of Wiener amalgam spaces associated with some exact decomposition methods.

Suppose $0 \leq \epsilon \leq \infty, 0 < \epsilon \leq 1$ and $s \in \mathbb{R}$. Let $\{a^\sigma_k\}_{\sigma,k\in\mathbb{Z}^n}$ denote a sequence of complex numbers. Set

$$\| \{a^\sigma_k\} \|_{l^{s,1-\epsilon}_{1+\epsilon}} = \begin{cases} \left(\sum_{k\in\mathbb{Z}^n} \sum_{\sigma} \langle k\rangle^{\frac{s(1+\epsilon)}{\epsilon}} |a^\sigma_k|^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}} & \text{if } 0 \leq \epsilon < \infty \\ \sup_{k\in\mathbb{Z}^n} \sum_{\sigma} \left(\langle k\rangle^{\frac{s}{\epsilon}} |a^\sigma_k|\right) & \text{if } \epsilon = \infty. \end{cases} \tag{2.8}$$

We use $l^{s,1-\epsilon}_{1+\epsilon}$ to denote the set of all sequences $\{a^\sigma_k\}_{\sigma,k\in\mathbb{Z}^n}$ such that $\| \{a^\sigma_k\} \|_{l^{s,1-\epsilon}_{1+\epsilon}} < \infty$. Similarly, we use $l^{s,1}_{1+\epsilon}$ to denote the set of all sequences $\left\{a^\sigma_j\right\}_{\sigma,j\in\mathbb{N}}$ such that

$$\| \left\{a^\sigma_j\right\} \|_{l^{s,1}_{1+\epsilon}} = \begin{cases} \left(\sum_{j\in\mathbb{N}} \sum_{\sigma} 2^{js}\left|a^\sigma_j\right|^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}} & \text{if } 0 \leq \epsilon < \infty \\ \sup_{j\in\mathbb{N}} \sum_{\sigma} \left(2^{js}\left|a^\sigma_j\right|\right) & \text{if } \epsilon = \infty \end{cases} \tag{2.9}$$

is finite. Denote by $\dot{l}^{s,1}_{1+\epsilon}$ the set of all sequence $\left\{a^\sigma_j\right\}_{\sigma,j\in\mathbb{N}}$ such that

$$\| \{a_j^\sigma\} \|_{l_{1+\epsilon}^{s,1}} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} \sum_{\sigma} 2^{js} |a_j^\sigma|^{1+\epsilon} \right)^{\frac{1}{1+\epsilon}} & \text{if } 0 \le \epsilon < \infty \\ \sup_{j\in\mathbb{Z}} \sum_{\sigma} \left(2^{js}|a_j^\sigma|\right) & \text{if } \epsilon = \infty \end{cases} \quad (2.10)$$

is finite.

Next, we define the space of pointwise multipliers between sequence spaces. For $0 < \epsilon \le 1$, we set

$$\mathcal{M}_{1+\epsilon}\left(l_{1+\epsilon}^{s_1,1-\epsilon}, l_{1+2\epsilon}^{s_2,1-\epsilon}\right) = \left\{ \{a_k^\sigma\}_{\sigma,k\in\mathbb{Z}^n} : \| \{a_k^\sigma \lambda_k\} \|_{l_{1+2\epsilon}^{s_2,1-\epsilon}} \le C \| \{\lambda_k\} \|_{l_{1+\epsilon}^{s_1,1-\epsilon}} \right\} \quad (2.11)$$

and denote the norm of $\{a_k^\sigma\} \in \mathcal{M}_{1+\epsilon}\left(l_{1+\epsilon}^{s_1,1-\epsilon}, l_{1+2\epsilon}^{s_2,1-\epsilon}\right)$ by

$$\| \{a_k^\sigma\} | \mathcal{M}_{1+\epsilon}\left(l_{1+\epsilon}^{s_1,1-\epsilon}, l_{1+2\epsilon}^{s_2,1-\epsilon}\right) \| = \| \{a_k^\sigma\} \|_{l_{1+\epsilon}^{s_1,1-\epsilon} \to l_{1+2\epsilon}^{s_2,1-\epsilon}} = \sup_{\|\{\lambda_k\}| l_{1+\epsilon}^{s_1,1-\epsilon}\| = 1} \| \{a_k^\sigma \lambda_k\} \|_{l_{1+2\epsilon}^{s_2,1-\epsilon}}. \quad (2.12)$$

For a tempered distribution m, we denote

$$\begin{aligned} &\left\| m | W^{1-\epsilon} \left(\mathcal{M}_{\mathfrak{F}}\left(L^{1+\epsilon}, L^{1+2\epsilon}\right), \mathcal{M}_{1+\epsilon}\left(l_{1+\epsilon}^{s_1}, l_{1+2\epsilon}^{s_2}\right)\right) \right\| \\ &= \left\| \left\{ \| \square_k^{1-\epsilon} T_m \|_{L^{1+\epsilon} \to L^{1+2\epsilon}} \right\} \right\|_{l_{1+\epsilon}^{s_1,1-\epsilon} \to l_{1+2\epsilon}^{s_2,1-\epsilon}} \end{aligned} \quad (2.13)$$

for $0 < \epsilon \le 1$. The space $W^{1-\epsilon}\left(\mathcal{M}_{\mathfrak{F}}\left(L^{1+\epsilon}, L^{1+2\epsilon}\right), \mathcal{M}_{1+\epsilon}\left(l_{1+\epsilon}^{s_1}, l_{1+2\epsilon}^{s_2}\right)\right)$ is the set of all tempered distributions m satisfying

$$\left\| m | W^{1-\epsilon} \left(\mathcal{M}_{\mathfrak{F}}\left(L^{1+\epsilon}, L^{1+2\epsilon}\right), \mathcal{M}_{1+\epsilon}\left(l_{1+\epsilon}^{s_1}, l_{1+2\epsilon}^{s_2}\right)\right) \right\| < \infty.$$

This space is called the Wiener amalgam space associated to the $(1-\epsilon)$-modulation space $M_{1+\epsilon,1+\epsilon}^{s,1-\epsilon}$.

Similarly, we can define the Wiener amalgam spaces associated to the inhomogeneous Besov space $B_{1+\epsilon,1+\epsilon}^{s}$ and homogeneous Besov space $\dot{B}_{1+\epsilon,1+\epsilon}^{s}$, respectively, by defining their norms

$$\left\| m | W^{1} \left(\mathcal{M}_{\mathfrak{F}}\left(L^{1+\epsilon}, L^{1+2\epsilon}\right), \mathcal{M}_{1+\epsilon}\left(l_{1+\epsilon}^{s_1}, l_{1+2\epsilon}^{s_2}\right)\right) \right\| = \| \{ \| \Delta_j T_m \|_{(L^{1+\epsilon} \to L^{1+2\epsilon}}\} \|_{l_{1+\epsilon}^{s_1,1} \to l_{1+2\epsilon}^{s_2,1}} \quad (2.14)$$

and

$$\left\| m | \dot{W}^{1} \left(\mathcal{M}_{\mathfrak{F}}\left(L^{1+\epsilon}, L^{1+2\epsilon}\right), \mathcal{M}_{1+\epsilon}\left(l_{1+\epsilon}^{s_1}, l_{1+2\epsilon}^{s_2}\right)\right) \right\| = \| \{ \| \dot{\Delta}_j T_m \|_{(L^{1+\epsilon} \to L^{1+2\epsilon}}\} \|_{l_{1+\epsilon}^{s_1,1} \to l_{1+2\epsilon}^{s_2,1}}. \quad (2.15)$$

3. Fourier Multipliers on Function Spaces

To study the unimodular multipliers by a unified approach on frequency decomposition spaces, we give characterizations of Fourier multipliers by means of the corresponding Wiener amalgam spaces. We give an elementary proof without the aid of advanced theorems such as open mapping theorem used in [10]. The results fall into Theorem 2.11 in [9], in which Feichtinger obtained the characterization of the Fourier multiplier on an abstract Banach space frame by a contradiction argument. We use an alternative, but more elementary proof to obtain our result. More importantly, our simple method allows us to extend a corresponding result in [10] on the modulation space to the $(1-\epsilon)$-modulation spaces for all $0 \le \epsilon \le 1$.

Theorem 3.1 (see [27]) (Characterization of Fourier Multiplier on Function Spaces). Let $0 \le \epsilon \le \infty, s \in \mathbb{R}, 0 \le \epsilon \le 1$, and $m \in S'$. Then we have

$$\mathcal{M}_{\mathfrak{F}}\left(M_{1+\epsilon,1+\epsilon}^{s_1,1-\epsilon}, M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon}\right) = W^{1-\epsilon}\left(\mathcal{M}_{\mathfrak{F}}\left(L^{1+\epsilon}, L^{1+2\epsilon}\right), \mathcal{M}_{1+\epsilon}\left(l_{1+\epsilon}^{s_1}, l_{1+2\epsilon}^{s_2}\right)\right) \quad (3.1)$$

and

$$\mathcal{M}_{\mathfrak{F}}\left(\dot{B}_{1+\epsilon,1+\epsilon}^{s_1}, \dot{B}_{1+2\epsilon,1+2\epsilon}^{s_2}\right) = \dot{W}^{1}\left(\mathcal{M}_{\mathfrak{F}}\left(L^{1+\epsilon}, L^{1+2\epsilon}\right), \mathcal{M}_{1+\epsilon}\left(l_{1+\epsilon}^{s_1}, l_{1+2\epsilon}^{s_2}\right)\right). \quad (3.2)$$

Proof. We only state the proof for $(1-\epsilon)$-modulation spaces for $0 < \epsilon \le 1$, since the proof for other case is similar.

Firstly, assume $m \in W^{1-\epsilon}\left(\mathcal{M}_{\mathfrak{F}}\left(L^{1+\epsilon}, L^{1+2\epsilon}\right), \mathcal{M}_{1+\epsilon}\left(l_{1+\epsilon}^{s_1}, l_{1+2\epsilon}^{s_2}\right)\right)$. Then, for $k \in \mathbb{Z}^n, \square_k^{1-\epsilon} T_m \in \mathcal{L}\left(L^{1+\epsilon}, L^{1+2\epsilon}\right)$ and $\{\| \square_k^{1-\epsilon} T_m \|_{L^{1+\epsilon} \to L^{1+2\epsilon}}\} \in M_{1+\epsilon}\left(l_{1+\epsilon}^{s_1,1-\epsilon}, l_{1+2\epsilon}^{s_2,1-\epsilon}\right)$. For any $f_\sigma \in S$, we have

$$\begin{aligned} \| \sum_\sigma \square_k^{1-\epsilon} T_m f_\sigma \|_{L^{1+2\epsilon}} &= \| \sum_\sigma \square_k^{1-\epsilon} T_m \square_k^{1-\epsilon,*} f_\sigma \|_{L^{1+2\epsilon}} \\ &\lesssim \sum_\sigma \| \square_k^{1-\epsilon} T_m \|_{L^{1+\epsilon} \to L^{1+2\epsilon}} \| \square_k^{1-\epsilon,*} f_\sigma \|_{L^{1+\epsilon}}. \end{aligned} \quad (3.3)$$

Hence,

$$\| \sum_\sigma T_m f_\sigma \|_{M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon}} = \| \sum_\sigma \left\{ \| \square_k^{1-\epsilon} T_m f_\sigma \|_{L^{1+2\epsilon}} \right\} \|_{l_{1+2\epsilon}^{s_2,1-\epsilon}}$$

$$\lesssim \sum_{\sigma} \| \left\{ \| \square_k^{1-\epsilon} T_m \|_{L^{1+\epsilon} \to L^{1+2\epsilon}} \| \square_k^{1-\epsilon,*} f_\sigma \|_{L^{1+\epsilon}} \right\} \|_{l_{1+2\epsilon}^{s_2,1-\epsilon}}$$

$$\lesssim \sum_{\sigma} \| \left\{ \| \square_k^{1-\epsilon} T_m \|_{L^{1+\epsilon} \to L^{1+2\epsilon}} \right\} \|_{l_{1+\epsilon}^{s_1,1-\epsilon} \to l_{1+2\epsilon}^{s_2,1-\epsilon}} \| \left\{ \| \square_k^{1-\epsilon,*} f_\sigma \|_{L^{1+\epsilon}} \right\} \|_{l_{1+\epsilon}^{s_1,1-\epsilon}}$$

$$\lesssim \sum_{\sigma} \left\| m | W^{1-\epsilon} \left(\mathcal{M}_{\mathfrak{F}} (L^{1+\epsilon}, L^{1+2\epsilon}), \mathcal{M}_{1+\epsilon}\left(l_{1+\epsilon}^{s_1}, l_{1+2\epsilon}^{s_2}\right)\right) \right\| \ \| f_\sigma \|_{M_{1+\epsilon,1+2\epsilon}^{s_1,1-\epsilon}}, \qquad (3.4)$$

which implies

$$\left\| m | \mathcal{M}_{\mathfrak{F}}\left(M_{1+\epsilon,1+\epsilon}^{s_1,1-\epsilon}, M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon}\right) \right\| \lesssim \left\| m | W^{1-\epsilon} \left(\mathcal{M}_{\mathfrak{F}} (L^{1+\epsilon}, L^{1+2\epsilon}), \mathcal{M}_{1+\epsilon}\left(l_{1+\epsilon}^{s_1}, l_{1+2\epsilon}^{s_2}\right)\right) \right\|. \qquad (3.5)$$

Next, we assume $m \in \mathcal{M}_{\mathfrak{F}}\left(M_{1+\epsilon,1+\epsilon}^{s_1,1-\epsilon}, M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon}\right)$. We obtain that

$$\| \sum_{\sigma} \square_k^{1-\epsilon} T_m f_\sigma \|_{L^{1+2\epsilon}} \sim \langle k \rangle^{\frac{-s_2}{\epsilon}} \| \sum_{\sigma} \square_k^{1-\epsilon} T_m f_\sigma \|_{M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon}}$$

$$= \langle k \rangle^{\frac{-s_2}{\epsilon}} \| \sum_{\sigma} T_m \square_k^{1-\epsilon} f_\sigma \|_{M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon}}$$

$$\lesssim \sum_{\sigma} \langle k \rangle^{\frac{-s_2}{\epsilon}} \left\| m | \mathcal{M}_{\mathfrak{F}}\left(M_{1+\epsilon,1+\epsilon}^{s_1,1-\epsilon}, M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon}\right) \right\| \ \| \square_k^{1-\epsilon} f_\sigma \|_{M_{1+\epsilon,1+\epsilon}^{s_1,1-\epsilon}}$$

$$\lesssim \sum_{\sigma} \langle k \rangle^{\frac{s_1-s_2}{\epsilon}} \left\| m | \mathcal{M}_{\mathfrak{F}}\left(M_{1+\epsilon,1+\epsilon}^{s_1,1-\epsilon}, M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon}\right) \right\| \| f_\sigma \|_{L^{1+\epsilon}}. \qquad (3.6)$$

Thus, we have $\square_k^{1-\epsilon} T_m \in \mathcal{L}(L^{1+\epsilon}, L^{1+2\epsilon})$.

By the spirit of the disjointization lemma of decomposition (see Lemma 2 in [10]), one can find a subset of $\mathbb{Z}^n$ denoted by Γ_m, which depends on exact m, such that

$$\Lambda_k^{1-\epsilon,*} \cap \Lambda_l^{1-\epsilon,*} = \emptyset \qquad (3.7)$$

for any $k, l \in \Gamma_m, k \neq l$, and

$$\left\| m | W^{1-\epsilon} \left(\mathcal{M}_{\mathfrak{F}} (L^{1+\epsilon}, L^{1+2\epsilon}), \mathcal{M}_{1+\epsilon}\left(l_{1+\epsilon}^{s_1}, l_{1+2\epsilon}^{s_2}\right)\right) \right\|$$
$$\leq C \left\| \left\{ \chi_{\Gamma_m} \| \square_k^{1-\epsilon} T_m \|_{L^{1+\epsilon} \to L^{1+2\epsilon}} \right\} \right\|_{l_{1+\epsilon}^{s_1,1-\epsilon} \to l_{1+2\epsilon}^{s_2,1-\epsilon}}, \qquad (3.8)$$

where the constant C is independent of the exact m. For every $k \in \Gamma_m$, one can find $(f_\sigma)_k \in \mathcal{S}, (f_\sigma)_k \neq 0$ such that

$$\sum_{\sigma} \| \square_k^{1-\epsilon} T_m (f_\sigma)_k \|_{L^{1+2\epsilon}} \gtrsim \| \sum_{\sigma} \square_k^{1-\epsilon} T_m \|_{L^{1+\epsilon} \to L^{1+2\epsilon}} \| (f_\sigma)_k \|_{L^{1+\epsilon}}. \qquad (3.9)$$

It then follows

$$\sum_{\sigma} \| \square_k^{1-\epsilon} T_m \square_k^{1-\epsilon,*} (f_\sigma)_k \|_{L^{1+2\epsilon}} \gtrsim \| \sum_{\sigma} \square_k^{1-\epsilon} T_m \|_{L^{1+\epsilon} \to L^{1+2\epsilon}} \| \square_k^{1-\epsilon,*} (f_\sigma)_k \|_{L^{1+\epsilon}}. \qquad (3.10)$$

For any nonnegative sequence $\{a_k^\sigma\}_{\sigma, k \in \Gamma_m}$, we have

$$\| \sum_{\sigma} \{ a_k^\sigma \| \square_k^{1-\epsilon} T_m \|_{L^{1+\epsilon} \to L^{1+2\epsilon}} \| \square_k^{1-\epsilon,*} (f_\sigma)_k \|_{L^{1+\epsilon}} \} | l_{1+2\epsilon}^{s_2,1-\epsilon} (\Gamma_m) \|$$

$$\lesssim \sum_{\sigma} \| \{ a_k^\sigma \| \square_k^{1-\epsilon} T_m (f_\sigma)_k \|_{L^{1+2\epsilon}} \} | l_{1+2\epsilon}^{s_2,1-\epsilon} (\Gamma_m) \|$$

$$\lesssim \left\| \sum_{k \in \Gamma_m} \sum_{\sigma} a_k^\sigma \square_k^{1-\epsilon,*} T_m (f_\sigma)_k \right\|_{M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon}}$$

$$= \left\| T_m \left(\sum_{k \in \Gamma_m} \sum_{\sigma} a_k^\sigma \square_k^{1-\epsilon,*} (f_\sigma)_k \right) \right\|_{M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon}}$$

$$\lesssim \sum_{\sigma} \left\| m | \mathcal{M}_{\mathfrak{F}}\left(M_{1+\epsilon,1+\epsilon}^{s_1,1-\epsilon}, M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon}\right) \right\| \left\| \sum_{k \in \Gamma_m} a_k^\sigma \square_k^{1-\epsilon,*} (f_\sigma)_k \right\|_{M_{1+\epsilon,1+\epsilon}^{s_1,1-\epsilon}}$$

$$\sim \left\| m | \mathcal{M}_{\mathfrak{F}}\left(M_{1+\epsilon,1+\epsilon}^{s_1,1-\epsilon}, M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon}\right) \right\| \sum_{\sigma} \| \{ a_k^\sigma \| \square_k^{1-\epsilon,*} (f_\sigma)_k \|_{L^{1+\epsilon}} \} | l_{1+\epsilon}^{s_1,1-\epsilon} (\Gamma_m) \|. \qquad (3.11)$$

Since the sequence $\{a_k^\sigma\}_{\sigma,k\in\Gamma_u}$ is arbitrary, we have that

$$\left\|\left\{\| \square_k^{1-\epsilon} T_m \|_{L^{1+\epsilon}\to L^{1+2\epsilon}}\right\}\right\|_{l_{1+\epsilon}^{s_1,1-\epsilon}(\Gamma_m)\to l_{1+2\epsilon}^{s_2,1-\epsilon}(\Gamma_m)} \lesssim \left\|m\right| \mathcal{M}_{\mathcal{J}}\left(M_{1+\epsilon,1+\epsilon}^{s_1,1-\epsilon}, M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon}\right)\right\|. \quad (3.12)$$

By the fact

$$\left\|\left\{\chi_{\Gamma_m} \| \square_k^{1-\epsilon} T_m \|_{L^{1+\epsilon}\to L^{1+2\epsilon}}\right\}\right\|_{l_{1+\epsilon}^{s_1,1-\epsilon}\to l_{1+2\epsilon}^{s_2,1-\epsilon}}$$
$$= \left\|\left\{\| \square_k^{1-\epsilon} T_m \|_{L^{1+\epsilon}\to L^{1+2\epsilon}}\right\}\right\|_{l_{1+\epsilon}^{s_1,1-\epsilon}(\Gamma_m)\to l_{1+2\epsilon}^{s_2,1-\epsilon}(\Gamma_m)}, \quad (3.13)$$

we deduce the desired inequality

$$\left\|m\right| W^{1-\epsilon}\left(\mathcal{M}_{\mathcal{J}}\left(L^{1+\epsilon}, L^{1+2\epsilon}\right), \mathcal{M}_{1+\epsilon}\left(l_{1+\epsilon}^{s_1}, l_{1+2\epsilon}^{s_2}\right)\right)\right\| \lesssim \left\|m\right| \mathcal{M}_{\mathcal{J}}\left(M_{1+\epsilon,1+\epsilon}^{s_1,1-\epsilon}, M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon}\right)\right\|. \quad (3.14)$$

4. Unimodular Multipliers on Function Spaces

By using Theorem 3.1, we study the unimodular multipliers on function spaces. We only take $(1-\epsilon)$-modulation spaces as examples to illustrate our idea. The same method works also for the homogeneous Besov spaces. We recall the following proposition as a simple application of Theorem 3.1.

Proposition 4.1 (see [27]) (Embedding, [20,14]). Let $0 \le \epsilon \le \infty, s_i \in \mathbb{R}$, for $i = 1,2$, and $0 \le \epsilon \le 1$. Then

$$M_{1+\epsilon,1+\epsilon}^{s_1,1-\epsilon} \subseteq M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon} \quad (4.1)$$

if and only if

$$\begin{cases} \epsilon \ge 0 \\ s_2 - \dfrac{n(1-\epsilon)}{1+2\epsilon} \le s_1 - \dfrac{n(1-\epsilon)}{1+\epsilon} \end{cases} \quad (4.2)$$

or

$$\begin{cases} \epsilon \ge 0 \\ s_2 - \dfrac{n(1-\epsilon)+n(\epsilon)}{1+2\epsilon} \le s_1 - \dfrac{n(1-\epsilon)+n(\epsilon)}{1+\epsilon} \end{cases} \quad (4.3)$$

holds.

Proof. This result actually can be found in Triebel's book for the case $\epsilon = 0$, and in a recent paper by Han and Wang for the case $0 \le \epsilon < 1$. Here, as an application of Theorem 3.1, we reprove the result by a simpler method. We will only show the proof for $0 < \epsilon \le 1$, since the proof for $\epsilon = 0$ is similar. It is obvious that the embedding relations between α-modulation spaces can be viewed as the boundedness of the identity operator between the same $(1-\epsilon)$-modulation spaces. By the viewpoint of Theorem 3.1, we need to obtain the asymptotic estimates for $\| \square_k^{1-\epsilon} \|_{L^{1+\epsilon}\to L^{1+2\epsilon}}$. In fact, by Young's inequality and a scaling argument, one can easily verify

$$\| \square_k^{1-\epsilon} \|_{L^{1+\epsilon}\to L^{1+2\epsilon}} \sim \langle k\rangle^{\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}}. \quad (4.4)$$

With the help of (4.4) and Theorem 3.1, we reduce the embedding between $(1-\epsilon)$-modulation spaces to the boundedness

$$\left\|\left\{\| \square_k^{1-\epsilon} \|_{L^{1+\epsilon}\to L^{1+2\epsilon}}\right\}\right\|_{l_{1+\epsilon}^{s_1,1-\epsilon}\to l_{1+2\epsilon}^{s_2,1-\epsilon}} = \left\|\left\{\langle k\rangle^{\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}}\right\}\right\|_{l_{1+\epsilon}^{s_1,1-\epsilon}\to l_{1+2\epsilon}^{s_2,1-\epsilon}}. \quad (4.5)$$

To prove the last identity, we divide the proof into two cases: $\epsilon \ge 0$ and $\epsilon < 0$.

Case 1: $\epsilon \ge 0$.

In this case, we have $\mathcal{M}_{1+\epsilon}(l_{1+\epsilon}^{s_1,1-\epsilon}, l_{1+2\epsilon}^{s_2,1-\epsilon}) = l_\infty^{s_2-s_1,1-\epsilon}$, and

$$\left\|\left\{\langle k\rangle^{\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}}\right\}\right\|_{l_\infty^{s_2-s_1,1-\epsilon}} = \sup_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{s_2-s_1}{\epsilon}+\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}}. \quad (4.6)$$

Obviously, $s_2 - \frac{n(1-\epsilon)}{1+2\epsilon} \le s_1 - \frac{n(1-\epsilon)}{1+\epsilon}$ is the sufficient and necessary condition for the boundedness of (4.6).

Case 2: $\epsilon < 0$.

In this case, we have that $\mathcal{M}_{1+\epsilon}(l_{1+\epsilon}^{s_1,1-\epsilon}, l_{1+2\epsilon}^{s_2,1-\epsilon}) = l_r^{s_2-s_1,1-\epsilon}$, where $\frac{1}{r} = \frac{-\epsilon}{(1+2\epsilon)(1+\epsilon)}$. Thus it is easy to see that

$$\left\|\left\{\langle k\rangle^{\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}}\right\}\right\|_{l_r^{s_2-s_1,1-\epsilon}} = \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{r\left[\frac{s_2-s_1}{\epsilon}+\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}\right]}\right)^{\frac{1}{r}}. \quad (4.7)$$

We now easily verify that $s_2 - \frac{n(1-\epsilon)+n(\epsilon)}{1+2\epsilon} < s_1 - \frac{n(1-\epsilon)+n(\epsilon)}{1+\epsilon}$ is the sharp condition for the boundedness of (4.7). The proposition is proved.

Since the estimates of $\| \square_k^{1-\epsilon} \|_{L^{1+\epsilon}\to L^{1+2\epsilon}}$ is easy, the above proof is not difficult. However, to study the unimodular multipliers, we need to face a more complicated situation. We will study the boundedness properties of the unimodular Fourier multiplier $e^{i\mu_\sigma(D)}$ where μ_σ is a real-valued function satisfying some derivative assumptions of order $\beta > 0$. Again, we only consider the $(1-\epsilon)$-modulation case.

It is known that $e^{i|D|^\beta}$ is not bounded on any Lebesgue space $L^{1+\epsilon}$ and Besov space, except for $\epsilon = 1$ or $\beta = 1$ and $n = 1$, (see [15, 19, 24]). However, in [2], the authors proved that if $0 \le \beta \le 2, e^{i|D|^\beta}$ is bounded on $M^s_{1+\epsilon,1+\epsilon}$ for all $0 \le \epsilon \le \infty, s \in \mathbb{R}$. Furthermore, in the case $\beta > 2$, Miyachi–Nicola–Rivetti–Tabacco–Tomita [18] showed that, for $0 \le \epsilon \le \infty$ and $s_1, s_2 \in \mathbb{R}, e^{i|D|^\beta}$ is bounded from $M^{s_1}_{1+\epsilon,1+\epsilon}$ to $M^{s_2}_{1+\epsilon,1+\epsilon}$ if and only if $s_1 - s_2 \ge (\beta-2)n|\frac{1}{2(1+\epsilon)}|$. In [3, 6], the authors obtained some asymptotic estimates for certain unimodular Fourier multipliers on the modulation spaces. In [25], we study the boundedness of $e^{i\mu_\sigma(D)}$ on $(1-\epsilon)$-modulation spaces and establish a sharp theorem by assuming some radial conditions on μ_σ. Now, we revisit this topic as an application of Theorem 3.1. Using a new idea, we will extend some results in [18] by a different proof.

Firstly, we give a conclusion about the necessity of boundedness of unimodular Fourier multipliers on $(1-\epsilon)$-modulation spaces.

Theorem 4.2 [27]. Let $\beta > 0$ and μ_σ be a real-valued smooth function on $\mathbb{R}^n \setminus \{0\}$ which is homogeneous of degree β. Suppose there exists a point $\xi_0 \ne 0$ at which the rank of the Hessian matrix is at least r. Let $0 \le \epsilon \le \infty, s_i \in \mathbb{R}, 0 \le \epsilon \le 1$ for $i = 1, 2$. Suppose that the Fourier multiplier $e^{i\mu_\sigma(D)}$ is bounded from $M^{s_1,1-\epsilon}_{1+\epsilon,1+\epsilon}$ to $M^{s_2,1-\epsilon}_{1+\epsilon,1+\epsilon}$. Then we have

$$s_2 + |\left(\frac{1}{2(1+\epsilon)}\right)| \max\{r\beta - 2\epsilon n, 0\} \le s_1. \quad (4.8)$$

To prove the theorem, by the spirit of Theorem 3.1, we need to obtain the lower bound estimates for $\| \square_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^{1+\epsilon}\to L^{1+\epsilon}}$ and $\| \Delta_j e^{i\mu_\sigma(D)} \|_{L^{1+\epsilon}\to L^{1+\epsilon}}$, respectively. For this purpose, we will invoke the following Lemmas 4.3, 4.4, and 4.5. The first lemma is the following classical result due to Littman [16].

Lemma 4.3 [27]. Let Ω be a bounded open set of $\mathbb{R}^n$. Assume that g_σ is a smooth function and $\operatorname{supp} g_\sigma \subset \Omega$. Let ϕ_σ be a real-valued C^∞ function on $\mathbb{R}^n$ satisfying that the rank of $\left(\frac{\partial^2\phi_\sigma(\xi)}{\partial\xi_i\partial\xi_j}\right)^n_{i,j=1}$ is at least $r > 0$.

Then there exists a constant $C = C(n, g_\sigma, \phi_\sigma)$ such that

$$\| \sum_\sigma \mathcal{F}^{-1}\left[g_\sigma(\xi)e^{i(1+\epsilon)\phi_\sigma(\xi)}\right] \|_{L^\infty} \le C(1 + |1+\epsilon|)^{-\frac{r}{2}}. \quad (4.9)$$

We use this lemma to further get the following partial dual estimates (see [27]).

Lemma 4.4 (Partial Dual Estimates for $B^s_{1+\epsilon,1+\epsilon}$). Suppose that $\beta > 0, \mu_\sigma$ satisfies the assumptions of Theorem 4.2. Then there exists a smooth function sequence $\{h_j\}_{j\in\mathbb{N}}$ such that

(1) $\operatorname{\mathbf{supp}} h_j \subset B(2^j e_0, 2^j\delta)$,

(2) $h_j = 1$ on $B(2^j e_0, 2^{j-1}\delta)$,

(3) $$\| \sum_\sigma \mathcal{F}^{-1}\left[h_j(\xi)e^{i\mu_\sigma(\xi)}\right] \|_{L^\infty} \le C2^{jn}(1 + 2^{j\beta})^{-\frac{r}{2}} \quad (4.10)$$

for all $j \in \mathbb{N}$, where e_0 is a unit vector, δ is a small positive constant, C is a positive constant independent of j.

Proof. By the rank condition in Theorem 4.2 and the homogeneous of μ_σ, we can find an appropriate C^∞ function h satisfying $\operatorname{\mathbf{supp}} h \subset B(e_0, \delta)$ and $h = 1$ on $B(e_0, 2^{-1}\delta)$ for some unit vector e_0 and sufficiently small constant δ, such that the rank of Hessian matrix of μ_σ is at least r on the support of h.

We use Lemma 4.3 to deduce

$$\| \sum_\sigma \mathcal{F}^{-1}\left[h(\xi)e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^\infty} \lesssim (1 + |1+\epsilon|)^{-\frac{r}{2}}. \quad (4.11)$$

Denote

$$h_j(\xi) = h(\xi/2^j). \tag{4.12}$$

Then by a scaling argument the final conclusion can be verified by

$$\| \sum_\sigma \mathcal{F}^{-1}\left[h_j(\xi)e^{i\mu_\sigma(\xi)}\right] \|_{L^\infty} = 2^{jn} \| \sum_\sigma \mathcal{F}^{-1}\left[h(\xi)e^{i2^{j\beta}\mu_\sigma(\xi)}\right] \|_{L^\infty} \lesssim 2^{jn}\left(1+2^{j\beta}\right)^{-\frac{r}{2}}. \tag{4.13}$$

Lemma 4.5 (see [27]) (Partial Dual Estimates for $M^{s,1-\epsilon}_{1+\epsilon,1+\epsilon}, 0<\epsilon\le 1$). Suppose $\beta > 0, \mu_\sigma$ satisfies the assumptions of Theorem 4.2. Then there exists a nonempty open cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin, and a sufficiently large constant $\mathcal{R}$ such that

$$\| \sum_\sigma \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i\mu_\sigma(\xi)}\right] \|_{L^\infty} \le C\langle k\rangle^{\frac{n}{\epsilon}}\langle k\rangle^{\frac{\beta}{\epsilon}\left(-\frac{r}{2}\right)} \tag{4.14}$$

for all $k \in \mathbb{Z}^n$ with $\langle k\rangle^{\frac{1-\epsilon}{\epsilon}} k \in \Gamma \setminus B(0,R)$, where the constant C is independent of k.

Proof. In fact, we will use the continuous version of Lemma 4.4. Taking h as the function we found in Lemma 4.4, one can easily verify that

$$\| \sum_\sigma \mathcal{F}^{-1}\left[h_{(1+\epsilon)}(\xi)e^{i\mu_\sigma(\xi)}\right] \|_{L^\infty} \lesssim (1+\epsilon)^n\left(1+(1+\epsilon)^\beta\right)^{-\frac{r}{2}} \tag{4.15}$$

for $r \ge 1$, where $h_{(1+\epsilon)}(\xi) = h(\frac{\xi}{(1+\epsilon)})$.

Notice that the support of $h_{(1+\epsilon)}$ expands faster than the support of $\eta_k^{1-\epsilon}$. We can find a nonempty open cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin, and a sufficiently large constant $\mathcal{R}$, such that for all $k \in \mathbb{Z}^n$ with $\langle k\rangle^{\frac{1-\epsilon}{\epsilon}} k \in \Gamma \setminus B(0,R)$, there exists an appropriate $(1+\epsilon)$ satisfying

$$h_{(1+\epsilon)}(\xi)\eta_k^{1-\epsilon}(\xi) = \eta_k^{1-\epsilon}(\xi) \tag{4.16}$$

with the relationship

$$(1+\epsilon) \sim \langle k\rangle^{\frac{1}{\epsilon}}. \tag{4.17}$$

Then

$$\begin{aligned}\| \sum_\sigma \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i\mu_\sigma(\xi)}\right] \|_{L^\infty} &\lesssim \sum_\sigma \| \mathcal{F}^{-1}\left[\psi_{(1+\epsilon)}(\xi)\eta_k^{1-\epsilon}(\xi)e^{i\mu_\sigma(\xi)}\right] \|_{L^\infty} \\ &\lesssim \sum_\sigma \| \mathcal{F}^{-1}\left[\psi_{(1+\epsilon)}(\xi)e^{i\mu_\sigma(\xi)}\right] \|_{L^\infty} \\ &\lesssim (1+\epsilon)^n\left(1+(1+\epsilon)^\beta\right)^{-\frac{r}{2}} \sim \langle k\rangle^{\frac{n}{\epsilon}}\langle k\rangle^{\frac{\beta}{\epsilon}\left(-\frac{r}{2}\right)}. \end{aligned} \tag{4.18}$$

Proof of Theorem 4.2 (see [27]). We first give the proof for the case $0<\epsilon\le 1, -1 \le \epsilon \le 1$. By the spirit of Theorem 3.1, if $e^{i\mu_\sigma(D)}$ is bounded from $M^{s_1,1-\epsilon}_{1+\epsilon,1+\epsilon}$ to $M^{s_2,1-\epsilon}_{1+\epsilon,1+\epsilon}$, we have

$$\langle k\rangle^{\frac{s_2-s_1}{\epsilon}} \| \sum_\sigma \square_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^{1+\epsilon}\to L^{1+\epsilon}} \lesssim 1 \tag{4.19}$$

for all $k \in \mathbb{Z}^n$. But for $k \in \mathbb{Z}^n$ with $\langle k\rangle^{\frac{1-\epsilon}{\epsilon}} k \in \Gamma \setminus B(0,R)$, where $\Gamma \subset \mathbb{R}^n$ is a nonempty open cone chosen in Lemma 4.5, denote $(f_\sigma)_k = \mathcal{F}^{-1}\eta_k^{1-\epsilon,*}$, we have

$$\begin{aligned}\sum_\sigma \| \square_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^{1+\epsilon}\to L^{1+\epsilon}} &\gtrsim \sum_\sigma \frac{\| \square_k^{1-\epsilon} e^{i\mu_\sigma(D)}(f_\sigma)_k \|_{L^{1+\epsilon}}}{\| (f_\sigma)_k \|_{L^{1+\epsilon}}} \\ &= \sum_\sigma \frac{\| \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i\mu_\sigma(\xi)}\right] \|_{L^{1+\epsilon}}}{\| \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon,*}(\xi)e^{i\mu_\sigma(\xi)}\right] \|_{L^{1+\epsilon}}}. \end{aligned} \tag{4.20}$$

Recalling

$$\| \sum_\sigma \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i\mu_\sigma(\xi)}\right] \|_{L^2} = \| \sum_\sigma \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)\right] \|_{L^2} \sim \langle k\rangle^{\frac{(1-\epsilon)n}{\epsilon}\left(\frac{1}{2}\right)} \tag{4.21}$$

and

$$\| \sum_\sigma \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i\mu_\sigma(\xi)}\right] \|_{L^\infty} \lesssim \langle k\rangle^{\frac{n}{\epsilon}}\langle k\rangle^{\frac{\beta}{\epsilon}\left(-\frac{r}{2}\right)}, \tag{4.22}$$

we use

$$\| \sum_\sigma \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i\mu_\sigma(\xi)}\right] \|_{L^2} \lesssim \sum_\sigma \| \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i\mu_\sigma(\xi)}\right] \|_{L^{1+\epsilon}}^{\frac{1+\epsilon}{2}} \| \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i\mu_\sigma(\xi)}\right] \|_{L^\infty}^{\frac{1-\epsilon}{2}} \tag{4.23}$$

to deduce

$$\| \sum_{\sigma} \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i\mu_\sigma(\xi)}\right] \|_{L^{1+\epsilon}} \gtrsim \langle k\rangle^{\frac{(1-\epsilon)n}{\epsilon}\left(\frac{1}{1+\epsilon}\right)} \langle k\rangle^{\frac{1}{\epsilon}\left(\frac{1-\epsilon}{2(1+\epsilon)}\right)(r\beta-2n)}. \tag{4.24}$$

So we obtain

$$\| \sum_{\sigma} \Box_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^{1+\epsilon}\to L^{1+\epsilon}} \gtrsim \sum_{\sigma} \frac{\| \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i\mu_\sigma(\xi)}\right] \|_{L^{1+\epsilon}}}{\| \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon,*}(\xi)\right] \|_{L^{1+\epsilon}}}$$
$$\gtrsim \langle k\rangle^{\frac{1}{\epsilon}\left(\frac{1-\epsilon}{2(1+\epsilon)}\right)(r\beta-2n+2(1-\epsilon)n)}. \tag{4.25}$$

Then we use (4.19) to deduce

$$\langle k\rangle^{\frac{s_2-s_1}{\epsilon}} \langle k\rangle^{\frac{1}{\epsilon}\left(\frac{1-\epsilon}{2(1+\epsilon)}\right)(r\beta-2\epsilon n)} \lesssim 1 \tag{4.26}$$

for $\langle k\rangle^{\frac{1-\epsilon}{\epsilon}} k \in \Gamma \setminus B(0,R)$. Letting k tend to infinity, we obtain

$$s_2 + \left(\frac{1-\epsilon}{2(1+\epsilon)}\right)(r\beta - 2\epsilon n) \le s_1. \tag{4.27}$$

Next, we use the Hausdorff–Young inequality to deduce

$$\| \sum_{\sigma} \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i\mu_\sigma(\xi)}\right] \|_{L^\infty} \lesssim \sum_{\sigma} \| \eta_k^{1-\epsilon}(\xi)e^{i\mu_\sigma(\xi)} \|_{L^1} \lesssim \langle k\rangle^{\frac{(1-\epsilon)n}{\epsilon}}. \tag{4.28}$$

Repeating the above argument, we deduce

$$\sum_{\sigma} \| \Box_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^{1+\epsilon}\to L^{1+\epsilon}} \gtrsim 1. \tag{4.29}$$

Then (4.19) yields $s_2 \le s_1$. Combining with (4.27) and the fact $s_2 \le s_1$, we conclude

$$s_2 + \left(\frac{1-\epsilon}{2(1+\epsilon)}\right) \max\{r\beta - 2\epsilon n, 0\} \le s_1. \tag{4.30}$$

The proof for the case $-1 \le \epsilon \le 1$ is completed.
For the case of $\epsilon > 1$, if $e^{i\mu_\sigma(D)}$ is bounded from $M_{1+\epsilon,1+\epsilon}^{s_1,1-\epsilon}$ to $M_{1+\epsilon,1+\epsilon}^{s_2,1-\epsilon}$, we also have

$$\langle k\rangle^{\frac{s_2-s_1}{\epsilon}} \| \sum_{\sigma} \Box_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^{1+\epsilon}\to L^{1+\epsilon}} \lesssim 1 \tag{4.31}$$

for all $k \in \mathbb{Z}^n$. By duality, we deduce

$$\| \sum_{\sigma} \Box_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^{1+\epsilon}\to L^{1+\epsilon}} = \| \sum_{\sigma} \Box_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^{\frac{1+\epsilon}{\epsilon}}\to L^{\frac{1+\epsilon}{\epsilon}}}, \tag{4.32}$$

where $\frac{1}{1+\epsilon} + \frac{\epsilon}{1+\epsilon} = 1, \frac{1+\epsilon}{\epsilon} \in [1,2)$. So we have

$$\langle k\rangle^{\frac{s_2-s_1}{\epsilon}} \| \sum_{\sigma} \Box_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^{\frac{1+\epsilon}{\epsilon}}\to L^{\frac{1+\epsilon}{\epsilon}}} \lesssim 1 \tag{4.33}$$

in this case, where $\frac{1+\epsilon}{\epsilon} \in [1,2)$. By the same method as in the case of $0 \le \epsilon \le 1$, we conclude

$$s_2 + \left(\frac{\epsilon-1}{2(1+\epsilon)}\right) \max\{r\beta - 2\epsilon n, 0\} \le s_1. \tag{4.34}$$

Noticing $\left(\frac{\epsilon-1}{2(1+\epsilon)}\right) \max\{r\beta - 2\epsilon n, 0\} = |\left(\frac{1-\epsilon}{2(1+\epsilon)}\right)| \max\{r\beta - 2\epsilon n, 0\}$, we conclude

$$s_2 + |\left(\frac{1-\epsilon}{2(1+\epsilon)}\right)| \max\{r\beta - 2\epsilon n, 0\} \le s_1. \tag{4.35}$$

We complete the proof of Theorem 4.2.
Checking the proof of Theorem 4.2, one may observe that the role of L^∞-estimates is to provide a lower bound estimate of the operator norm of $e^{i\mu_\sigma(D)}$. In order to get an upper bound of the operator norm of $e^{i\mu_\sigma(D)}$, we may need some L^1-estimates. We recall an estimate in [26]. Since [26] is published in Chinese, for convenience to the reader, we present the proofs in this paper.
Lemma 4.6 (See [26]). Let $\epsilon \ge 0, \beta > 0$ and $\epsilon > 0$. Assume that μ_σ is a real-valued function of class $C^{\left[\frac{n}{2}\right]+3}$ on $\mathbb{R}^n \setminus \{0\}$ which satisfies

$$|\sum_{\sigma} \partial^\gamma \mu_\sigma(\xi)| \le C_\gamma |\xi|^{\epsilon-|\gamma|}, \quad 0 < |\xi| \le 1, \quad |\gamma| \le [n/2] + 1, \tag{4.36}$$

and

$$|\sum_{\sigma} \partial^{\gamma}\mu_{\sigma}(\xi)| \leq C_{\gamma}|\xi|^{\beta-|\gamma|}, \quad |\xi| > 1, \quad 2 \leq |\gamma| \leq [n/2] + 3. \quad (4.37)$$

Then we have

$$\begin{cases} \| \sum_{\sigma} \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^1} \lesssim \left(1 + (1+\epsilon)\langle k\rangle^{\frac{\beta-2\epsilon}{\epsilon}}\right)^{\frac{n}{2}}, \quad 0<\epsilon\leq 1, \quad k \in \mathbb{Z}^n; \\ \| \sum_{\sigma} \mathcal{F}^{-1}\left[\psi_j(\xi)e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^1} \lesssim \left(1+(1+\epsilon)2^{j(\beta-2+2(1-\epsilon))}\right)^{\frac{n}{2}}, \quad \epsilon = 2, \quad k \in \mathbb{N}. \end{cases} \quad (4.38)$$

Proof. We only show the proof for $0 < \epsilon \leq 1$. By the property of L^1 norm, we have

$$\| \sum_{\sigma} \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^1} = \| \sum_{\sigma} \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}\left(\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}\xi + \langle k\rangle^{\frac{1-\epsilon}{\epsilon}}k\right)e^{i(1+\epsilon)\mu_\sigma\left(\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}\xi + \langle k\rangle^{\frac{1-\epsilon}{\epsilon}}k\right)}\right] \|_{L^1}. \quad (4.39)$$

By Taylor's formula, we write

$$\mu_\sigma\left(\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}\xi + \langle k\rangle^{\frac{1-\epsilon}{\epsilon}}k\right) = \mu_\sigma\left(\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}k\right) + \langle k\rangle^{\frac{1-\epsilon}{\epsilon}}(\nabla u)\left(\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}k\right)\cdot\xi$$
$$+ 2\langle k\rangle^{\frac{2(1-\epsilon)}{\epsilon}} \sum_{|\rho|=2}\frac{\xi^\rho}{\rho!}\int_0^1 (1-\theta)(\partial^\rho u)\left(\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}k + (1+\epsilon)\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}\xi\right)d\theta \quad (4.40)$$

and denote

$$\tau_k^{1-\epsilon}(\xi) = \mu_\sigma\left(\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}\xi + \langle k\rangle^{\frac{1-\epsilon}{\epsilon}}k\right) - \mu_\sigma\left(\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}k\right)$$
$$- \langle k\rangle^{\frac{1-\epsilon}{\epsilon}}(\nabla u)\left(\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}k\right)\cdot\xi. \quad (4.41)$$

Then we have

$$\| \sum_{\sigma} \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^1} = \| \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}\left(\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}\xi + \langle k\rangle^{\frac{1-\epsilon}{\epsilon}}k\right)e^{i(1+\epsilon)\tau_k^{1-\epsilon}(\xi)}\right] \|_{L^1}. \quad (4.42)$$

For sufficiently large $|k|$, small ξ and $0 \leq \theta \leq 1$, we have

$$\left|\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}k + \theta\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}\xi\right| \sim \langle k\rangle^{\frac{1}{\epsilon}}$$

and

$$\left|\sum_{\sigma}(\partial^\gamma\mu_\sigma)\left(\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}k + \theta\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}\xi\right)\right| \lesssim \langle k\rangle^{\frac{\beta-|\gamma|}{\epsilon}}$$

for $2 \leq |\gamma| \leq [n/2] + 3$. So

$$|\partial^\gamma \tau_k^{1-\epsilon}(\xi)| = \Bigg|2\langle k\rangle^{\frac{2(1-\epsilon)}{\epsilon}}\sum_{|\rho|=2}\sum_{\gamma_1+\gamma_2=\gamma} C_{\gamma_1,\gamma_2}\frac{\partial^{\gamma_1}(\xi^\rho)}{\rho!}$$
$$\cdot\int_0^1 (1-\theta)\left(\theta\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}\right)^{|\gamma_2|}(\partial^{\rho+\gamma_2}u)\left(\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}k + \theta\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}\xi\right)d\theta\Bigg|$$
$$\lesssim \langle k\rangle^{\frac{2(1-\epsilon)}{\epsilon}}\langle k\rangle^{\frac{1-\epsilon}{\epsilon}|\gamma_2|}\langle k\rangle^{\frac{1}{\epsilon}|(\beta-2-|\gamma_2|)} \lesssim \langle k\rangle^{\frac{2(1-\epsilon)}{\epsilon}}\langle k\rangle^{\frac{\beta-2}{\epsilon}} \quad (4.43)$$

for $|\gamma| \leq [n/2] + 1$. We now have

$$|\partial^\gamma e^{i(1+\epsilon)\tau_k^{1-\epsilon}(\xi)}| \lesssim \left(1+(1+\epsilon)\langle k\rangle^{\frac{\beta-2\epsilon}{\epsilon}}\right)^{|\gamma|} \quad (4.44)$$

for $|\gamma| \leq [n/2] + 1$. Moreover, we conclude that

$$\left|\partial^\gamma\left(\eta_k^{1-\epsilon}\left(\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}\xi + \langle k\rangle^{\frac{1-\epsilon}{\epsilon}}k\right)e^{i(1+\epsilon)\tau_k^{1-\epsilon}(\xi)}\right)\right| \lesssim \left(1+(1+\epsilon)\langle k\rangle^{\frac{\beta-2\epsilon}{\epsilon}}\right)^{|\gamma|} \quad (4.45)$$

for $|\gamma| \leq [n/2] + 1$.

Combining the above estimate with the support condition of $\eta_k^{1-\epsilon}\left(\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}\xi + \langle k\rangle^{\frac{1-\epsilon}{\epsilon}}k\right)$, we have

$$\left\|\partial^\gamma\left(\eta_k^{1-\epsilon}\left(\langle k\rangle^{\frac{1-\epsilon}{\epsilon}}\xi + \langle k\rangle^{\frac{1-\epsilon}{\epsilon}}k\right)e^{i(1+\epsilon)\tau_k^{1-\epsilon}(\xi)}\right)\right\|_{L^2} \lesssim \left(1+(1+\epsilon)\langle k\rangle^{\frac{\beta-2\epsilon}{\epsilon}}\right)^{|\gamma|} \quad (4.46)$$

for $|\gamma| \leq [n/2] + 1$. Bernstein's theorem and a dilation argument then yield

$$\| \sum_{\sigma} \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^1} \lesssim \left(1+(1+\epsilon)\langle k\rangle^{\frac{\beta-2\epsilon}{\epsilon}}\right)^{\frac{n}{2}}. \quad (4.47)$$

For small k ($k = 0$), we have

$$\| \sum_{\sigma} \mathcal{F}^{-1}\left[\eta_0^{1-\epsilon}(\xi)e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^1} \lesssim \sum_{\sigma} \| \mathcal{F}^{-1}\left[\eta_0^{1-\epsilon}\left(e^{i(1+\epsilon)\mu_\sigma(\xi)}-1\right)\right] \|_{L^1} + \| \mathcal{F}^{-1}\eta_0^{1-\epsilon} \|_{L^1}$$

$$\lesssim \sum_{\sigma} \| \mathcal{F}^{-1}\left[\eta_0^{1-\epsilon}\left(e^{i(1+\epsilon)\mu_\sigma(\xi)}-1\right)\right] \|_{L^1} + 1$$

$$\lesssim \sum_{j\le c}\sum_{\sigma} \| \mathcal{F}^{-1}\left[\psi_j\left(e^{i(1+\epsilon)\mu_\sigma(\xi)}-1\right)\right] \|_{L^1} + 1. \quad (4.48)$$

By the property of L^1 norm, we have

$$\sum_{j\le c}\sum_{\sigma} \| \mathcal{F}^{-1}\left[\psi_j\left(e^{i(1+\epsilon)\mu_\sigma(\xi)}-1\right)\right] \|_{L^1} = \sum_{j\le c}\sum_{\sigma} \| \mathcal{F}^{-1}\left[\psi\left(e^{i(1+\epsilon)\mu_\sigma(2^j\xi)}-1\right)\right] \|_{L^1}. \quad (4.49)$$

On the other hand, by the fact that $|\xi| \sim 1, 2^j \lesssim 1$, one can verify that

$$|\sum_{\sigma} \partial^\gamma(e^{i(1+\epsilon)\mu_\sigma(2^j\xi)} - 1)| \lesssim \left((1+\epsilon) + (1+\epsilon)^{|\gamma|}\right)2^{j\epsilon} \quad (4.50)$$

for $|\gamma| \le [\frac{n}{2}] + 1$. So we can verify that

$$\left|\sum_{\sigma} \partial^\gamma\left[\psi(\xi)\left(e^{i(1+\epsilon)\mu_\sigma(2^j\xi)} - 1\right)\right]\right| \lesssim \left(1 + (1+\epsilon)\right)^{|\gamma|}2^{j\epsilon}. \quad (4.51)$$

Then

$$\| \sum_{\sigma} \partial^\gamma\left[\psi(\xi)\left(e^{i(1+\epsilon)\mu_\sigma(2^j\xi)} - 1\right)\right] \|_{L^2} \lesssim \left(1 + (1+\epsilon)\right)^{|\gamma|}2^{j\epsilon} \quad (4.52)$$

follows. We use Bernstein's theorem and a dilation argument to deduce

$$\| \sum_{\sigma} \mathcal{F}^{-1}\left[\psi\left(e^{i(1+\epsilon)\mu_\sigma(2^j\xi)} - 1\right)\right] \|_{L^1} \lesssim \left(1 + (1+\epsilon)\right)^{\frac{n}{2}}2^{j\epsilon}. \quad (4.53)$$

Hence

$$\| \sum_{\sigma} \mathcal{F}^{-1}\left[\eta_0^{1-\epsilon}(\xi)e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^1} \lesssim \sum_{j\le c}\sum_{\sigma} \| \mathcal{F}^{-1}\left[\psi(\xi)\left(e^{i(1+\epsilon)\mu_\sigma(2^j\xi)} - 1\right)\right] \|_{L^1} + 1$$

$$\lesssim \sum_{j\le c}\left(1 + (1+\epsilon)\right)^{\frac{n}{2}}2^{j\epsilon} + 1 \lesssim \left(1 + (1+\epsilon)\right)^{\frac{n}{2}}. \quad (4.54)$$

Using Lemma 4.6, one can verify the following theorem which we obtained in [26].

Theorem 4.7 ([26]). Let $\epsilon \ge 0, \beta > 0$ and $\epsilon > 0$, and

$$S_{1+\epsilon} = S_{1+\epsilon}(\beta) = n\left(\frac{1-\epsilon}{2(1+\epsilon)}\right)\max\{\beta - 2\epsilon, 0\}.$$

Assume that μ_σ is a real-valued function of class $C^{[\frac{n}{2}]+3}$ on $\mathbb{R}^n \setminus \{0\}$ which satisfies

$$|\sum_{\sigma} \partial^\gamma\mu_\sigma(\xi)| \le C_\gamma|\xi|^{\epsilon-|\gamma|}, \quad 0 < |\xi| \le 1, \quad |\gamma| \le [n/2] + 1, \quad (4.55)$$

and

$$|\sum_{\sigma} \partial^\gamma\mu_\sigma(\xi)| \le C_\gamma|\xi|^{\beta-2}, \quad |\xi| > 1, \quad 2 \le |\gamma| \le [n/2] + 3. \quad (4.56)$$

Suppose $1 \le \epsilon \le \infty, s_i \in \mathbb{R}, 0 \le \epsilon \le 1$ for $i = 1, 2$, and they satisfy $s_1 - s_2 \ge |S_{1+\epsilon}|$. Then we have

$$\| \sum_{\sigma} e^{i(1+\epsilon)\mu_\sigma(D)}f_\sigma \|_{M^{s,1-\epsilon}_{1+\epsilon,1+\epsilon}} \lesssim \sum_{\sigma} \| f_\sigma \|_{M^{s,1-\epsilon}_{1+\epsilon,1+\epsilon}} + (1+\epsilon)^{n\left|\frac{1-\epsilon}{2(1+\epsilon)}\right|}\sum_{\sigma} \| f_\sigma \|_{M^{s+|S_{1+\epsilon}|,1-\epsilon}_{1+\epsilon,1+\epsilon}}. \quad (4.57)$$

The following theorem shows that if the L^1-estimates and L^∞-estimates have some uniform relationship, the operator norm of $e^{i\mu_\sigma(D)}$ is equivalent to its norm of the corresponding modulation space.

Theorem 4.8 (see [27]). Let $1 \le \epsilon \le \infty, s_i \in \mathbb{R}, 0 \le \epsilon \le 1$ for $i = 1, 2, \epsilon \ge 0$. Suppose that the unimodular Fourier multiplier $e^{i\mu_\sigma(D)}$ satisfies the following uniform inverse Hölder conditions:

$$\begin{cases} \| \sum_{\sigma} \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^1} \| \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^\infty} \le C\sum_{\sigma} \| \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^2}^2, & if\ \epsilon < 2 \\ \| \sum_{\sigma} \mathcal{F}^{-1}\left[\psi_j(\xi)e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^1} \| \mathcal{F}^{-1}\left[\psi_j(\xi)e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^\infty} \le C\sum_{\sigma} \| \mathcal{F}^{-1}\left[\psi_j(\xi)e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^2}^2, & if\ \epsilon = 2 \end{cases} \quad (4.58)$$

where the constant C is independent of k in the case $\epsilon < 2$, and independent of j in the case $\epsilon = 0$. Then we have

$$\|e^{i\mu_\sigma}|\ \mathcal{M}_{\mathcal{F}}(M^{s_1,1-\epsilon}_{1+\epsilon,1+\epsilon}, M^{s_2,1-\epsilon}_{1+2\epsilon,1+2\epsilon})\| \sim \|\mathcal{F}^{-1}e^{i\mu_\sigma(1+\epsilon)}\|_{M^{s,1-\epsilon}_{1+\epsilon,r}}, \tag{4.59}$$

Where

$$(1+\epsilon) = \begin{cases} 1+2\epsilon & if \dfrac{1}{1+2\epsilon} \geq \dfrac{\epsilon}{1+\epsilon} \\ \dfrac{1+\epsilon}{\epsilon} & if \dfrac{1}{1+2\epsilon} \leq \dfrac{\epsilon}{1+\epsilon} \end{cases},$$

$$r = \begin{cases} \infty & if \epsilon \geq 0 \\ \dfrac{-\epsilon}{(1+\epsilon)(1+2\epsilon)} & if \epsilon < 0' \end{cases} \tag{4.60}$$

$s = s_2 - s_1 + (1-\epsilon)n\left(\dfrac{-\epsilon}{(1+2\epsilon)(1+\epsilon)}\right) - (1-\epsilon)n\left(\dfrac{\epsilon}{1+\epsilon}\right).$

Proof. We only show the proof for $0 < \epsilon \leq 1$, since the proof for $\epsilon = 0$ is similar. By the spirit of Theorem 3.1, we need the asymptotic estimate for $\| \square_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^{1+\epsilon} \to L^{1+2\epsilon}}$. In fact, we want to verify

$$\| \sum_\sigma \square_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^{1+\epsilon}\to L^{1+2\epsilon}} \sim \langle k\rangle^{\left(\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}\right)} \sum_\sigma \frac{\| \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i\mu_\sigma(\xi)}\right] \|_{L^{1+\epsilon}}}{\langle k\rangle^{\frac{(1-\epsilon)n}{\epsilon}\left(\frac{\epsilon}{1+\epsilon}\right)}}. \tag{4.61}$$

We denote $A_{1+\epsilon,k} = \sum_\sigma \dfrac{\|\mathcal{F}^{-1}[\eta_k^{1-\epsilon}(\xi)e^{i\mu_\sigma(\xi)}]\|_{L^{1+\epsilon}}}{\langle k\rangle^{\left(\frac{(1-\epsilon)n}{1+\epsilon}\right)}}$, then $A_{2,k} \sim 1$. By the assumption, we have $A_{1,k}A_{\infty,k} \lesssim 1$. Moreover, one can deduce

$$A_{1+\epsilon,k} \sim A^{1-\theta}_{1+2\epsilon,k}A^{\theta}_{1+3\epsilon,k}, \tag{4.62}$$

for any $0 \leq \epsilon \leq \infty, \theta \in [0,1], (1+\epsilon) = (1-\theta)(1+2\epsilon) + \theta(1+3\epsilon)$. In order to estimate (4.61), we divide the condition $\epsilon \geq 0$ into four cases.

Case 1. $\frac{1}{1+2\epsilon} \geq \frac{\epsilon}{1+\epsilon}, \frac{1}{2} \leq \frac{1}{1+2\epsilon} \leq \frac{1}{1+\epsilon}$.

Firstly, we use the interpolation argument to deduce

$$\begin{aligned} \| \sum_\sigma \square_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^{1+2\epsilon}\to L^{1+2\epsilon}} &\lesssim \sum_\sigma \| \square_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^1\to L^1}^{\frac{1-2\epsilon}{1+2\epsilon}} \| \square_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^2\to L^2}^{\frac{4\epsilon}{1+2\epsilon}} \\ &\lesssim \sum_\sigma \| \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^1}^{\frac{1-2\epsilon}{1+2\epsilon}} \\ &= A_{1,k}^{\frac{1-2\epsilon}{1+2\epsilon}} = A_{1,k}^{\frac{1-2\epsilon}{1+2\epsilon}} A_{2,k}^{\frac{4\epsilon}{1+2\epsilon}} = A_{1+2\epsilon,k}. \end{aligned} \tag{4.63}$$

Then, we have

$$\begin{aligned} \| \sum_\sigma \square_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^{1+\epsilon}\to L^{1+2\epsilon}} &\lesssim \langle k\rangle^{\left(\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}\right)} \sum_\sigma \| \square_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^{1+2\epsilon}\to L^{1+2\epsilon}} \\ &\lesssim \langle k\rangle^{\left(\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}\right)} A_{1+2\epsilon,k}. \end{aligned} \tag{4.64}$$

On the other hand, we denote $(f_\sigma)_k = \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon,*}(\xi)\right]$. Then

$$\begin{aligned} \sum_\sigma \| \square_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^{1+\epsilon}\to L^{1+2\epsilon}} &\gtrsim \sum_\sigma \frac{\| \square_k^{1-\epsilon} e^{i\mu_\sigma(D)}(f_\sigma)_k \|_{L^{1+2\epsilon}}}{\| (f_\sigma)_k \|_{L^{1+\epsilon}}} \\ &= \sum_\sigma \frac{\| \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i\mu_\sigma(\xi)}\right] \|_{L^{1+2\epsilon}}}{\| \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon,*}(\xi)\right] \|_{L^{1+\epsilon}}} \\ &\sim \langle k\rangle^{\left(\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}\right)} A_{1+2\epsilon,k}. \end{aligned} \tag{4.65}$$

Case 2. $\frac{1}{1+2\epsilon} \geq \frac{\epsilon}{1+\epsilon}, \frac{1}{1+2\epsilon} \leq \frac{1}{2} \leq \frac{1}{1+\epsilon}$.

Firstly, we use an interpolation argument to deduce

$$\begin{aligned} \| \sum_\sigma \square_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^{\frac{1+\epsilon}{\epsilon}}\to L^{1+2\epsilon}} &\lesssim \sum_\sigma \| \square_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^1\to L^\infty}^{\frac{2\epsilon-1}{1+2\epsilon}} \| \square_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^2\to L^2}^{\frac{2}{1+2\epsilon}} \\ &\lesssim \sum_\sigma \| \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^\infty}^{\frac{2\epsilon-1}{1+2\epsilon}} \end{aligned}$$

$$= \langle k\rangle^{\frac{(1-\epsilon)n}{\epsilon}\left(\frac{2\epsilon-1}{1+2\epsilon}\right)} A_{\infty,k}^{\left(\frac{2\epsilon-1}{1+2\epsilon}\right)}$$

$$= \langle k\rangle^{\frac{(1-\epsilon)n}{\epsilon}\left(\frac{2\epsilon-1}{1+2\epsilon}\right)} A_{\infty,k}^{\left(\frac{2\epsilon-1}{1+2\epsilon}\right)} A_{2,k}^{\frac{2}{1+2\epsilon}} = \langle k\rangle^{\left(\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}\right)} A_{1+2\epsilon,k}. \quad (4.66)$$

Then, we have

$$\|\sum_{\sigma} \square_k^{1-\epsilon} e^{i\mu_\sigma(D)}\|_{L^{1+\epsilon}\to L^{1+2\epsilon}} \lesssim \langle k\rangle^{\frac{(1-\epsilon)n}{\epsilon}\left(\frac{1}{1+\epsilon}-\frac{2\epsilon}{1+2\epsilon}\right)} \sum_{\sigma} \|\square_k^{1-\epsilon} e^{i\mu_\sigma(D)}\|_{L^{\frac{1+2\epsilon}{2\epsilon}}\to L^{1+2\epsilon}}$$

$$\lesssim \langle k\rangle^{\frac{(1-\epsilon)n}{\epsilon}\left(\frac{1}{1+\epsilon}-\frac{2\epsilon}{1+2\epsilon}\right)} \langle k\rangle^{\frac{(1-\epsilon)n}{\epsilon}\left(\frac{2\epsilon-1}{1+2\epsilon}\right)} A_{1+2\epsilon,k}$$

$$= \langle k\rangle^{\left(\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}\right)} A_{1+2\epsilon,k}. \quad (4.67)$$

On the other hand, we use the same method as in Case 1 to deduce

$$\|\sum_{\sigma} \square_k^{1-\epsilon} e^{i\mu_\sigma(D)}\|_{L^{1+\epsilon}\to L^{1+2\epsilon}} \gtrsim \langle k\rangle^{\left(\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}\right)} A_{1+2\epsilon,k}. \quad (4.68)$$

Case 3. $\frac{1}{1+2\epsilon} \leq \frac{\epsilon}{1+\epsilon}, \frac{1}{1+2\epsilon} \leq \frac{1}{1+\epsilon} \leq \frac{1}{2}$.

By the symmetry of $(1+\epsilon)$ and $(1+2\epsilon)$, we reduce this case to Case 1.

Case 4. $\frac{1}{1+2\epsilon} \leq \frac{\epsilon}{1+\epsilon}, \frac{1}{1+2\epsilon} \leq \frac{1}{2} \leq \frac{1}{1+\epsilon}$.

By the symmetry of $(1+\epsilon)$ and $(1+2\epsilon)$, we reduce this case to Case 2.

By Theorem 3.1, we have

$$\left\|\sum_{\sigma} e^{i\mu_\sigma} | \mathcal{M}_{\mathfrak{F}}\left(M_{1+\epsilon,1+\epsilon}^{s_1,1-\epsilon}, M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon}\right)\right\| = \left\|\sum_{\sigma} e^{i\mu_\sigma} | W^{1-\epsilon}\left(\mathcal{M}_{\mathfrak{F}}\left(L^{1+\epsilon}, L^{1+2\epsilon}\right), \mathcal{M}_{1+\epsilon}\left(l_{1+\epsilon}^{s_1}, l_{1+2\epsilon}^{s_2}\right)\right)\right\|. \quad (4.69)$$

By the fact that $\mathcal{M}_{1+\epsilon}\left(l_{1+\epsilon}^{s_1,1-\epsilon}, l_{1+2\epsilon}^{s_2,1-\epsilon}\right) = l_r^{s_2-s_1,1-\epsilon}$, we have

$$\left\|\sum_{\sigma} e^{i\mu_\sigma} | \mathcal{M}_{\mathfrak{F}}\left(M_{1+\epsilon,1+\epsilon}^{s_1,1-\epsilon}, M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon}\right)\right\| = \left\|\sum_{\sigma} \left\{\|\square_k^{1-\epsilon} e^{i\mu_\sigma(D)}\|_{L^{1+\epsilon}\to L^{1+2\epsilon}}\right\}\right\|_{l_{1+\epsilon}^{s_1,1-\epsilon}\to l_{1+2\epsilon}^{s_2,1-\epsilon}}$$

$$\sim \left\|\left\{\langle k\rangle^{\left(\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}\right)} A_{1+\epsilon,k}\right\}\right\|_{l_r^{s_2-s_1,1-\epsilon}}$$

$$\sim \sum_{\sigma} \|\mathcal{F}^{-1} e^{i\mu_\sigma}\|_{M_{1+\epsilon,r}^{s,1-\epsilon}}. \quad (4.70)$$

By the spirit of Theorem 4.8, we give the following theorem (see [27]).

Theorem 4.9. Let $\beta > 0$ and let μ_σ be a real-valued $C^\infty(\mathbb{R}^n \setminus \{0\})$ function which is homogeneous of degree β. Suppose that the Hessian matrix of μ_σ is non-degenerate on $\mathbb{R}^n \setminus \{0\}$. Let $0 \leq \epsilon \leq \infty, s_i \in \mathbb{R}, 0 \leq \epsilon \leq 1$ for $i = 1, 2$,

$$S_{1+\epsilon} = S_{1+\epsilon}(\beta) = n\left(\frac{1-\epsilon}{2(1+\epsilon)}\right) \max\{\beta - 2\epsilon, 0\}.$$

Then the Fourier multiplier $e^{i\mu_\sigma(D)}$ is bounded from $M_{1+\epsilon,1+\epsilon}^{s_1,1-\epsilon}$ to $M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon}$ if and only if

$$\begin{cases} \epsilon \geq 0 \\ s_2 - \dfrac{n(1-\epsilon)}{1+2\epsilon} + \max\left\{S_{1+2\epsilon}, S_{\frac{1+\epsilon}{\epsilon}}\right\} \leq s_1 - \dfrac{n(1-\epsilon)}{1+\epsilon} \end{cases} \quad (4.71)$$

or

$$\begin{cases} \epsilon \geq 0 \\ s_2 - \dfrac{n(1-\epsilon)}{1+2\epsilon} + \max\left\{S_{1+2\epsilon}, S_{\frac{1+\epsilon}{\epsilon}}\right\} + \dfrac{n(\epsilon)}{1+2\epsilon} \leq s_1 - \dfrac{n(1-\epsilon)+n\epsilon}{1+\epsilon} \end{cases} \quad (4.72)$$

holds.

Proof. We will sketch the proof for case $0 < \epsilon \leq 1$, the proof of case $\epsilon = 0$ is similar. Using Lemma 4.3 and Lemma 4.6, we use the assumption of μ_σ to verify

$$\begin{cases} \|\sum_{\sigma} \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi) e^{i(1+\epsilon)\mu_\sigma(\xi)}\right]\|_{L^1} \lesssim \left(1 + \langle k\rangle^{\frac{\beta-2\epsilon}{\epsilon}}\right)^{\frac{n}{2}} \\ \|\sum_{\sigma} \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi) e^{i(1+\epsilon)\mu_\sigma(\xi)}\right]\|_{L^\infty} \lesssim \langle k\rangle^{\frac{(1-\epsilon)n}{\epsilon}} \left(1 + \langle k\rangle^{\frac{\beta-2\epsilon}{\epsilon}}\right)^{-\frac{n}{2}}, \end{cases} \quad (4.73)$$

for all $k \in \mathbb{Z}^n$. Then we have

$$\| \sum_{\sigma} \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i(1+\epsilon)\mu_\sigma(\xi)}\right]\|_{L^1}\| \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i(1+\epsilon)\mu_\sigma(\xi)}\right]\|_{L^\infty}$$
$$\lesssim \left(1 + \langle k\rangle^{\frac{\beta-2\epsilon}{\epsilon}}\right)^{\frac{n}{2}} \langle k\rangle^{\frac{(1-\epsilon)n}{\epsilon}}\left(1 + \langle k\rangle^{\frac{\beta-2\epsilon}{\epsilon}}\right)^{-\frac{n}{2}}$$
$$\lesssim \langle k\rangle^{\frac{(1-\epsilon)n}{\epsilon}} \sim \sum_{\sigma} \| \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i(1+\epsilon)\mu_\sigma(\xi)}\right]\|_{L^2}^2. \quad (4.74)$$

Denote

$$(1+\epsilon) = \begin{cases} 1+2\epsilon & \text{if} \frac{1}{1+2\epsilon} \geq \frac{\epsilon}{1+\epsilon} \\ \frac{1+\epsilon}{\epsilon} & \text{if} \frac{1}{1+2\epsilon} \leq \frac{\epsilon}{1+\epsilon} \end{cases}, \qquad A_{1+\epsilon,k} = \frac{\sum_\sigma \| \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi)e^{i\mu_\sigma(\xi)}\right]\|_{L^{1+\epsilon}}}{\langle k\rangle^{\left(\frac{(1-\epsilon)n}{1+\epsilon}\right)}}. \quad (4.75)$$

We have

$$A_{1+\epsilon,k} \sim \left(1 + \langle k\rangle^{\frac{\beta-2\epsilon}{\epsilon}}\right)^{n\left(\frac{1-\epsilon}{2(1+\epsilon)}\right)} \sim \langle k\rangle^{S_{1+\epsilon}}. \quad (4.76)$$

Using Theorem 4.8, we obtain

$$\left\|\sum_{\sigma} e^{i\mu_\sigma(D)}\right\|_{M^{s_1,1-\epsilon}_{1+\epsilon,1+\epsilon}\to M^{s_2,1-\epsilon}_{1+2\epsilon,1+2\epsilon}} \sim \left\|\left\{\langle k\rangle^{\left(\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}\right)}A_{1+\epsilon,k}\right\}\right\|_{l^{s_1,1-\epsilon}_{1+\epsilon}\to l^{s_2,1-\epsilon}_{1+2\epsilon}}. \quad (4.77)$$

So we have

$$\left\|\left\{\langle k\rangle^{\left(\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}\right)}A_{1+\epsilon,k}\right\}\right\|_{l^{s_1,1-\epsilon}_{1+\epsilon}\to l^{s_2,1-\epsilon}_{1+2\epsilon}} \sim \left\|\left\{\| \square_k^{1-\epsilon} \|_{L^{1+\epsilon}\to L^{1+2\epsilon}} A_{1+\epsilon,k}\right\}\right\|_{l^{s_1,1-\epsilon}_{1+\epsilon}\to l^{s_2,1-\epsilon}_{1+2\epsilon}}$$
$$\sim \left\|\left\{\langle k\rangle^{S_{1+\epsilon}} \| \square_k^{1-\epsilon} \|_{L^{1+\epsilon}\to L^{1+2\epsilon}}\right\}\right\|_{l^{s_1,1-\epsilon}_{1+\epsilon}\to l^{s_2,1-\epsilon}_{1+2\epsilon}}$$
$$\sim \|I\|_{M^{s_1,1-\epsilon}_{1+\epsilon,1+\epsilon}\to M^{s_2+S_{1+\epsilon},1-\epsilon}_{1+2\epsilon,1+2\epsilon}}. \quad (4.78)$$

Noticing $S_{1+\epsilon} = \max\{S_{1+2\epsilon}, S_{\frac{1+\epsilon}{\epsilon}}\}$, we use Proposition 4.1 to obtain the final conclusion.

By using Theorem 4.8, one can also obtain the asymptotic estimates about unimodular Fourier multipliers with parameters, if the inverse Hölder conditions is uniform with the parameters. We give the following theorem for the asymptotic estimates of free dispersive semigroups.

Theorem 4.10 (see [27]). Let $\beta > 0$ and let μ_σ be a real-valued $C^\infty(\mathbb{R}^n \setminus \{0\})$ function which is homogeneous of degree β. Suppose that the Hessian matrix of μ_σ is non-degenerate on $\mathbb{R}^n \setminus \{0\}$. Let $1 \leq \epsilon \leq \infty, s_i \in \mathbb{R}, 0 \leq \epsilon \leq 1$ for $i = 1, 2$. Denote

$$(1+\epsilon) = \begin{cases} 1+2\epsilon & \text{if} \frac{1}{1+2\epsilon} \geq \frac{\epsilon}{1+\epsilon} \\ \frac{1+\epsilon}{\epsilon} & \text{if} \frac{1}{1+2\epsilon} \leq \frac{\epsilon}{1+\epsilon} \end{cases}, \qquad S_{1+\epsilon} = S_{1+\epsilon}(\beta) = n(\frac{1-\epsilon}{2(1+\epsilon)}) \max\{\beta - 2\epsilon, 0\}. \quad (4.79)$$

Suppose

$$s_1 = s_2 + n(1-\epsilon)\left(\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}\right) + S_{1+\epsilon} + n(\epsilon)\max\{\frac{-\epsilon}{(1+2\epsilon)(1+\epsilon)}, 0\} + \delta, \quad (4.80)$$

where $\delta \geq 0$ if $\epsilon \geq 0, \delta > 0$ if $\epsilon < 0$. Denote

$$A = -\delta - S_{1+\epsilon} + (\beta - 2\epsilon)n\left(\frac{1-\epsilon}{2(1+\epsilon)}\right),$$
$$B = -(\beta - 2\epsilon), \qquad (1+\epsilon) = n\left(\frac{1-\epsilon}{2(1+\epsilon)}\right).$$

We have the following asymptotic estimates for $e^{i(1+\epsilon)\mu_\sigma(D)}, 0 < \epsilon < \infty$:

$$\left\|\sum_{\sigma} e^{i(1+\epsilon)\mu_\sigma} | \mathcal{M}_{\mathcal{J}}\left(M^{s_1,1-\epsilon}_{1+\epsilon,1+\epsilon}, M^{s_2,1-\epsilon}_{1+2\epsilon,1+2\epsilon}\right)\right\| \sim \begin{cases} (1+\epsilon)^{\frac{A}{B}+(1+\epsilon)}, & B < 0, \quad A + B(1+\epsilon) > 0 \\ (\ln (1+\epsilon)^{-1})^{\max\{\frac{-\epsilon}{(1+2\epsilon)(1+\epsilon)}, 0\}}, & B < 0, \quad A + B(1+\epsilon) = 0 \\ 1, & otherwise \end{cases} \quad (4.81)$$

as $\epsilon \to 0^+$, and

$$\left\|\sum_{\sigma} e^{i(1+\epsilon)\mu_\sigma} | \mathcal{M}_{\mathcal{J}}\left(M^{s_1,1-\epsilon}_{1+\epsilon,1+\epsilon}, M^{s_2,1-\epsilon}_{1+2\epsilon,1+2\epsilon}\right)\right\| \sim \begin{cases} (1+\epsilon)^{\frac{A}{B}+(1+\epsilon)}, & B > 0, \quad A > 0 \\ t^{(1+\epsilon)}(\ln (1+\epsilon)^{-1})^{\max\{\frac{-\epsilon}{(1+2\epsilon)(1+\epsilon)}, 0\}}, & B > 0, \quad A = 0 \\ t^{(1+\epsilon)}, & otherwise \end{cases} \quad (4.82)$$

as $\epsilon \to \infty$.

Proof. We only sketch the proof for $\epsilon < 1$, since the case $\epsilon = 0$ can be handled similarly. We divide this proof into two cases.

Case 1. $\epsilon \geq 0$.

In this case, we have

$$\left\|\sum_{\sigma} e^{i(1+\epsilon)\mu_\sigma} | \mathcal{M}_{\mathcal{F}}\left(M^{s_1,1-\epsilon}_{1+\epsilon,1+\epsilon}, M^{s_2,1-\epsilon}_{1+2\epsilon,1+2\epsilon}\right)\right\| \sim \sup_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{s_2-s_1}{\epsilon}} \sum_{\sigma} \| \square_k^{1-\epsilon} e^{i(1+\epsilon)\mu_\sigma(D)} \|_{L^{1+\epsilon}\to L^{1+2\epsilon}}. \quad (4.83)$$

We denote

$$A_{1+\epsilon,k}(1+\epsilon) = \frac{\sum_\sigma \| \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi) e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^{1+\epsilon}}}{\langle k\rangle^{\left(\frac{(1-\epsilon)n}{1+\epsilon}\right)}}. \quad (4.84)$$

Using Lemmas 4.3 and 4.6, we use the assumption of μ_σ to verify

$$\begin{cases} \| \sum_{\sigma} \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi) e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^1} \lesssim \left(1 + (1+\epsilon)\langle k\rangle^{\frac{\beta-2\epsilon}{\epsilon}}\right)^{\frac{n}{2}} \\ \| \sum_{\sigma} \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi) e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^\infty} \lesssim \langle k\rangle^{\frac{(1-\epsilon)n}{\epsilon}}\left(1 + (1+\epsilon)\langle k\rangle^{\frac{\beta-2\epsilon}{\epsilon}}\right)^{-\frac{n}{2}}, \end{cases} \quad (4.85)$$

for all $k \in \mathbb{Z}^n$. Observing that

$$\| \sum_{\sigma} \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi) e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^1} \| \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi) e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^\infty} \lesssim \| \sum_{\sigma} \mathcal{F}^{-1}\left[\eta_k^{1-\epsilon}(\xi) e^{i(1+\epsilon)\mu_\sigma(\xi)}\right] \|_{L^2}^2, \quad (4.86)$$

we deduce

$$A_{1+\epsilon,k} \sim A^{1-\theta}_{1+2\epsilon,k} A^{\theta}_{1+3\epsilon,k}, \quad (4.87)$$

for any $0 \leq \epsilon \leq \infty, \theta \in [0,1], (1+\epsilon) = (1-\theta)(1+2\epsilon) + \theta(1+3\epsilon)$, it follows that

$$A_{1+\epsilon,k}(1+\epsilon) \sim \left(1 + (1+\epsilon)\langle k\rangle^{\frac{\beta-2\epsilon}{\epsilon}}\right)^{n\left(\frac{1-\epsilon}{2(1+\epsilon)}\right)}. \quad (4.88)$$

On the other hand, by the proof of Theorem 4.8, we have

$$\| \sum_{\sigma} \square_k^{1-\epsilon} e^{i(1+\epsilon)\mu_\sigma(D)} \|_{L^{1+\epsilon}\to L^{1+2\epsilon}} \sim \langle k\rangle^{\left(\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}\right)} A_{1+\epsilon,k}(1+\epsilon). \quad (4.89)$$

By a direct calculation, we have

$$\begin{aligned} \langle k\rangle^{\frac{s_2-s_1}{\epsilon}} \| \sum_{\sigma} \square_k^{1-\epsilon} e^{i(1+\epsilon)\mu_\sigma(D)} \|_{L^{1+\epsilon}\to L^{1+2\epsilon}} &\sim \langle k\rangle^{\frac{s_2-s_1}{\epsilon}} \langle k\rangle^{\left(\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}\right)} A_{1+\epsilon,k}(1+\epsilon) \\ &\sim \langle k\rangle^{\frac{s_2-s_1}{\epsilon}} \langle k\rangle^{\left(\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}\right)} \left((2+\epsilon)\langle k\rangle^{\frac{\beta-2\epsilon}{\epsilon}}\right)^{n\left(\frac{1-\epsilon}{2(1+\epsilon)}\right)} \\ &= \langle k\rangle^{\frac{A}{\epsilon}} \left(\langle k\rangle^{\frac{\beta}{\epsilon}} + (1+\epsilon)\right)^{1+\epsilon}. \end{aligned} \quad (4.90)$$

Thus,

$$\left\|\sum_{\sigma} e^{i(1+\epsilon)\mu_\sigma} | \mathcal{M}_{\mathcal{F}}\left(M^{s_1,1-\epsilon}_{1+\epsilon,1+\epsilon}, M^{s_2,1-\epsilon}_{1+2\epsilon,1+2\epsilon}\right)\right\| \sim \sup_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{A}{\epsilon}} \left(\langle k\rangle^{\frac{\beta}{\epsilon}} + (1+\epsilon)\right)^{1+\epsilon}. \quad (4.91)$$

For the proof of (4.81). If $B \geq 0$, we have $\langle k\rangle^{\frac{\beta}{\epsilon}} + (1+\epsilon) \sim \langle k\rangle^{\frac{\beta}{\epsilon}}$ as $\epsilon \to 0^+$. Then

$$\left\|\sum_{\sigma} e^{i(1+\epsilon)\mu_\sigma} | \mathcal{M}_{\mathcal{F}}\left(M^{s_1,1-\epsilon}_{1+\epsilon,1+\epsilon}, M^{s_2,1-\epsilon}_{1+2\epsilon,1+2\epsilon}\right)\right\| \sim \sup_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{A+B(1+\epsilon)}{\epsilon}} = 1. \quad (4.92)$$

If $B < 0$, we have $A \leq 0$,

$$\begin{aligned} &\sup_{\langle k\rangle^{\frac{\beta}{\epsilon}} \leq \frac{(1+\epsilon)}{2}} \langle k\rangle^{\frac{A}{\epsilon}} \left(\langle k\rangle^{\frac{\beta}{\epsilon}} + (1+\epsilon)\right)^{1+\epsilon} \\ &\sim \sup_{\langle k\rangle^{\frac{\beta}{\epsilon}} \leq \frac{(1+\epsilon)}{2}} \langle k\rangle^{\frac{A}{\epsilon}} t^{1+\epsilon} \sim (1+\epsilon)^{\frac{A}{B}+(1+\epsilon)}, \end{aligned} \quad (4.93)$$

$$\sup_{\langle k\rangle^{\frac{\beta}{\epsilon}}\sim(1+\epsilon)} \langle k\rangle^{\frac{A}{\epsilon}}\left(\langle k\rangle^{\frac{\beta}{\epsilon}}+(1+\epsilon)\right)^{1+\epsilon} \sim \sup_{\langle k\rangle^{\frac{\beta}{\epsilon}}\sim(1+\epsilon)} \langle k\rangle^{\frac{A}{\epsilon}} t^{1+\epsilon} \sim (1+\epsilon)^{\frac{A}{B}+(1+\epsilon)}, \tag{4.94}$$

and

$$\sup_{\langle k\rangle^{\frac{\beta}{\epsilon}}\geq 2(1+\epsilon)} \langle k\rangle^{\frac{A}{\epsilon}}\left(\langle k\rangle^{\frac{\beta}{\epsilon}}+(1+\epsilon)\right)^{1+\epsilon} \sim \sup_{\langle k\rangle^{\frac{\beta}{\epsilon}}\geq 2(1+\epsilon)} \langle k\rangle^{\frac{A+B(1+\epsilon)}{\epsilon}} \sim \begin{cases} 1, & A+B(1+\epsilon)\leq 0 \\ (1+\epsilon)^{\frac{A}{B}+(1+\epsilon)}, & A+B(1+\epsilon)>0. \end{cases} \tag{4.95}$$

Then we have

$$\sup_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{A}{\epsilon}}\left(\langle k\rangle^{\frac{\beta}{\epsilon}}+(1+\epsilon)\right)^{1+\epsilon} = \begin{cases} (1+\epsilon)^{\frac{A}{B}+(1+\epsilon)}, & B<0,\quad A+B(1+\epsilon)>0, \\ 1, & otherwise. \end{cases} \tag{4.96}$$

For the proof of (4.82). If $B\leq 0$, we have $\langle k\rangle^{\frac{\beta}{\epsilon}}+(1+\epsilon)\sim(1+\epsilon)$ as $\epsilon\to\infty$. Then

$$\left\|\sum_{\sigma} e^{i(1+\epsilon)\mu_\sigma}|\,\mathcal{M}_{\mathfrak{F}}\left(M^{s_1,1-\epsilon}_{1+\epsilon,1+\epsilon}, M^{s_2,1-\epsilon}_{1+2\epsilon,1+2\epsilon}\right)\right\| \sim \sup_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{A}{\epsilon}} t^{1+\epsilon} = t^{1+\epsilon}. \tag{4.97}$$

If $B>0$, we have

$$\sup_{\langle k\rangle^{\frac{\beta}{\epsilon}}\leq\frac{(1+\epsilon)}{2}} \langle k\rangle^{\frac{A}{\epsilon}}\left(\langle k\rangle^{\frac{\beta}{\epsilon}}+(1+\epsilon)\right)^{1+\epsilon} \sim \sup_{\langle k\rangle^{\frac{\beta}{\epsilon}}\leq\frac{(1+\epsilon)}{2}} \langle k\rangle^{\frac{A}{\epsilon}} t^{1+\epsilon} \sim \begin{cases} t^{1+\epsilon}, & A\leq 0 \\ (1+\epsilon)^{\frac{A}{B}+(1+\epsilon)}, & A>0 \end{cases} \tag{4.98}$$

and

$$\sup_{\langle k\rangle^{\frac{\beta}{\epsilon}}\sim(1+\epsilon)} \langle k\rangle^{\frac{A}{\epsilon}}\left(\langle k\rangle^{\frac{\beta}{\epsilon}}+(1+\epsilon)\right)^{1+\epsilon} \sim \sup_{\langle k\rangle^{\frac{\beta}{\epsilon}}\sim(1+\epsilon)} \langle k\rangle^{\frac{A}{\epsilon}} t^{1+\epsilon} \sim (1+\epsilon)^{\frac{A}{B}+(1+\epsilon)}. \tag{4.99}$$

In addition, observing that $A+B(1+\epsilon)=-\delta\leq 0$ in this case, we obtain

$$\sup_{\langle k\rangle^{\frac{\beta}{\epsilon}}\geq 2(1+\epsilon)} \langle k\rangle^{\frac{A}{\epsilon}}\left(\langle k\rangle^{\frac{\beta}{\epsilon}}+(1+\epsilon)\right)^{1+\epsilon} \sim \sup_{\langle k\rangle^{\frac{\beta}{\epsilon}}\geq 2(1+\epsilon)} \langle k\rangle^{\frac{A+B(1+\epsilon)}{\epsilon}} \sim (1+\epsilon)^{\frac{A}{B}+(1+\epsilon)}. \tag{4.100}$$

Then we have

$$\sup_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{A}{\epsilon}}\left(\langle k\rangle^{\frac{\beta}{\epsilon}}+(1+\epsilon)\right)^{1+\epsilon} = \begin{cases} (1+\epsilon)^{\frac{A}{B}+(1+\epsilon)}, & B>0,\quad A>0, \\ t^{1+\epsilon}, & otherwise. \end{cases} \tag{4.101}$$

Case 2. $\epsilon<0$.

In this case, we denote $\frac{1}{r}=\frac{-\epsilon}{(1+2\epsilon)(1+\epsilon)}$. Using Theorem 3.1, we deduce

$$\left\|\sum_{\sigma} e^{i(1+\epsilon)\mu_\sigma}|\,\mathcal{M}_{\mathfrak{F}}\left(M^{s_1,1-\epsilon}_{1+\epsilon,1+\epsilon}, M^{s_2,1-\epsilon}_{1+2\epsilon,1+2\epsilon}\right)\right\| \sim \sum_{\sigma} \left\|\left\{\|\,\square_k^{1-\epsilon} e^{i\mu_\sigma(D)}\,\|_{L^{1+\epsilon}\to L^{1+2\epsilon}}\right\}\right\|_{l^{s_1,1-\epsilon}_{1+\epsilon}\to l^{s_2,1-\epsilon}_{1+2\epsilon}}$$

$$\sim \sum_{\sigma} \left\| \left\{ \langle k \rangle^{\frac{s_2 - s_1}{\epsilon}} \| \square_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^{1+\epsilon} \to L^{1+2\epsilon}} \right\} \right\|_{l_r}. \quad (4.102)$$

As in Case 1, we obtain

$$\| \sum_{\sigma} \square_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^{1+\epsilon} \to L^{1+2\epsilon}} \sim \langle k \rangle^{\left(\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}\right)} \left((2+\epsilon) \langle k \rangle^{\frac{\beta - 2\epsilon}{\epsilon}} \right)^{n\left(\frac{1-\epsilon}{2(1+\epsilon)}\right)}. \quad (4.103)$$

Write

$$\langle k \rangle^{\frac{s_2 - s_1}{\epsilon}} \| \sum_{\sigma} \square_k^{1-\epsilon} e^{i\mu_\sigma(D)} \|_{L^{1+\epsilon} \to L^{1+2\epsilon}} \sim \langle k \rangle^{\frac{s_2 - s_1}{\epsilon}} \langle k \rangle^{\left(\frac{(1-\epsilon)n}{(1+\epsilon)(1+2\epsilon)}\right)} \left((2+\epsilon) \langle k \rangle^{\frac{\beta - 2\epsilon}{\epsilon}} \right)^{n\left(\frac{1-\epsilon}{2(1+\epsilon)}\right)}$$

$$= \langle k \rangle^{-\frac{n}{r} + \frac{A}{\epsilon}} \left(\langle k \rangle^{\frac{\beta}{\epsilon}} + (1+\epsilon) \right)^{1+\epsilon}. \quad (4.104)$$

Thus, we have

$$\left\| \sum_{\sigma} e^{i(1+\epsilon)\mu_\sigma} | \mathcal{M}_{\mathfrak{F}}\left(M_{1+\epsilon,1+\epsilon}^{s_1,1-\epsilon}, M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon} \right) \right\|$$

$$\sim \left\| \left\{ \langle k \rangle^{-\frac{n}{r} + \frac{A}{\epsilon}} \left(\langle k \rangle^{\frac{\beta}{\epsilon}} + (1+\epsilon) \right)^{1+\epsilon} \right\} \right\|_{l_r}. \quad (4.105)$$

For the proof of (4.81). If $B \geq 0$, we have $\langle k \rangle^{\frac{\beta}{\epsilon}} + (1+\epsilon) \sim \langle k \rangle^{\frac{\beta}{\epsilon}}$ as $\epsilon \to 0^+$. In addition, we have $A + B(1+\epsilon) = -\delta < 0$ by the fact $S_{1+\epsilon} = 0$ in this case. Thus

$$\left\| \sum_{\sigma} e^{i(1+\epsilon)\mu_\sigma} | \mathcal{M}_{\mathfrak{F}}\left(M_{1+\epsilon,1+\epsilon}^{s_1,1-\epsilon}, M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon} \right) \right\| \sim \left\| \left\{ \langle k \rangle^{-\frac{n}{r} + \frac{A+B(1+\epsilon)}{\epsilon}} \right\} \right\|_{l_r} \sim 1. \quad (4.106)$$

If $B < 0$, we have $A = -\delta < 0$. By a direct calculation, we obtain

$$\sum_{\langle k \rangle^{\frac{\beta}{\epsilon}} \leq \frac{(1+\epsilon)}{2}} \langle k \rangle^{-n + \frac{Ar}{\epsilon}} \left(\langle k \rangle^{\frac{\beta}{\epsilon}} + (1+\epsilon) \right)^{(1+\epsilon)r}$$

$$\sim \sum_{\langle k \rangle^{\frac{\beta}{\epsilon}} \leq \frac{(1+\epsilon)}{2}} \langle k \rangle^{-n + \frac{Ar}{\epsilon}} (1+\epsilon)^{(1+\epsilon)r} \sim (1+\epsilon)^{\left(\frac{A}{B} + (1+\epsilon)\right)r}, \quad (4.107)$$

$$\sum_{\frac{(1+\epsilon)}{2} < \langle k \rangle^{\frac{\beta}{\epsilon}} < 2(1+\epsilon)} \langle k \rangle^{-n + \frac{Ar}{\epsilon}} \left(\langle k \rangle^{\frac{\beta}{\epsilon}} + (1+\epsilon) \right)^{(1+\epsilon)r}$$

$$\sim \sum_{(1+\epsilon)/2 < \langle k \rangle^{\frac{\beta}{\epsilon}} < 2(1+\epsilon)} \langle k \rangle^{-n + \frac{Ar}{\epsilon}} (1+\epsilon)^{(1+\epsilon)r} \sim (1+\epsilon)^{\left(\frac{A}{B} + (1+\epsilon)\right)r}, \quad (4.108)$$

We also have

$$\sum_{\langle k \rangle^{\frac{\beta}{\epsilon}} \geq 2(1+\epsilon)} \langle k \rangle^{-n + \frac{Ar}{\epsilon}} \left(\langle k \rangle^{\frac{\beta}{\epsilon}} + (1+\epsilon) \right)^{(1+\epsilon)r}$$

$$\sim \sum_{\langle k \rangle^{\frac{\beta}{\epsilon}} \geq 2(1+\epsilon)} \langle k \rangle^{-n + \frac{(A+B(1+\epsilon))r}{\epsilon}}, \quad (4.109)$$

Where

$$\sum_{\langle k \rangle^{\frac{\beta}{\epsilon}} \geq 2(1+\epsilon)} \langle k \rangle^{-n + \frac{(A+B(1+\epsilon))r}{\epsilon}} \sim \begin{cases} 1, & A + B(1+\epsilon) < 0, \\ \ln (1+\epsilon)^{-1}, & A + B(1+\epsilon) = 0, \\ (1+\epsilon)^{\left(\frac{A}{B} + (1+\epsilon)\right)r}, & A + B(1+\epsilon) > 0. \end{cases} \quad (4.110)$$

Thus, we have

$$\left\| \sum_{\sigma} e^{i(1+\epsilon)\mu_\sigma} | \mathcal{M}_{\mathfrak{F}}\left(M_{1+\epsilon,1+\epsilon}^{s_1,1-\epsilon}, M_{1+2\epsilon,1+2\epsilon}^{s_2,1-\epsilon} \right) \right\|$$

$$\sim \begin{cases} 1, & A + B(1+\epsilon) < 0, \\ (\ln (1+\epsilon)^{-1})^{1/r}, & A + B(1+\epsilon) = 0, \\ (1+\epsilon)^{\left(\frac{A}{B}+(1+\epsilon)\right)}, & A + B(1+\epsilon) > 0. \end{cases} \quad (4.111)$$

as $\epsilon \to 0^+$ for $B < 0$. We get the desired conclusion by the estimates of the cases $B \geq 0$ and $B < 0$.

For the proof of (4.82). If $B \leq 0$, we have $\langle k \rangle^{\frac{\beta}{\epsilon}} + 1 + \epsilon \sim 1 + \epsilon$ as $\epsilon \to \infty$. In addition, we have $A = -\delta < 0$ in this case. Thus

$$\left\| \sum_{\sigma} e^{i(1+\epsilon)\mu_\sigma} | \mathcal{M}_{\mathfrak{F}}\left(M^{s_1,1-\epsilon}_{1+\epsilon,1+\epsilon}, M^{s_2,1-\epsilon}_{1+2\epsilon,1+2\epsilon}\right) \right\| \sim \left\| \left\{ \langle k \rangle^{-\frac{n}{r}+\frac{A}{\epsilon}} t^{1+\epsilon} \right\} \right\|_{l_r} \sim t^{1+\epsilon}. \quad (4.112)$$

If $B > 0$, we have $A + B(1+\epsilon) = -\delta < 0$. By a direct calculation, we obtain

$$\sum_{\langle k \rangle^{\frac{\beta}{\epsilon}} \leq \frac{(1+\epsilon)}{2}} \langle k \rangle^{-n+\frac{Ar}{\epsilon}} \left(\langle k \rangle^{\frac{\beta}{\epsilon}} + (1+\epsilon) \right)^{(1+\epsilon)r} \sim \sum_{\langle k \rangle^{\frac{\beta}{\epsilon}} \leq (1+\epsilon)/2} \langle k \rangle^{-n+\frac{Ar}{\epsilon}} (1+\epsilon)^{(1+\epsilon)r}, \quad (4.113)$$

where

$$\sum_{\langle k \rangle^{\frac{\beta}{\epsilon}} \leq \frac{(1+\epsilon)}{2}} \langle k \rangle^{-n+\frac{Ar}{\epsilon}} \left(\langle k \rangle^{\frac{\beta}{\epsilon}} + (1+\epsilon) \right)^{(1+\epsilon)r} \sim \begin{cases} (1+\epsilon)^{(1+\epsilon)r}, & A < 0, \\ (1+\epsilon)^{(1+\epsilon)r} \ln (1+\epsilon), & A = 0, \\ (1+\epsilon)^{\left(\frac{A}{B}+(1+\epsilon)\right)r}, & A > 0. \end{cases} \quad (4.114)$$

We also have

$$\sum_{\frac{(1+\epsilon)}{2} < \langle k \rangle^{\frac{\beta}{\epsilon}} < 2(1+\epsilon)} \langle k \rangle^{-n+\frac{Ar}{\epsilon}} \left(\langle k \rangle^{\frac{\beta}{\epsilon}} + (1+\epsilon) \right)^{(1+\epsilon)r} \sim \sum_{(1+\epsilon)/2 < \langle k \rangle^{\frac{\beta}{\epsilon}} < 2(1+\epsilon)} \langle k \rangle^{-n+\frac{Ar}{\epsilon}} (1+\epsilon)^{(1+\epsilon)r} \sim (1+\epsilon)^{\left(\frac{A}{B}+(1+\epsilon)\right)r}, \quad (4.115)$$

and

$$\sum_{\langle k \rangle^{\frac{\beta}{\epsilon}} \geq 2(1+\epsilon)} \langle k \rangle^{-n+\frac{Ar}{\epsilon}} \left(\langle k \rangle^{\frac{\beta}{\epsilon}} + (1+\epsilon) \right)^{(1+\epsilon)r} \sim \sum_{\langle k \rangle^{\frac{\beta}{\epsilon}} \geq 2(1+\epsilon)} \langle k \rangle^{-n+\frac{(A+B(1+\epsilon))r}{\epsilon}} \sim (1+\epsilon)^{\left(\frac{A}{B}+(1+\epsilon)\right)r}. \quad (4.116)$$

Thus, we have

$$\left\| \sum_{\sigma} e^{i(1+\epsilon)\mu_\sigma} | \mathcal{M}_{\mathfrak{F}}\left(M^{s_1,(1-\epsilon)}_{1+\epsilon,1+\epsilon}, M^{s_2,(1-\epsilon)}_{1+2\epsilon,1+2\epsilon}\right) \right\| \sim \begin{cases} t^{1+\epsilon}, & A < 0, \\ t^{1+\epsilon}\left(\ln (1+\epsilon)\right)^{\frac{1}{r}}, & A = 0, \\ (1+\epsilon)^{\left(\frac{A}{B}+(1+\epsilon)\right)}, & A > 0. \end{cases} \quad (4.117)$$

as $\epsilon \to \infty$ for $B > 0$. We get our desired conclusion by the above estimates of the cases $B \leq 0$ and $B > 0$.

Comments [27]:

1. In the special case $\epsilon = 0$, the result of Theorem 4.9 was obtained in [26]. In [26], we also obtain the following asymptotic estimate

$$\left\|\sum_{\sigma} e^{i\mu_\sigma(D)}\right\|_{M^{s_1,1-\epsilon}_{1+\epsilon,1+\epsilon}\to M^{s_2,1-\epsilon}_{1+\epsilon,1+\epsilon}} \sim (1 + |1+\epsilon|)^{n\left|\frac{1-\epsilon}{2(1+\epsilon)}\right|} \quad (4.118)$$

for $s_1 - s_2 \geq S_{1+\epsilon}, (1+\epsilon) \in \mathbb{R}$.

2. A radial version of Theorem 4.9, when $\mu_\sigma(\xi) = r(|\xi|)$, can be founded in [25]. Using Theorem 3.1 or 4.8, one can also give a simple proof for the corresponding results in [25].

3. The assumption $\epsilon \geq 0$ is based on the observation that if $e^{i\mu_\sigma(D)}$ is bounded from $M^{s_1,1-\epsilon}_{1+\epsilon,1+\epsilon}$ to $M^{s_2,1-\epsilon}_{1+2\epsilon,1+2\epsilon}$, then $\square_0^{1-\epsilon} e^{i\mu_\sigma(D)}$ is bounded from $L^{1+\epsilon}$ to $L^{1+2\epsilon}$. Since $\square_0^{1-\epsilon} e^{i\mu_\sigma(D)}$ is a non-zero translation invariant operator, it is easy to deduce $\epsilon \geq 0$.

4. Using more tedious symbols and techniques, all the conclusions in this paper can be extended to a wider range $0 < \epsilon \leq \infty$ by the same methods. In order to make the discussion more clear and concise, we only choose the more interesting case $0 \leq \epsilon \leq \infty$ to discuss in this paper.

5. The result obtained in Theorem 4.2 is not sharp in the degenerate case $r < n$. By an example of separation of variables $\mu_\sigma(\xi) = \sum_{i=1}^{r} |\xi|^\beta$ where $r < n$, the expected sharp result may be

$$s_2 + r\left|\left(\frac{1-\epsilon}{2(1+\epsilon)}\right)\right| \max\{\beta - 2\epsilon, 0\} \leq s_1. \quad (4.119)$$

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